$$\int_A K dM = \int_{\varphi(A)} K dM.$$

Making use of the Gauss-Bonnet theorem we have

$$\int_A K dM + \oint k_g ds = 2\pi \quad \text{and} \quad \int_{\varphi(A)} K dM - \oint k_g ds = 2\pi.$$

Comparing all three relations we get $\oint k_g ds = 0$.

Theorem 6. *A line of shadow on an ovaloid M of constant width divides M onto two regions with equal areas if and only if the total mean curvatures of these regions are equal.*

Proof. Let $C \in LS(M)$. We denote by $S(A)$ and $S(\varphi(A))$ the surface area of the regions A and $\varphi(A)$, respectively. From the proof of Theorem 5 it follows that $\int_A K dM = 2\pi$. The formulas (15), (17) and (19) imply the relation

$$S(A) - S(\varphi(A)) = \int_A (1 - K/K(\varphi)) dM = 2h \int_A H dM - h^2 \int_A K dM =$$

$$= 2h \int_A H dM - h^2 2\pi = h(2\int_A H dM - \int_M H dM) = h(\int_A H dM - \int_{\varphi(A)} H dM).$$

It ends the proof.

We recall a definition of constant brightness.

Definition 2. *An ovaloid M is said to be of constant brightness if the area of orthogonal projection of M onto the plane does not depend on the choice of the plane.*

G. Herglotz [1] has proved the following theorem. *An ovaloid is of constant brightness if and only if*

$$1/K + 1/K(\varphi) = c = \text{const.} \tag{22}$$

Theorem 7. *The surface area S of an avaloid M of constant brightness satisfying* (22) *is given by the formula*

$$S = 2\pi c, \tag{23}$$

Proof. We have $\int_M K dM = 4\pi$ and $\int_M \frac{K}{K(\varphi)} dM = S$ (Theorem 2). From (22) we get

$$Kc = 1 + K/K(\varphi). \tag{24}$$

The integration of both sides of (24) leads us to the formula (23).

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Institut Matematyki UMCS
Pl. M. C. Sklodowskiej 1
20—031 Lublin, Poland

An. şt. Univ. „Al. I. Cuza" Iaşi
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A PSEUDO-RIEMANNIAN STRUCTURE ON THE COTANGENT BUNDLE

BY

V. OPROIU and N. PAPAGHIUC

In [2] the first author has studied the properties of a certain pseudo-Riemannian structure on the tangent bundle of a Lagrange space, using a nonlinear connection defined by the Euler-Lagrange equations. In [3] the authors have considered the same problem in the case of the cotangent bundle of a Lagrange space by using the image of the above nonlinear connection by the Legendre transformation. Remark that the obtained pseudo-Riemannian structure is very much similar to the Riemann extension considered in [4] (see also [5] pp. 258, [6]).

In the present paper we study the properties of a similar pseudo-Riemannian metric G on the cotangent bundle T^*M of a manifold M by using an arbitrary symmetric nonlinear connection on this bundle. We find the Levi Civita connection $\tilde{\nabla}$ of this pseudo-Riemannian metric. It follows that the vertical distribution on the cotangent bundle is preserved by the parallel transport defined by $\tilde{\nabla}$, while the deviation from the parallel transport of the horizontal distribution is strongly related to the nonholonomy of this distribution. We study some problems related to the curvature of the connection $\tilde{\nabla}$: Bianchi identities, the conditions under which $\tilde{\nabla}$ is flat or conformally flat and the conditions under which (T^*M, G) is a locally symmetric pseudo-Riemannian space. In all these cases we find that the connection coefficients of the nonlinear connection on T^*M are expressed as polynomials of degree at most two in the cotangential coordinates and we get the conditions which are fulfilled by the coefficients.

The manifolds, tensor fields and connections we consider in this paper are assumed to be smooth (i.e. C^∞). We use the well known summation convention, the range for the indices a, b, c, h, i, j, k, l being always $\{1, ..., n\}$. $\mathfrak{X}(M)$ denotes the set of the smooth vector fields defined on M.

I. The nonlinear connection on T^*M. Let M be an n-dimensional manifold and denote by $\pi: T^*M \to M$ its cotangent bundle. Then T^*M is a $2n$-dimensional manifold and some special local charts on T^*M, induced from local charts on M may be used. Namely, if (U, x^i), $i = 1, ..., n$, is a local chart on M then the local chart $(\pi^{-1}(U), (q, p)) = (\pi^{-1}(U), q^i, p_i)$; $i = 1, ..., n$, is induced on T^*M where $q^i = x^i \circ \pi$ and p_i are the vector space coordinates of an element from $\pi^{-1}(U)$ with respect to the local natural frame $(dx^1, ..., dx^n)$ of T^*M defined by the local chart (U, x^i). The tangent bundle TT^*M of T^*M has an integrable vector subbundle $VT^*M = \text{Ker}\, \pi_*$, called the vertical distribution on T^*M. A nonlinear connection on T^*M is defined by a distribution HT^*M, complementary to VT^*M in TT^*M (horizontal distribution)

$$TT^*M = VT^*M \oplus HT^*M. \tag{1}$$

The system of the local vector fields $(\partial/\partial p_1, \ldots, \partial/\partial p_n)$ is a local frame in VT^*M while in HT^*M we may consider the local frame $(\delta/\delta q^1, \ldots, \delta/\delta q^n)$ where:

$$\delta/\delta q^i = \partial/\partial q^i - N_{ij}\, \partial/\partial p_j. \tag{2}$$

The N_{ij}'s are the connection coefficients of the nonlinear connection defined by HT^*M on T^*M and they determine completely the chosen nonlinear connection on T^*M.

A change $(U, x^i) \to (\tilde{U}, \tilde{x}^i)$ of local charts on M determines the following change rule for the coefficients N_{ij}:

$$N_{ij} = \tilde{N}_{kh} \frac{\partial \tilde{x}^k}{\partial x^i} \frac{\partial \tilde{x}^h}{\partial x^j} - \tilde{p}_k \frac{\partial^2 \tilde{x}^k}{\partial x^i \partial x^j}. \tag{3}$$

It follows that the symmetry property $N_{ij} = N_{ji}$ is independent of the local chart used to express it. A nonlinear connection on T^*M such that the connection coefficients satisfy the symmetry property $N_{ij} = N_{ji}$ is called a symmetric nonlinear connection on T^*M. A symmetric nonlinear connection on T^*M there always exists. If $\overset{\circ}{N}_{ij}$ are the connection coefficients of an arbitrary nonlinear connection on T^*M then $N_{ij} = \frac{1}{2}(\overset{\circ}{N}_{ij} + \overset{\circ}{N}_{ji})$ are the connection coefficients of a symmetric nonlinear connection on T^*M. From now on we shall assume that the nonlinear connection we use is symmetric.

