Proof. The conditions for (T^*M, G) to be conformally flat are

$$\text{(50)}\qquad \begin{aligned} &\text{(i)}\ \delta_i^h L_j^k-\delta_j^h L_i^k-\delta_i^k L_j^h+\delta_j^k L_i^h=0, \qquad \text{(ii)}\ K^{ik}_{jh}-\delta^i_h L^k_j-\delta^k_j L^i_h=0\\ &\text{(iii)}\ K^h_{kij}+\delta^h_i L_{jk}-\delta^h_j L_{ik}=0, \qquad \text{(iv)}\ K_{hkij}=0. \end{aligned}$$

From (50) (i) we get by usual contractions $L^i_j=0$ for $n>2$. Using (50) (ii) we get $K^{ik}_{jh}=0$. Then from (13) we get (48), next

$$\text{(51)}\qquad R_{kij}=\nabla_i t_{jk}-\nabla_j t_{ik}-p_h R^h_{kij},$$

where R^h_{kij} denotes the curvature tensor field of the linear connection ∇ on M defined by the coefficients Γ^k_{ij}. Using again (13) we get

$$\text{(52)}\qquad K^h_{kij}=R^h_{kij}$$

and from (50) (iii) obtain

$$\text{(53)}\qquad R^h_{kij}=-\delta^h_i L_{jk}+\delta^h_j L_{ik}$$

then

$$\text{(54)}\qquad R_{jk}=-(n-1)L_{jk}.$$

It follows that the linear connection ∇ is projectively flat and has its Ricci tensor field symmetric. From the condition (50) (iv) we get (49) by using (13), (51), (53), (54).

Remark. In the case $t_{ij}=0$, the condition (49) is automatically verified and we get a known result in the theory of Riemann extensions that a projectively flat linear connection with symmetric Ricci tensor field determines a conformally flat Riemann extension.

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Received 15. III. 1989

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Matematică, 36 (1990)

SOME THEOREMS ON CODAZZI TENSORS

BY

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1. Preliminaries. A Codazzi tensor L on a manifold M with a connection ∇ is, by definition, a smooth-symmetric tensor field on M of type (1, 1), such that $(\nabla_X L)Y=(\nabla_Y L)X$ for any vector fields X, Y on M. The class of manifolds admitting Codazzi tensors is large (see [2] and references therein).

Let M be a Riemannian manifold with metric tensor $<,>$ and Riemannian connection ∇. Let L be a positive definite Codazzi tensor on M. At first we prove the following

Lemma 1. *For any vector fields X, Y, Z, W on M and for any integer k we have:*

$$\text{(1)}\qquad \begin{aligned} &<R(X,Y)L^{k+1}Z-LR(X,Y)L^kZ, W>=<\nabla_X((\nabla_Y L)L^kZ), W>-\\ &<-\nabla_Y((\nabla_X L)W), L^kZ>-<(\nabla_X L^k)Z, (\nabla_Y L)W>+<\nabla_Y W,\\ &(\nabla_X L)L^kZ>-<\nabla_X Z, L^k(\nabla_Y L)W>-<[X,Y], (\nabla_W L)L^kZ> \end{aligned}$$

and

$$\text{(2)}\qquad \begin{aligned} &<L^{-1}R(X,Y)L^{k+1}Z-R(X,Y)L^kZ, W>=<\nabla_X(L^{-1}(\nabla_Y L)L^kZ), W>\\ &-<\nabla_Y((\nabla_X L)L^{-1}W), L^kZ>+<L^{-1}(\nabla_Y L)L^kZ, (\nabla_X L)L^{-1}W>\\ &-<L^{-1}(\nabla_X L^{k+1})Z, (\nabla_Y L)L^{-1}W>+<\nabla_Y W, L^{-1}(\nabla_X L)L^kZ>-\\ &-<\nabla_X Z, L^k(\nabla_Y L)L^{-1}W>-<L^kZ, (\nabla_{[X,Y]}L)L^{-1}W>, \end{aligned}$$

where R is the curvature tensor of $<,>$, L^{-1} is the inverse of L and $L^k=L\ldots L$ (k-times), ($L^1=L$ and $L^0=\mathrm{Id}$).

Proof. Let V be a vector field on M. Using $(\nabla_X L)V=\nabla_X(LV)-L\nabla_X V$ and

$$R(X,Y)Z=\nabla_X\nabla_Y Z-\nabla_Y\nabla_X Z-\nabla_{[X,Y]}Z$$

we easily obtain

$$\text{(3)}\qquad \begin{aligned} R(X,Y)LV=LR(X,Y)V&+\nabla_X((\nabla_Y L)V)-\nabla_Y((\nabla_X L)V+\\ &+(\nabla_X L)\nabla_Y V-(\nabla_Y L)\nabla_X V-(\nabla_{[X,Y]}L)V. \end{aligned}$$

Putting $V=L^kZ$ in (3) and using L^k, $\nabla_X L^k$ are self adjoint we easily get

$$\begin{aligned} &<R(X,Y)L^{k+1}Z-LR(X,Y)L^kZ, W>=<\nabla_X(\nabla_Y L)L^kZ, W>-\\ &-\nabla_Y<Z, L^k(\nabla_X L)W>+<\nabla_Y W, (\nabla_X L)L^kZ)>+<(\nabla_Y L^k)Z, (\nabla_X L)W>+\\ &+<\nabla_Y Z, L^k(\nabla_X L)W>-<(\nabla_X L^k)Z, (\nabla_Y L)W>-<\nabla_X Z, L^k(\nabla_Y L)W>-\\ &-<[X,Y], (\nabla_W L)L^kZ>. \end{aligned}$$

But

$$=Y<Z,\ L^k(\nabla_X L)W>+<(\nabla_Y L^k)Z,\ (\nabla_X L)W>+<\nabla_Y Z,\ L^k(\nabla_X L)W>=$$

$$=-<Z,\ \nabla_Y(L^k(\nabla_X L)W)>+<Z,\ (\nabla_Y L^k)(\nabla_X L)W>=-<L^kZ,\ \nabla_Y((\nabla_X L)W>.$$

This completes the proof of (1).

Using $(\nabla L^{-1})L=-L^{-1}\nabla L$, we immediately find from (3)

$$(4)\qquad L^{-1}R(X,\ Y)V-R(X,\ Y)V=\nabla_X(L^{-1}(\nabla_Y L)V-\nabla_Y(L^{-1}(\nabla_X L)V)+$$

$$+L^{-1}\{(\nabla_X L)(L^{-1}(\nabla_Y L)V)-(\nabla_Y L)(L^{-1}(\nabla_X L)V)+(\nabla_X L)\nabla_Y V-(\nabla_Y L)\nabla_X V-$$

$$-(\nabla_Y L)[X,\ Y]\}.$$

Putting $V=L^kZ$ in (4) and using $(\nabla_X L^{k+1})Y=L(\nabla_X L^k)Y+(\nabla_X L)L^kY$ we conclude

$$<L^{-1}R(X,\ Y)L^{k+1}Z-R(X,\ Y)L^kZ,\ W>=<\nabla_X(L^{-1}(\nabla_Y L)L^kZ),\ W>-$$

$$-Y<L^kZ,\ (\nabla_X L)L^{-1}W>+<\nabla_Y W,\ L^{-1}(\nabla_X L)L^kZ>+<L^{-1}(\nabla_Y L)L^kZ,$$

$$(\nabla_X L)L^{-1}W>-<L^{-1}(\nabla_X L)L^kZ,\ (\nabla_Y L)L^{-1}W>+<\nabla_Y(L^kZ),\ (\nabla_X L)L^{-1}W>-$$

$$-<\nabla_X L^k)Z,\ (\nabla_Y L)L^{-1}W>-<\nabla_X Z,\ L^k(\nabla_Y L)L^{-1}W>-<(\nabla_{[X,Y]}L)L^{-1}W,L^kZ>.$$

But

$$-Y<L^kZ,\ (\nabla_X L)L^{-1}W>+<\nabla_Y(L^kZ),\ (\nabla_X L)L^{-1}W>=$$

$$=-<L^kZ,\ \nabla_Y((\nabla_X L)L^{-1}W)>$$

and

$$L^{-1}(\nabla_X L)L^kZ+(\nabla_X L^k)Z=L^{-1}(\nabla_X L^{k+1})Z.$$

This completes the proof of (2).