The complete integrability (or the holonomy) of the differential system on T^*M defined by HT^*M is equivalent to the vanishing of the torsion of HT^*M defined by R_{kij} where

$$[\delta/\delta q^i, \delta/\delta q^j] = -R_{kij}\, \partial/\partial p_k. \tag{4}$$

We have

$$R_{kij} = \delta N_{kj}/\delta q^i - \delta N_{ki}/\delta q^j. \tag{4'}$$

Then $R_{kij} + R_{kji} = 0$ and, due to the symmetry property of the chosen nonlinear connection we have

$$\sum_{(i,j,k)} R_{kij} = 0 \tag{5}$$

where $\sum_{(i,j,k)}$ denotes the sum consisting of three terms determined by the cyclic permutations of (i, j, k).

An M-tensor field of type (k, h) on T^*M (sometimes called a Finsler tensor field) is defined by sets of local coordinate components $T^{i_1 \ldots i_k}_{j_1 \ldots j_h}$ assigned to every induced local chart $(\pi^{-1}(U), (q, p))$ such that the transformation rule is that of the local coordinate components of a tensor field of type (k, h) on M. E.g. for an M-tensor of type $(1, 1)$ on T^*M defined by $T^i_j(q, p)$ we have the following rule $\tilde{T}^k_h(q, p) = \dfrac{\partial \tilde{x}^k}{\partial x^i} \dfrac{\partial x^j}{\partial \tilde{x}^h} T^i_j(q, p)$, where the local coordinate transformation $\tilde{x}^i = \tilde{x}^i(x^1, \ldots, x^n)$ on M induces the local coordinate transformation $\tilde{q}^i = \tilde{x}^i(q^1, \ldots, q^n)$, $\tilde{p}_i = \dfrac{\partial x^j}{\partial \tilde{x}^i} p_j$ on T^*M. In a similar way we may define the notion of a linear M-connection on T^*M as being defined by the connection coefficients obeying the transformation rule of the connection coefficients on M (see [1] for details).

Remark. A tensor field on M can be thought of as an M-tensor field on T^*M in an obvious way. Similarly a linear connection on M defines a linear M-connection on T^*M.

The local coordinate components R_{kij} given by (4') define an M-tensor field of type $(0, 3)$ on T^*M. We have also

$$[\partial/\partial p_i, \delta/\delta q^j] = \Phi^i_{jk}\, \partial/\partial p_k, \quad \Phi^i_{jk} = -\partial N_{jk}/\partial p_i. \tag{6}$$

From the change rule (3) of the connection coefficients N_{ij} of the nonlinear connection on T^*M we may get the change rule for Φ^i_{jk} and we conclude that they define a linear M-connection on T^*M.

2. The pseudo-Riemannian metric on T^*M. Consider the following pseudo-Riemannian metric G on T^*M of Riemann extension type

$$G = 2\delta p_i dq^i \tag{7}$$

where

$$\delta p_i = dp_i + N_{ij} dq^j. \tag{8}$$

It follows

$$G\left(\frac{\partial}{\partial p_i}, \frac{\partial}{\partial p_j}\right) = 0, \quad G\left(\frac{\delta}{\delta q^i}, \frac{\delta}{\delta q^j}\right) = 0,$$
$$G\left(\frac{\partial}{\partial p_i}, \frac{\delta}{\delta q^j}\right) = G\left(\frac{\delta}{\delta q^j}, \frac{\partial}{\partial p_i}\right) = \delta^i_j \tag{7'}$$

i.e. the distributions HT^*M, VT^*M on T^*M are both isotropic with respect to G. Moreover, G determines an isomorphism of VT^*M and HT^*M, such that HT^*M may be considered in a certain sense as the dual vector bundle of VT^*M. The signature of G is (n, n). The system of local 1-forms $(\delta p_i, dq^i)$; $i = 1, \ldots, n$, is the local dual frame of the local frame $(\partial/\partial p_i, \delta/\delta q^i)$; $i = 1, \ldots, n$, on T^*M adapted to the direct sum decomposition (1).

Denote by $\tilde{\nabla}$ the Levi Civita connection of the pseudo-Riemannian metric G on T^*M. We shall use the following notations

$$\tilde{\nabla}^i = \tilde{\nabla}_{\partial/\partial p_i}, \qquad \tilde{\nabla}_i = \tilde{\nabla}_{\delta/\delta q^i}$$

for the covariant derivatives of $\tilde{\nabla}$ with respect to $\partial/\partial p_i$ and $\delta/\delta q^i$. Then the following result is proved by a straightforward computation:

Proposition 1. *The local coordinate expression of* $\tilde{\nabla}$ *is*

$$\text{(9)}\quad \begin{aligned} &\tilde{\nabla}^i \frac{\partial}{\partial p_j} = 0,\ \tilde{\nabla}_i \frac{\partial}{\partial p_j} = -\Phi^j_{ik}\frac{\partial}{\partial p_k},\ \tilde{\nabla}^i \frac{\delta}{\delta q^j} = 0,\\ &\tilde{\nabla}_i \frac{\delta}{\delta q^j} = \Phi^k_{ij}\frac{\delta}{\delta q^k} + R_{ijk}\frac{\partial}{\partial p_k}. \end{aligned}$$

Remark. The vertical distribution VT^*M is preserved by the parallel transport defined by $\tilde{\nabla}$ on T^*M. From the last formula (9) we see that the coefficients R_{ijk} expressing the nonholonomy of HT^*M give also the deviation from the parallel transport of HT^*M by $\tilde{\nabla}$ on T^*M.

Remark. The covariant differentiation with respect to the cotangential coordinates p on T^*M does coincide with the usual partial differentiation.

By projecting the connection $\tilde{\nabla}$ on VT^*M and HT^*M we get the Schouten connection $\bar{\nabla}$ defined by $\tilde{\nabla}$ and the direct sum decomposition (1). The local coordinate expression of $\bar{\nabla}$ is

$$\text{(10)}\quad \bar{\nabla}^i \frac{\partial}{\partial p_j} = 0,\ \bar{\nabla}_i \frac{\partial}{\partial p_j} = -\Phi^j_{ik}\frac{\partial}{\partial p_k},\ \bar{\nabla}^i\frac{\delta}{\delta q^j} = 0,\ \bar{\nabla}_i \frac{\delta}{\delta q^j} = \Phi^k_{ij}\frac{\delta}{\delta q^k}.$$

It follows once again that HT^*M may be viewed as the dual vector bundle of VT^*M. Moreover, the covariant derivative defined by the Schouten connection $\bar{\nabla}$ may be interpreted as being determined by the linear M-connection defined by (6). Then, in the case of the M-tensor field on T^*M defined by R_{kij} we may get a new M-tensor field on T^*M defined by $\bar{\nabla}_h R_{kij}$ where

$$\text{(11)}\quad \bar{\nabla}_h R_{kij} = \frac{\delta R_{kij}}{\delta q^h} - \Phi^l_{hk}R_{lij} - \Phi^l_{hi}R_{klj} - \Phi^l_{hj}R_{kil}.$$

In a similar way we may get from any M-tensor field on T^*M a new M-tensor field on T^*M by using $\bar{\nabla}$.

2. The curvature tensor field. The curvature tensor field K of the connection $\tilde{\nabla}$ is defined by the formula $K(X, Y)Z = \tilde{\nabla}_X\tilde{\nabla}_Y Z - \tilde{\nabla}_Y\tilde{\nabla}_X Z - \tilde{\nabla}_{[X,Y]}Z$; $X, Y, Z \in \mathcal{X}(T^*M)$. From this definition we get by a straightforward computation.

Proposition 2. *The local coordinate expression of* K *is given by*

$$\text{(12)}\quad \begin{aligned} &K\left(\frac{\partial}{\partial p_i}, \frac{\partial}{\partial p_j}\right)\frac{\partial}{\partial p_k} = K\left(\frac{\partial}{\partial p_i}, \frac{\partial}{\partial p_j}\right)\frac{\delta}{\delta q^k} = 0,\ K\left(\frac{\partial}{\partial p_i}, \frac{\delta}{\delta q^j}\right)\frac{\partial}{\partial p_k} =\\ &= -K^{ik}_{jh}\frac{\partial}{\partial p_h},\ K\left(\frac{\delta}{\delta q^i}, \frac{\delta}{\delta q^j}\right)\frac{\partial}{\partial p_k} = -K^k_{hij}\frac{\partial}{\partial p_h},\ K\left(\frac{\partial}{\partial p_i}, \frac{\delta}{\delta q^j}\right)\frac{\delta}{\delta q^k} =\\ &= K^{ih}_{jk}\frac{\delta}{\delta q^h} - K^i_{jkh}\frac{\partial}{\partial p_h},\ K\left(\frac{\delta}{\delta q^i}, \frac{\delta}{\delta q^j}\right)\frac{\delta}{\delta q^k} = K^h_{kij}\frac{\delta}{\delta q^h} + K_{hkij}\frac{\partial}{\partial p_h}, \end{aligned}$$

where

$$\text{(13)}\quad \begin{aligned} &K^{ik}_{jh} = \frac{\partial \Phi^i_{jh}}{\partial p_k} = -\frac{\partial^2 N_{jh}}{\partial p_k \partial p_i},\ K_{hkij} = -\bar{\nabla}_i R_{jhk} + \bar{\nabla}_j R_{ihk},\\ &K^h_{kij} = \frac{\delta \Phi^h_{kj}}{\delta q^i} - \frac{\delta \Phi^h_{ki}}{\delta q^j} + \Phi^h_{il}\Phi^l_{kj} - \Phi^h_{jl}\Phi^l_{ki} = -\frac{\partial R_{kij}}{\partial p_h}. \end{aligned}$$

It follows that K^{ik}_{jh}, K^h_{kij}, K_{hkij} define M-tensor fields on T^*M.

Remark. The pseudo-Riemannian space (T^*M, G) cannot have nonvanishing constant sectional curvature.

Theorem 3. *The connection* $\tilde{\nabla}$ *is flat if and only if the symmetric nonlinear connection defined by* N_{ij} *is given by*

$$\text{(14)}\quad N_{ij} = t_{ij} - \Gamma^k_{ij}p_k$$

where

a) *the coefficients* t_{ij}, Γ^k_{ij} *are independent of* p,

b) Γ^k_{ij} *are the connection coefficients of a flat linear connection* ∇ *on* M *and* t_{ij} *are the local coordinate components of a symmetric tensor field of type* (0, 2) *on* M, *satisfying*

$$\text{(15)}\quad \nabla_i(\nabla_h t_{jk} - \nabla_k t_{jh}) - \nabla_j(\nabla_h t_{ik} - \nabla_k t_{ih}) = 0.$$

Proof. From the condition $K^{ik}_{jh} = 0$ for $\tilde{\nabla}$ to be flat we get the expression (14) of N_{ij} where Γ^k_{ij}, t_{ij} are independent of p. Then Γ^k_{ij} are the connection coefficients of a linear connection ∇ on M and from the condition $K^h_{kij} = 0$ we get that ∇ is flat. Next, from the condition $K_{hkij} = 0$ we get (15).

Remark. If N_{ij} are given by (14) we have $\Phi^k_{ij} = \Gamma^k_{ij}$, thus the Schouten connection $\bar{\nabla}$ is induced from the connection ∇ on M, thought of as an M-connection on T^*M.

Remark. The system (15) of (homogeneous) partial differential equations is verified identically in the following cases: i) $t_{ij} = 0$, ii) $\nabla_k t_{ij} = 0$, i.e. the tensor field defined by t_{ij} is parallel with respect to ∇, iii) $\nabla_k t_{ij} - \nabla_i t_{kj} = 0$, i.e. t_{ij} define a Codazzi tensor field with respect to ∇.