Now, for any $P_0\in M$ we choose a field of orthonormal frames $e_1, e_2, \dots, e_n$ in a neighbourhood U of P_0. Putting $X=W=e_i$, $Y=Z=e_j$ in (1) and (2) we get in U:

$$\sum_{i,j}<R(e_i,\ e_j)L^{k+1}e_j-LR(e_i,\ e_j)L^ke_j,\ e_i>=\sum_{i,j}\{<\nabla_{e_i}((\nabla_{e_j}L)L^ke_j),e_i>$$

$$(5)\qquad -<\nabla_{e_j}((\nabla_{e_i}L)e_i),\ L^ke_j>-<(\nabla_{e_i}L^k)e_j,\ (\nabla_{e_i}L)e_j>\}.$$

$$\sum_{i,j}<L^{-1}R(e_i,\ e_j)L^{k+1}e_j-R(e_i,\ e_j)L^ke_j,\ e_i>=\sum_{i,j}\{\nabla_{e_i}(L^{-1}(\nabla_{e_j}L)L^ke_j),\ e_i>$$

$$(6)\qquad -<\nabla_{e_j}((\nabla_{e_i}L)L^{-1}e_i),\ L^ke_j>+<L^{-1}(\nabla_{e_j}L)L^ke_j,\ (\nabla_{e_i}L)L^{-1}e_i>-$$

$$-<L^{-1}(\nabla_{e_i}L^{k+1})e_j,\ (\nabla_{e_j}L)L^{-1}e_i>\}.$$

To prove (5) and (6) we must show that

$$(7)\qquad \sum_{i,j}\{2<\nabla_{e_j}e_i,\ (\nabla_{e_i}L)L^ke_j>-<\nabla_{e_i}e_j,\ L^k(\nabla_{e_j}L)e_i+(\nabla_{e_i}L)L^ke_j>\}=0$$

and

$$(8)\qquad \sum_{i,j}\{<\nabla_{e_j}e_i,\ L^{-1}(\nabla_{e_i}L)L^ke_j>-<\nabla_{e_i}e_j,\ L^k(\nabla_{e_j}L)L^{-1}e_i>-$$

$$-<L^ke_j,\ (\nabla_{[e_i,e_j]})L^{-1}e_i>\}=0.$$

For the proof of (7) we choose, for any $P\in U$, a field of orthonormal frames $\bar e_i$ $(i=1,\dots,n)$ in a neighbourhood of P such that, at P, $\bar e_i$ are eigenvectors of L and $\nabla_{\bar e_i}\bar e_j=0$. For this field, the left-hand side of (7) is 0, at P. Let λ_i be the eigenvalues of L at P.

$$e_i=\sum_j a_{ij}\bar e_j,\quad \textit{and}\ (\nabla_{\bar e_i}L)\bar e_j=\sum_\rho b_{ij\rho}\bar e_\rho,\ (b_{ij\rho}=b_{ji\rho}=b_{i\rho j}).$$

Since

$$\sum_\rho a_{i\rho}a_{j\rho}=\delta_{ij},\qquad \sum_\rho a_{\rho i}a_{\rho j}=\delta_{ij}$$

$$\sum_\rho a_{j\rho}\bar e_l(a_{i\rho})=-\sum_\rho a_{i\rho}\bar e_l(a_{j\rho})\ \text{and}\ \sum_\rho a_{\rho i}\bar e_l(a_{\rho j})=-\sum_\rho a_{\rho j}\bar e_l(a_{\rho i})$$