Now we may prove by a straightforward computation

Proposition 4. *The first Bianchi identity* $K(X, Y)Z + K(Y, Z)X + K(Z, X)Y = 0$; $X, Y, Z \in \mathcal{X}(T^*M)$ *is determined by the following local coordinate relations*

$$\text{(16)}\quad K^{ih}_{jk} = K^{hi}_{jk} = K^{ih}_{kj},\ \sum_{(i,j,k)} K^h_{ijk} = 0,\ \sum_{(i,j,k)} K_{hkij} = -\sum_{(i,j,k)} \bar{\nabla}_k R_{hij} = 0.$$

Remark. From the expression (13) of K_{hkij} we have $K_{hkij} = -K_{hkji} = -K_{khij}$. If we take into account the last identity (16) we get just like in the classical Riemannian geometry the following symmetry property

$$\text{(17)}\quad K_{hkij} = K_{ijhk}.$$

Thus the M-tensor field K_{hkij} is very much similar to the Riemann-Christoffel tensor field from Riemannian geometry.

Using a straightforward computation we have

Proposition 5. *The local coordinate expressions of the covariant derivative of K with respect to $\tilde{\nabla}$ are given by*

$$(18)\qquad \begin{aligned}
&(\tilde{\nabla}^l K)\left(\frac{\partial}{\partial p_i}, \frac{\partial}{\partial p_j}\right)\frac{\partial}{\partial p_k} = (\tilde{\nabla}^l K)\left(\frac{\partial}{\partial p_i}, \frac{\partial}{\partial p_j}\right)\frac{\delta}{\delta q^k} = 0,\\
&(\tilde{\nabla}^l K)\left(\frac{\partial}{\partial p_i}, \frac{\delta}{\delta q^j}\right)\frac{\partial}{\partial p_k} = -\frac{\partial K^{ki}_{jh}}{\partial p_l}\frac{\partial}{\partial p_h},\\
&(\tilde{\nabla}^l K)\left(\frac{\delta}{\delta q^i}, \frac{\delta}{\delta q^j}\right)\frac{\partial}{\partial p_k} = -\frac{\partial K^k_{hij}}{\partial p_l}\frac{\partial}{\partial p_h},\\
&(\tilde{\nabla}^l K)\left(\frac{\partial}{\partial p_i}, \frac{\delta}{\delta q^j}\right)\frac{\delta}{\delta q^k} = \frac{\partial K^{ih}_{jk}}{\partial p_l}\frac{\delta}{\delta q^h} - \frac{\partial K^i_{jkh}}{\partial p_l}\frac{\partial}{\partial p_h},\\
&(\tilde{\nabla}^l K)\left(\frac{\delta}{\delta q^i}, \frac{\delta}{\delta q^j}\right)\frac{\delta}{\delta q^k} = \frac{\partial K^h_{kij}}{\partial p_l}\frac{\delta}{\delta q^h} + \frac{\partial K_{hkij}}{\partial p_l}\frac{\partial}{\partial p_h},
\end{aligned}$$

and

$$(19)\qquad \begin{aligned}
&(\tilde{\nabla}_l K)\left(\frac{\partial}{\partial p_i}, \frac{\partial}{\partial p_j}\right)\frac{\partial}{\partial p_k} = (\tilde{\nabla}_l K)\left(\frac{\partial}{\partial p_i}, \frac{\partial}{\partial p_j}\right)\frac{\delta}{\delta q^k} = 0,\\
&(\tilde{\nabla}_l K)\left(\frac{\partial}{\partial p_i}, \frac{\delta}{\delta q^j}\right)\frac{\partial}{\partial p_k} = -\bar{\nabla}_l K^{ik}_{jh}\frac{\partial}{\partial p_h},\\
&(\tilde{\nabla}_l K)\left(\frac{\delta}{\delta q^i}, \frac{\delta}{\delta q^j}\right)\frac{\partial}{\partial p_k} = (-\bar{\nabla}_l K^k_{hij} + R_{lia}K^{ak}_{jh} - R_{lja}K^{ak}_{ih})\frac{\partial}{\partial p_h},\\
&(\tilde{\nabla}_l K)\left(\frac{\partial}{\partial p_i}, \frac{\delta}{\delta q^j}\right)\frac{\delta}{\delta q^k} = \bar{\nabla}_l K^{ih}_{jk}\frac{\delta}{\delta q^h} + (-\bar{\nabla}_l K^i_{jkh} + R_{lka}K^{ai}_{jh} -\\
&\quad - R_{lha}K^{ai}_{jk})\frac{\partial}{\partial p_h}, \quad (\tilde{\nabla}_l K)\left(\frac{\delta}{\delta q^i}, \frac{\delta}{\delta q^j}\right)\frac{\delta}{\delta q^k} = (\bar{\nabla}_l K^h_{kij} -\\
&\quad - R_{lia}K^{ah}_{jk} + R_{lja}K^{ah}_{ik})\frac{\delta}{\delta q^h} + (\bar{\nabla}_l K_{hkij} - R_{lha}K^a_{kij} + R_{lka}K^a_{hij} -\\
&\quad - R_{lia}K^a_{jhk} + R_{lja}K^a_{ihk})\frac{\partial}{\partial p_h}.
\end{aligned}$$

Consider the second Bianchi identity $(\tilde{\nabla}_X K)(Y, Z) + (\tilde{\nabla}_Y K)(Z, X) + (\tilde{\nabla}_Z K)(X, Y) = 0$; $X, Y, Z \in \mathcal{X}(T^*M)$ satisfied by K. Then

Proposition 6. *The second Bianchi identity satisfied by K is determined by the following local coordinate relations:*

$$(20)\qquad \begin{aligned}
&\frac{\partial K^{ih}_{jk}}{\partial p_l} = \frac{\partial K^{lh}_{jk}}{\partial p_i}, \quad \frac{\partial K^h_{kij}}{\partial p_l} = \frac{\partial K^l_{kij}}{\partial p_h}, \quad \frac{\partial K^h_{kij}}{\partial p_l} = \bar{\nabla}_i K^{lh}_{jk} - \bar{\nabla}_j K^{lh}_{ik},\\
&\frac{\partial K_{hkij}}{\partial p_l} = \bar{\nabla}_i K^l_{jhk} - \bar{\nabla}_j K^l_{ihk} + R_{ika}K^{al}_{jh} + R_{jha}K^{al}_{ik} - R_{iha}K^{al}_{jk} -\\
&\quad - R_{jka}K^{al}_{ih}, \quad \sum_{(i,j,k)}\{\bar{\nabla}_k K^l_{hij} + R_{aij}K^{al}_{kh}\} = 0,\\
&\sum_{(i,j,k)}\{\bar{\nabla}_k K_{hlij} - R_{kha}K^a_{lij} + R_{kla}K^a_{hij} + R_{aij}K^a_{khl}\} = 0.
\end{aligned}$$