it is easy to see that, at P,

$$\sum_{i,j}<\nabla_{e_j}e_i,\ (\nabla_{e_i}L)L^ke_j>=\sum_{i,j,\sigma,l,\rho,\tau}a_{i\sigma}a_{jl}a_{j\tau}\bar e_l(a_{i\rho})\lambda_\tau^k b_{\sigma\tau\rho}=0$$

and

$$\sum_{i,j}<L^ke_j,\ L^k(\nabla_{e_j}L)e_i+(\nabla_{e_i}L)L^ke_j>=\sum_{i,j,l,\rho,\sigma,\tau}\{a_{jl}\bar e_l(a_{j\rho})a_{j\sigma}a_{i\tau}\lambda_\rho^k b_{\tau\sigma\rho}+$$

$$+a_{jl}\bar e_l(a_{j\rho})a_{i\sigma}a_{j\tau}\lambda_\tau^k b_{\sigma\tau\rho}\}=0.$$

This completes the proof of (7). The proof of (8) is analogous.

Remark. (5) and (6) are identities for $k=0$ and $k=-1$ respectively. In order to compute the terms of (5) and (6) we need the following

Lemma 2. *For any orthonormal frames $e_1,\dots,e_n$ in a neighbourhood of $P_0\in M$ it is valid*

$$\sum_i(\nabla_{e_i}L)L^ke_i=\begin{cases}\dfrac{1}{k+1}\ \text{grad Tr}\ L^{k+1},\ \textit{for any integer}\ k\neq-1\\ \text{grad log det}\ L,\ \textit{for}\ k=-1.\end{cases}$$

Proof. It is easy to prove, by induction, that

$$(9)\qquad \nabla_{e_j}L^k=\begin{cases}\sum_{\rho=0}^{k-1}L^\rho(\nabla_{e_j}L)L^{k-\rho-1}, & k=1,\ 2,\ \dots\\ -\sum_{\rho=0}^{-k-1}L^{-\rho-1}(\nabla_{e_j}L)L^{k+\rho}, & k=-1,\ -2,\ \dots,\end{cases}$$

Using $\text{Tr}(AB)=\text{Tr}(BA)$, $\text{Tr}\ \nabla_{e_j}L^k=e_j\text{Tr}\ L^k$ and (9) we have

$$\text{Tr}(L^k\nabla_{e_j}L)=\text{Tr}\ \nabla_{e_j}L^{k+1}-\text{Tr}(\nabla_{e_j}L^k)L=e_j\text{Tr}\ L^{k+1}-k\text{Tr}(L^k\nabla_{e_j}L).$$

So

$$(k+1)\text{Tr}(L^k\nabla_{e_j}L)=e_j\text{Tr}\ L^{k+1}.$$

Hence

$$\sum_i(\nabla_{e_i}L)L^ke_i=\sum_{i,j}<(\nabla_{e_i}L)L^ke_i,\ e_j>e_j=\sum_{i,j}<L^k(\nabla_{e_j}L)e_i,\ e_i>e_j=$$

$$=\sum_j(\text{Tr}(L^k\nabla_{e_j}L))e_j=\frac{1}{k+1}\text{grad Tr}\ L^{k+1},\ k\neq-1.$$

The proof for the case $k=-1$ is also simple, if we use the formula $X(\det L)=\sum_{i,j} X(l_{ij})L_{ij}$, $([l_{ij}]=L$ and L_{ij} is the cofactor of $l_{ij})$.

Let K_{ij} be the sectional curvature of M for the plane $e_i \wedge e_j$. Using Lemma 2 and the formula $\operatorname{div} X=\sum_i <\nabla_{e_i} X, e_i>$ for the divergence of a vector field X on M, we get from (5) and (6) the following formulas at any point $P \subset U$.

$$\text{(10)} \quad \frac{1}{k+1}\Delta \operatorname{Tr} L^{k+1}-\sum_j <\nabla_{e_j}\operatorname{grad} \operatorname{Tr} L, L^k e_j>=Q_k+P_k, \text{ for any integer } k\neq -1.$$

$$\text{(11)} \quad \Delta \log \det L-\sum_j <\nabla_{e_j}\operatorname{grad} \operatorname{Tr} L, L^{-1}e_j>=Q_{-1}+P_{-1},$$

$$\text{(12)} \quad \begin{aligned}&\frac{1}{k+1}\operatorname{div}(L^{-1}\operatorname{grad} \operatorname{Tr} L^{k+1})-\sum_j<\nabla_{e_j}\operatorname{grad}\log\det L, L^k e_j>+\\ &+\frac{1}{k+1}<L^{-1}\operatorname{grad} \operatorname{Tr} L^{k+1}, \operatorname{grad}\log\det L>=\overline{Q}_k+\overline{P}_k,\quad k\neq -1\end{aligned}$$

where $\Delta=\operatorname{div}\operatorname{grad}$ is the Laplacian operator.

$$\text{(13)} \quad Q_k=\sum_{i,j}<R(e_i, e_j)L^{k+1}e_j-LR(e_i, e_j)L^k e_j, e_i>,$$

$$\text{(14)} \quad P_k=\sum_{i,j}<(\Delta_{e_i}L^k)e_j, (\nabla_{e_i}L)e_j>,$$

$$\text{(15)} \quad \overline{Q}_k=\sum_{i,j}<L^{-1}R(e_i, e_j)L^{k+1}e_j-R(e_i, e_j)L^k e_j, e_i>,$$

$$\text{(16)} \quad \overline{P}_k=\sum_{i,j}<L^{-1}(\nabla_{e_i}L^{k+1})e_j, (\nabla_{e_j}L)L^{-1}e_i>.$$

If we choose the local field of orthonormal frames $e_1, \dots, e_n$ such that at $P_0\in U$, $Le_i=\lambda_i e_i$, $\nabla_{e_i}e_j=0$ and use (9), we find

$$\text{(17)} \quad Q_k(P_0)=\sum_{i<j}(\lambda_i-\lambda_j)(\lambda_i^k-\lambda_j^k)k_{ij}|_{P_0},$$

$$\text{(18)} \quad P_k(P_0)=\begin{cases}\sum\limits_{i,j}\sum\limits_{\rho=0}^{k-1}\lambda_j^{k-\rho-1}<L^\rho(\nabla_{e_i}L)e_j, (\nabla_{e_i}L)e_j>|_{P_0}\geq 0, & k=1, 2, \dots\\ -\sum\limits_{i,j}\sum\limits_{\rho=0}^{-k-1}\lambda_j^{k+\rho}<L^{-\rho-1}(\nabla_{e_i}L)e_j, (\nabla_{e_i}L)e_j>|_{P_0}\leq 0, & k=-1, -2, \dots\end{cases}$$

$$\text{(19)} \quad \overline{Q}_k(P_0)=\sum_{i<j}(\lambda_i-\lambda_j)(\lambda_i^{k+1}-\lambda_j^{k+1})(\lambda_i\lambda_j)^{-1}K_{ij}|_{P_0}$$

$$\text{(20)} \quad \overline{P}_k(P_0)=\begin{cases}\sum\limits_{i,j}\sum\limits_{\rho=0}^{K}\lambda_j^{k-\rho}\lambda_i^{-1}<L^{\rho-1}(\nabla_{e_i}L)e_j, (\nabla_{e_i}L)e_j>|_{P_0}\geq 0, & k=0, 1, 2,\dots\\ -\sum\limits_{i,j}\sum\limits_{\rho=0}^{-K}\lambda_j^{k+\rho+1}\lambda_i^{-1}<L^{-\rho-2}(\nabla_{e_i}L)e_j, (\nabla_{e_i}L)e_j>|_{P_0}\leq 0, & k=-2, -3\dots\end{cases}$$

$$<\nabla_{e_j}\operatorname{grad} \operatorname{Tr} L, L^k e_j>|_{P_0}=\lambda_j^k e_j e_j \operatorname{Tr} L|_{P_0}$$

$$<\nabla_{e_j}\operatorname{grad}\log\det L, L^k e_j>|_{P_0}=\lambda_j^k e_j e_j \operatorname{grad}\log\det L|_{P_0}.$$