4. Locally symmetric pseudo-Riemannian space (T^*M, G). We shall use the formulas (18), (19) to find the conditions under which the pseudo-Riemannian space (T^*M, G) is a locally symmetric space, i.e. $\tilde{\nabla}K = 0$. We must have

$$(21)\qquad \begin{aligned}
&\text{(i)}\quad \frac{\partial K^{hk}_{ij}}{\partial p_l} = 0, \qquad \text{(ii)}\quad \frac{\partial K^h_{kij}}{\partial p_l} = 0, \qquad \text{(iii)}\quad \frac{\partial K_{hkij}}{\partial p_l} = 0,\\
&\text{(iv)}\quad \bar{\nabla}_l K^{hk}_{ij} = 0, \qquad \text{(v)}\quad \bar{\nabla}_l K^a_{kij} - R_{lih}K^{ha}_{jk} + R_{ljh}K^{ha}_{ik} = 0,\\
&\text{(vi)}\quad \bar{\nabla}_l K_{hkij} - R_{lha}K^a_{kij} + R_{lka}K^a_{hij} - R_{lia}K^a_{jhk} + R_{lja}K^a_{ihk} = 0.
\end{aligned}$$

Firstly, from the condition (21) (i) we get by using (13)

$$(22)\qquad N_{ij} = t_{ij} - \Gamma^k_{ij}p_k - \tfrac{1}{2}K^{hk}_{ij}p_h p_k,$$

where t_{ij}, Γ^k_{ij}, K^{hk}_{ij} are independent of p. Moreover $t_{ij} = t_{ji}$, $K^{hk}_{ij} = K^{hk}_{ji} = K^{kh}_{ij}$ define tensor fields on M thought of as M-tensor fields on T^*M. The coefficients Γ^k_{ij} define a torsion free linear connection ∇ on M. Then, from (6) we have the following formula relating the linear connections ∇, $\bar{\nabla}$:

$$(23)\qquad \Phi^k_{ij} = \Gamma^k_{ij} + K^{kh}_{ij}p_h.$$

From the condition (21) (iv) we get by using (23):

$$(24)\qquad \nabla_l K^{hk}_{ij} = 0,$$

$$(25)\qquad L^{abc}_{ijk} = K^{ab}_{il}K^{lc}_{jk} + K^{ac}_{il}K^{lb}_{jk} - K^{bc}_{kl}K^{la}_{ij} - K^{bc}_{jl}K^{la}_{ik} = 0.$$

Then the condition (21) (ii) is automatically satisfied due to the following identities obtained from (24), (25): $\nabla_i K^{lh}_{jk} - \nabla_j K^{lh}_{ik} = 0$, $L^{abc}_{ijk} - L^{abc}_{jik} = 0$. It follows

$$(26)\qquad K^h_{kij} = R^h_{kij} - t_{il}K^{lh}_{jk} + t_{jl}K^{lh}_{ik}, \quad R_{kij} = \nabla_i t_{jk} - \nabla_j t_{ik} - p_h K^h_{kij},$$

where R^h_{kij} are the local coordinate components of the curvature tensor field R of the connection ∇ on M defined by the coefficients Γ^k_{ij}. From the condition (21) (v) we get

$$(27)\qquad L^a_{lkij} = \nabla_l K^a_{kij} - (\nabla_i t_{lh} - \nabla_h t_{li})K^{ha}_{jk} + (\nabla_j t_{lh} - \nabla_h t_{lj})K^{ha}_{ik} = 0$$

and

(28) $L^{ab}_{ikij}=K^{ab}_{ih}K^{h}_{kij}-K^{b}_{hij}K^{ha}_{ik}+K^{a}_{lih}K^{hb}_{jk}-K^{a}_{ljh}K^{hb}_{ik}+K^{b}_{kjh}K^{ha}_{il}-K^{b}_{kih}K^{ha}_{jl}=0.$

Using (13) and (26) we may compute the explicite expression of K_{hkij} and we get that the condition (21) (iii) is automatically satisfied due to the following identities deduced from (27), (28) : $L^{a}_{ikij}-L^{a}_{kiij}=0$, $L^{ab}_{ikij}-L^{ab}_{kiij}=0$. Then we have

(29) $K_{hkij}=-t_{il}K^{l}_{jhk}+t_{jl}K^{l}_{ihk}-\nabla_i(\nabla_h t_{kj}-\nabla_k t_{hj})+\nabla_j(\nabla_h t_{ki}-\nabla_k t_{hi}).$

Finally, from (21) (vi) we get the following conditions

$$\begin{aligned}\nabla_l K_{hkij}=&(\nabla_h t_{la}-\nabla_a t_{lh})K^{a}_{kij}-(\nabla_k t_{la}-\nabla_a t_{lk})K^{a}_{hij}+\\&+(\nabla_i t_{la}-\nabla_a t_{li})K^{a}_{jhk}-(\nabla_j t_{la}-\nabla_a t_{lj})K^{a}_{ihk}.\end{aligned}\tag{30}$$

$$\begin{aligned}K_{ijka}K^{ab}_{lh}-K_{ijha}K^{ab}_{lk}+K_{hkja}K^{ab}_{li}-K_{hkia}K^{ab}_{lj}=&-K^{b}_{lha}K^{a}_{kij}+\\&+K^{b}_{lka}K^{a}_{hij}-K^{b}_{lia}K^{a}_{jhk}+K^{b}_{lja}K^{a}_{ihk}.\end{aligned}\tag{31}$$

Thus we have proved

Theorem 7. *The pseudo-Riemannian space (T^*M, G) is locally symmetric if and only if the symmetric nonlinear connection on T^*M is given by (22) where t_{ij}, Γ^k_{ij}, K^{hk}_{ij} are independent of p and satisfy the conditions (24), (25), (27), (28), (30,) (31) with the tensor fields K^h_{kij}, K_{hkij} given by (26), (29).*