So from (10), (11) and (12) we get at P_0

$$\text{(21)} \quad \frac{1}{k+1}\Delta \operatorname{Tr} L^{k+1}(P_0)-\sum_j\lambda_j^k e_j e_j \operatorname{Tr} L(P_0)=Q_k(P_0)+P_k(P_0), \text{ for any } k\neq -1$$

$$\text{(22)} \quad \Delta\log\det L(P_0)-\sum_j\lambda_j^{-1}e_j e_j \operatorname{Tr} L(P_0)=Q_{-1}(P_0)+P_{-1}(P_0)$$

$$\text{(23)} \quad \begin{aligned}&\frac{1}{k+1}\operatorname{div} L^{-1}\operatorname{grad} \operatorname{Tr} L^{k+1}(P_0)-\sum_j\lambda_j^k e_j e_j\log\det L(P_0)+\\ &+\frac{1}{k+1}<L^{-1}\operatorname{grad} \operatorname{Tr} L^{k+1}, \operatorname{grad}\log\det L>(P_0)=\\ &=\overline{Q}_k(P_0)+\overline{P}_k(P_0), \text{ for any } k\neq -1.\end{aligned}$$

If the sectional curvature of M is positive, then $Q_k(P_0)$, $\overline{Q}_k(P_0)$ are nonnegative (nonpositive) if $k\geq 0$ $(k<0)$.

Remark. (21) was known only for positive intergers [5].

2. Main Results. To state our results we introduce the following notations: Denote by R^m_+ the open orthant

$$R^m_+=\{x=(x_1, \dots, x_m)|x_1>0, \dots, x_m>0\}.$$

Let $f: R^2_+\to R_+$ be a C^2 function satisfying

$$\text{(a)} \quad \frac{\partial f(t_1, t_2)}{\partial t_1}\geq 0, \qquad \frac{\partial f(t_1, t_2)}{\partial t_2}>0.$$

Let $F: R^m_+\to R$ be a C^2 function satisfying

(a$_1$) $\dfrac{\partial F(t_1, \dots, t_m)}{\partial t_a}\geq 0$, $1\leq a\leq m$, and there exists $\beta\geq 2$ such that $\partial F/\partial t_\beta>0$

(b$_1$) the matrix $\left(\dfrac{\partial^2 F}{\partial t_a \partial t_\beta}\right)$ is semi-positive definite.

The form $L_2(X, Y)=<LX, Y>$ defines a metric on M, which we also denote by $<, >_2$. Let div_2, grad_2 and $\overset{2}{\Delta}$ be the divergence, the gradient and the Laplacian operators with respect to $<, >_2$. We need the following

Lemma 3. *Let M be an n-dimensional Riemannian manifold M with a positive definite Codazzi tensor L. Then*

$$\operatorname{div}_2 X=\operatorname{div} X+\frac{1}{2}<X, \operatorname{grad}\log\det L>

for any vector field X on M.

Proof. Let $\overset{2}{\nabla}$ be the Riemannian connection of $<,>_2$. For any vector fields Y, Z on M, $Z<X, Y>_2 = Z<LX, Y>$ implies

$$(^*)\qquad <L(\overset{2}{\nabla}_Z X - \nabla_Z X), Y> + <L(\overset{2}{\nabla}_Z Y - \nabla_Z Y), X> = <(\nabla_X L)Y, Z>.$$

This identity remains true if X, Y, Z are replaced by Y, Z, X. So we get

$$<L(\overset{2}{\nabla}_Z Y - \nabla_Z Y), X> = <L(\overset{2}{\nabla}_X Y - \nabla_X Y), Z>.$$

Hence $(^*)$ can be written as

$$2<L(\overset{2}{\nabla}_Y X - \nabla_Y X), Z> = <(\nabla_X L)Y, Z>.$$

So

$$(^{**})\qquad \overset{2}{\nabla}_X Y = \nabla_X Y + \frac{1}{2} L^{-1}(\nabla_X L)Y.$$