We should remark that the conditions (28), (30), (31) may be transformed further to give explicite relations satisfied by the tensor fields t_{ij}, K^{hk}_{ij} and the curvature tensor field R of the connection ∇ on M. E.g. the condition (28) is equivalent to

$$\begin{aligned}K^{ab}_{lh}R^{h}_{kij}-R^{b}_{hij}K^{ha}_{lk}+R^{a}_{lih}K^{hb}_{jk}-R^{a}_{ljh}K^{hb}_{ik}-R^{b}_{kih}K^{ha}_{lj}+R^{b}_{kjh}K^{ha}_{li}+&\\+2t_{hc}(K^{ha}_{li}K^{bc}_{jk}-K^{ha}_{jl}K^{bc}_{ik})=0&\end{aligned}\tag{28'}$$

and from (27) we get

(27') $\nabla_l R^{a}_{kij}=(\nabla_i t_{lh}+\nabla_l t_{ih}-\nabla_h t_{li})K^{ha}_{jk}-(\nabla_l t_{jh}+\nabla_j t_{lh}-\nabla_h t_{lj})K^{ha}_{ik}$

Two other relations are obtained from (30), (31) by using the expressions (26), (29) of K^h_{kij} and K_{hkij}.

Of course, the conditions relating t_{ij}, Γ^k_{ij}, K^{hk}_{ij} in Theorem 7 are quite strong and it is natural to ask whether there are nontrivial geometric structures on M defining the tensor fields t_{ij}, K^{hk}_{ij} and the connection Γ^k_{ij} in order these conditions be fulfilled. In fact, we shall show that there are situations in which the answer is affirmative.

A. Take $t_{ij}=0$, $K^{hk}_{ij}=0$. Then we have $N_{ij}=-\Gamma^k_{ij}p_k$ and (24), (25), (28) become trivial. From (29) we get $K_{hkij}=0$ and the condition (30) becomes trivial too. From (26) we get $K^h_{kij}=R^h_{kij}$ and (27), (31) become, respectively

(27'') $\qquad \nabla_l R^h_{kij}=0,$

(31') $\qquad -R^b_{lha}R^a_{kij}+R^b_{lka}R^a_{hij}-R^b_{lia}R^a_{jhk}+R^b_{lja}R^a_{ihk}=0.$

We may show that the condition (31') is fulfilled due to the relation $(\nabla_i\nabla_j-\nabla_j\nabla_i)R^b_{lhk}+(\nabla_h\nabla_k-\nabla_k\nabla_h)R^b_{lij}=0$ obtained from (27''), next using the Ricci identity.

In the case discussed above the pseudo-Riemannian space (T^*M, G) si the Riemann extension of M with the linear connection ∇ and we have recovered a known result from the theory of Riemann extensions.

Theorem 8. *The pseudo-Riemannian space (T^*M, G) with $N_{ij}=-\Gamma^k_{ij}p_k$ is locally symmetric if and only if (M, ∇) is locally symmetric.*

B. Take

$$N_{ij}=t_{ij}-\Gamma^k_{ij}p_k.\tag{32}$$

i.e. $K^{hk}_{ij}=0$. In this case we have $\Phi^k_{ij}=\Gamma^k_{ij}$, $K^h_{kij}=R^h_{kij}$, the conditions (24) (25), (28) are trivial, the condition (27) becomes (27'') and (31) is transformed to (31'). The condition (31') is fulfilled by the same argument used in the case A. Finally, from (29), (30) we get

$$\begin{aligned}-\nabla_l t_{ia}R^a_{jhk}+\nabla_l t_{ja}R^a_{ihk}-\nabla_l\nabla_i(\nabla_h t_{kj}-\nabla_k t_{hj})+&\\+\nabla_l\nabla_j(\nabla_h t_{ki}-\nabla_k t_{hi})=(\nabla_h t_{la}-\nabla_a t_{lh})R^a_{kij}-&\\-(\nabla_k t_{la}-\nabla_a t_{lk})R^a_{hij}+(\nabla_i t_{la}-\nabla_a t_{li})R^a_{jhk}-(\nabla_j t_{la}-\nabla_a t_{lj})R^a_{ihk}&\end{aligned}\tag{33}$$

It follows

Theorem 9. *The pseudo-Riemannian space (T^*M, G) with N_{ij} given by (32) is locally symmetric if and only if (M, ∇) is locally symmetric and t_{ij} satisfies the relation (33). If the tensor field t_{ij} is parallel with respect to ∇ and (M, ∇) is locally symmetric then (T^*M, G) is locally symmetric.*

C. Consider

$$K^{hk}_{ij}=0(\delta^h_i\delta^k_j+\delta^k_j\delta^h_i)\tag{34}$$

where 0 is a constant. Then the condition (24) is trivially verified and a straightforward computation shows us that the condition (25) is fulfilled too. From the condition (28') (equivalent to (28)) we get

$$t_{ij}=\frac{-1}{20(n-1)}(R_{ij}+R_{ji}).\tag{35}$$

$$R^h_{kij}=\frac{1}{n^2-1}\{\delta^h_i(nR_{jk}+R_{kj})-\delta^h_j(nR_{ik}+R_{ki})-\frac{1}{n+1}(R_{ij}-R_{ji})\delta^h_k\},\tag{36}$$

where $R_{ij}=R^h_{jhi}$ are the local coordinate expressions of the Ricci tensor field $\rho(R)$ obtained from the curvature tensor field R. Remark that $\rho(R)$ is not necessarily symmetric. It follows that the connection ∇ must be projectively flat and t_{ij} is determined from the condition (28). Now we have to verify the conditions (27), (28), (30), (31). We shall use for convenience the following auxiliary notations : $P_{ij}=\frac{1}{n^2-1}(-nR_{ij}-R_{ji})$, $D_{ij}=\frac{1}{2}(P_{ij}-P_{ji})$. Then we have

$$t_{ij}=\frac{1}{20}(P_{ij}+P_{ji}).\tag{35'}$$

(36') $$R^h_{kij}=2\delta^h_k D_{ij}-\delta^h_i P_{jk}+\delta^h_j P_{ik}.$$

From the second Bianchi identity satisfied by R it follows that P_{ij} must verify the relation

(37) $$\nabla_i P_{jk}=\nabla_j P_{ik},$$

for $n>2$. Then

(38) $$\nabla_i t_{jk}-\nabla_j t_{ik}=\frac{1}{\theta}\nabla_k D_{ij}.$$

From (26), (29) we get by using (37) and the Ricci identity for P_{ij}:

(39) $$K^h_{kij}=2\delta^h_k D_{ij}-\delta^h_i D_{jk}+\delta^h_j D_{ik},$$

(40) $$K_{hkij}=\frac{2}{\theta}(2D_{hk}D_{ij}+D_{hi}D_{kj}-D_{hj}D_{ki}).$$

By using (38) we get by a straightforward computation that the conditions (27), (28), (30), (31) are fulfilled. Hence we may state

Theorem 10. *Consider the pseudo-Riemannian space (T^*M, G) with dim. $M>2$, where N_{ij} is given by (22), K^{hk}_{ij} is given by (34) and t_{ij} is given by (35). Then (T^*M, G) is locally symmetric if and only if (M, ∇) is projectively flat.*

Remark. Under the assumptions from Theorem 10, if ∇ has symmetric Ricci tensor field we get $K^h_{kij}=0$, $K_{hkij}=0$ and the conditions (27), (28), (30), (31) become trivial.

D. Assume that (M, g) is a (pseudo)-Riemannian space and let the local coordinate components of the metric g be g_{ij}. Consider

(41) $$K^{hk}_{ij}=\theta(\delta^h_i\delta^k_j+\delta^h_j\delta^k_i-g_{ij}g^{hk}),$$

where θ is a constant and g^{hk} are defined by $g^{hj}g_{ji}=\delta^h_i$. The condition (24) is fulfilled if ∇ is the Levi-Civita connection of g. Moreover, it follows by a straightforward computation that (25) is fulfilled too. We find from (28) by usual computations in Riemannian geometry

(42) $$t_{ij}=\frac{1}{\theta}C_{ij}=\frac{1}{\theta}\left\{\frac{-1}{n-2}R_{ij}+\frac{r}{2(n-1)(n-2)}g_{ij}\right\}$$

where $r=R_{ij}g^{ij}$ is the scalar curvature and C_{ij} is the local coordinate expression of the usual tensor field in the definition of the Weyl curvature tensor field in Riemannian geometry. By using again (28) we find

(43) $$R^h_{kij}=-\delta^h_i C_{jk}+\delta^h_j C_{ik}-g^{hl}(C_{il}g_{jk}-C_{jl}g_{ik}),$$

thus (M, g) must be conformally flat. For $n>3$, C_{jk} has the property

(44) $$\nabla_i C_{jk}=\nabla_j C_{ik},$$

i.e. it is a Codazzi tensor field. Using (26), (42), (43), (44), (29) we find

(45) $$K^h_{kij}=0,\quad K_{hkij}=0.$$

Then the conditions (27), (28), (30), (31) are fulfilled trivially. Hence

Theorem 11. *If (M, g) is a (pseudo)-Riemannian space of dimension $n>3$, then the pseudo-Riemannian space (T^*M, G) with N_{ij} given by (22) where t_{ij}, K^{hk}_{ij} are given by (42), (41) and Γ^k_{ij} are the Christoffel symbols of the Levi Civita connection ∇ of g, is locally symmetric if and only if (M, g) is conformally flat.*

5. **The conformal curvature tensor field of (T^*M, G).** In this section we study the conditions under which the pseudo-Riemannian space (T^*M, G) is conformally flat. Introduce the following M-tensor fields on T^*M

(46) $$L^i_j=\frac{1}{2n-2}K^{ik}_{jk}-\frac{1}{(2n-1)(2n-2)}K^{hk}_{hk}\delta^i_j,\quad L_{ij}=\frac{1}{2n-2}(K^k_{ijk}+K^k_{jik}).$$

From the usual definition of the conformal tensor field of a (pseudo-)Riemannian space we get by a straightforward computation

Proposition 12. *The local coordinate expression of the conformal curvature tensor field C of (T^*M, G) is given by*

(47)
$$C\left(\frac{\partial}{\partial p_i},\frac{\partial}{\partial p_j}\right)\frac{\partial}{\partial p_k}=0,\quad C\left(\frac{\partial}{\partial p_i},\frac{\delta}{\delta q^j}\right)\frac{\partial}{\partial p_k}=(-K^{ik}_{jh}+\delta^i_h L^k_j+\delta^k_j L^i_h)\frac{\partial}{\partial p_h},$$
$$C\left(\frac{\delta}{\delta q^i},\frac{\delta}{\delta q^j}\right)\frac{\partial}{\partial p_k}=(-K^k_{hij}-\delta^k_i L_{jh}+\delta^k_j L_{ih})\frac{\partial}{\partial p_h}+(\delta^h_i L^k_j-\delta^h_j L^k_i-$$
$$-\delta^k_i L^h_j+\delta^k_j L^h_i)\frac{\delta}{\delta q^h},\quad C\left(\frac{\partial}{\partial p_i},\frac{\partial}{\partial p_j}\right)\frac{\delta}{\delta q^k}=(\delta^j_h L^i_k-\delta^i_h L^j_k-\delta^j_k L^i_h+$$
$$+\delta^i_k L^j_h)\frac{\partial}{\partial p_h},\quad C\left(\frac{\partial}{\partial p_i},\frac{\delta}{\delta q^j}\right)\frac{\delta}{\delta q^k}=(K^{ih}_{jk}-\delta^i_k L^h_j-\delta^h_j L^i_h)\frac{\delta}{\delta q^h}+$$
$$+(-K^i_{jkh}+\delta^i_h L_{jk}-\delta^i_k L_{jh})\frac{\partial}{\partial p_h},\quad C\left(\frac{\delta}{\delta q^i},\frac{\delta}{\delta q^j}\right)\frac{\delta}{\delta q^k}=$$
$$=(K^h_{kij}+\delta^h_i L_{jk}-\delta^h_j L_{ik})\frac{\delta}{\delta q^h}+K_{hkij}\frac{\partial}{\partial p_h}.$$

Theorem 13. *The pseudo-Riemannian space (T^*M, G) with dim. $M>2$, is conformally flat if and only if the chosen nonlinear connection has the following expression*

(48) $$N_{ij}=t_{ij}-\Gamma^k_{ij}p_k,$$

where t_{ij}, Γ^k_{ij} are independent of p, Γ^k_{ij} define a projectively flat linear connection on M with symmetric Ricci tensor field R_{ij} and t_{ij} satisfies the condition

(49) $$\nabla_k(\nabla_i t_{jh}-\nabla_j t_{ih})-\nabla_h(\nabla_i t_{jk}-\nabla_j t_{ik})=\frac{1}{n-1}(t_{hi}R_{jk}+$$
$$+t_{jk}R_{hi}-t_{ki}R_{jh}-t_{jh}R_{ki}).$$

Proof. The conditions for (T^*M, G) to be conformally flat are

$$(50)\qquad \text{(i) } \delta_i^h L_j^k - \delta_j^h L_i^k - \delta_i^k L_j^h + \delta_j^k L_i^h = 0, \qquad \text{(ii) } K_{jh}^{ik} - \delta_h^i L_j^k - \delta_j^k L_h^i = 0$$
$$\text{(iii) } K_{kij}^h + \delta_i^h L_{jk} - \delta_j^h L_{ik} = 0, \qquad \text{(iv) } K_{hkij} = 0.$$

From (50) (i) we get by usual contractions $L_j^i = 0$ for $n > 2$. Using (50) (ii) we get $K_{jh}^{ik} = 0$. Then from (13) we get (48), next

$$(51)\qquad R_{kij} = \nabla_i t_{jk} - \nabla_j t_{ik} - p_h R_{kij}^h,$$

where R_{kij}^h denotes the curvature tensor field of the linear connection ∇ on M defined by the coefficients Γ_{ij}^k. Using again (13) we get

$$(52)\qquad K_{kij}^h = R_{kij}^h$$

and from (50) (iii) obtain

$$(53)\qquad R_{kij}^h = -\delta_i^h L_{jk} + \delta_j^h L_{ik}$$

then

$$(54)\qquad R_{jk} = -(n-1) L_{jk}.$$

It follows that the linear connection ∇ is projectively flat and has its Ricci tensor field symmetric. From the condition (50) (iv) we get (49) by using (13), (51), (53), (54).

Remark. In the case $t_{ij} = 0$, the condition (49) is automatically verified and we get a known result in the theory of Riemann extensions that a projectively flat linear connection with symmetric Ricci tensor field determines a conformally flat Riemann extension.

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Faculty of Mathematics,
University of Iaşi,
Romania

Department of Mathematics,
Polytechnic Institute of Iaşi,
Romania

An. şt. Univ. „Al. I. Cuza" Iaşi
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SOME THEOREMS ON CODAZZI TENSORS

BY

THEMIS KOYFOGIORGOS

1. Preliminaries. A Codazzi tensor L on a manifold M with a connection ∇ is, by definition, a smooth symmetric tensor field on M of type (1, 1), such that $(\nabla_X L)Y = (\nabla_Y L)X$ for any vector fields X, Y on M. The class of manifolds admitting Codazzi tensors is large (see [2] and references therein).

Let M be a Riemannian manifold with metric tensor $<,>$ and Riemannian connection ∇. Let L be a positive definite Codazzi tensor on M. At first we prove the following

Lemma 1. *For any vector fields X, Y, Z, W on M and for any integer k we have:*

$$(1)\qquad \begin{aligned} &<R(X, Y)L^{k+1}Z - LR(X, Y)L^kZ, W> = <\nabla_X((\nabla_Y L)L^kZ), W> - \\ &- <\nabla_Y((\nabla_X L)W), L^kZ> - <(\nabla_X L^k)Z, (\nabla_Y L)W> + <\nabla_Y W, \\ &(\nabla_X L)L^kZ> - <\nabla_X Z, L^k(\nabla_Y L)W> - <[X, Y], (\nabla_Y L)L^kZ> \end{aligned}$$

and

$$(2)\qquad \begin{aligned} &<L^{-1}R(X, Y)L^{k+1}Z - R(X, Y)L^kZ, W> = <\nabla_X(L^{-1}(\nabla_Y L)L^kZ), W> - \\ &- <\nabla_Y((\nabla_X L)L^{-1}W), L^kZ> + <L^{-1}(\nabla_Y L)L^kZ, (\nabla_X L)L^{-1}W> \\ &- <L^{-1}(\nabla_X L^{k+1})Z, (\nabla_Y L)L^{-1}W> + <\nabla_Y W, L^{-1}(\nabla_X L)L^kZ> - \\ &- <\nabla_X Z, L^k(\nabla_Y L)L^{-1}W> - <L^kZ, (\nabla_{[X, Y]}L)L^{-1}W>, \end{aligned}$$

where R is the curvature tensor of $<,>$, L^{-1} is the inverse of L and $L^k = L \ldots L$ (k-times). ($L^1 = L$ and $L^0 = \mathrm{Id}$).

Proof. Let V be a vector field on M. Using $(\nabla_X L)Y = \nabla_X(LY) - L\nabla_X Y$ and

$$R(X, Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X, Y]}Z$$

we easily obtain

$$(3)\qquad \begin{aligned} R(X, Y)LV = LR(X, Y)V + \nabla_X((\nabla_Y L)V) - \nabla_Y((\nabla_X L)V + \\ + (\nabla_X L)\nabla_Y V - (\nabla_Y L)\nabla_X V - (\nabla_{[X, Y]}L)V. \end{aligned}$$

Putting $V = L^kZ$ in (3) and using L^k, $\nabla_X L^k$ are self adjoint we easily get

$$\begin{aligned} &<R(X, Y)L^{k+1}Z - LR(X, Y)L^kZ, W> = <\nabla_X(\nabla_Y L)L^kZ, W> - \\ &- Y<Z, L^k(\nabla_X L)W> + <\nabla_Y W, (\nabla_X L)L^kZ)> + <(\nabla_Y L^k)Z, (\nabla_X L)W> + \\ &+ <\nabla_Y Z, L^k(\nabla_X L)W> - <(\nabla_X L^k)Z, (\nabla_Y L)W> - <\nabla_X Z, L^k(\nabla_Y L)W> - \\ &- <[X, Y], (\nabla_Y L)L^kZ>. \end{aligned}$$