If $e_1, \ldots, e_n$ is a field of orthonormal frames in a neighbourhood of M, then using $(^{**})$ and Lemma 2 we have

$$\mathrm{div}_2 X = \sum_i <\overset{k}{\nabla}_{e_i} X, e_i> = \sum_i <\nabla_{e_i} X, e_i> + \frac{1}{2}\sum_i <L^{-1}(\nabla_{e_i} L)X, e_i> =$$

$$= \mathrm{div}\, X + \frac{1}{2}\sum_i <(\nabla_{e_i} L)L^{-1}e_i, X> = \mathrm{div}\, X + \frac{1}{2}<X, \mathrm{grad}\log\det L>.$$

The following Theorems are consequences of the formulas $(2^1)-(23)$ and Lemma 3. We begin with a Theorem analogous of a classical Theorem of Hilbert.

Theorem 4. *Let M be an n-dimensional Riemannian manifold with positive sectional curvature and L be a positive definite Codazzi tensor on M. Then each of the following conditions (α), (β), implies that the eigenvalues λ_i of L are equal at P_0:*

(α) det L *or* Tr L *takes relative minimum at* $P_0 \in M$ *and* Tr L^k $(k \geqslant 1)$ *takes relative maximum at* P_0.

(β) det L *or* Tr L *takes relative maximum at* P_0 *and* Tr L^k $(k \leqslant -1)$ *takes relative maximum at* P_0.

Proof. If det L takes relative minimum at P_0 and Tr L^k $(k \geqslant 1)$ takes relative maximum at P_0 then

$$\mathrm{grad}\det L(P_0) = 0,\quad e_j e_j \det L(P_0) \geqslant 0 \text{ and } \overset{2}{\nabla}\mathrm{Tr}\, L^k(P_0) \leqslant 0.$$

Also it is easy to see that grad $= L$ grad_2. So using Lemma 3 we have

$$\mathrm{div}\, L^{-1}\mathrm{grad}\,\mathrm{Tr}\, L^k(P_0) = \mathrm{div}\,\mathrm{grad}_2\,\mathrm{Tr}\, L^k(P_0) = \overset{2}{\Delta}\mathrm{Tr}\, L^k(P_0) \leqslant 0.$$

Since $\overline{Q}_k$, $\overline{P}_k$ $(k \geqslant 0)$ are nonnegative at P_0, each term of (23) must be zero at P_0, for any $k \geqslant 0$. So from (19) follows that $\lambda_1(P_0) = \lambda_2(P_0) = \ldots = \lambda_n(P_0)$.

The proof for the rest of (α) as for the case (β) follows from (21) and (23) with a similar way.

Theorem 5. *Let M be an n-dimensional Riemannian manifold with positive sectional curvature. Let L be a positive definite Codazzi tensor on M and $F: R^\lambda \to R_+$ be a C^2 function satisfying (a_1) and (b_1). If*

$$F(\mathrm{Tr}\, L, \mathrm{Tr}\, L^2, \ldots, \mathrm{Tr}\, L^\lambda) = \mathrm{const.}$$

and det L *or* Tr L *attains its minimum at some interior point* $P_0 \in M$, *then* $\lambda_1 = \lambda_2 = \ldots = \lambda_n$ *on* M.

Proof. Since $\det L(P) = \min_{P\in M} \det L(P)$ we have

$$\mathrm{grad}\det L(P_0) = 0 \text{ and } e_i e_i \det L(P_0) \geqslant 0.$$

Putting $k+1=\lambda>0$ in (23) and using Lemma 3 and $\mathrm{grad}_2 = L^{-1}\mathrm{grad}$ we get

$$\mathrm{div}\, L^{-1}\,\mathrm{grad}\,\mathrm{Tr}\, L^\sigma(P_0) = \overset{2}{\Delta}\mathrm{Tr}\, L^\sigma(P_0) \geqslant 0,\quad \sigma = 1, 2, \ldots, \lambda.$$

Differentiation of the equality $F(\mathrm{Tr}\, L, \mathrm{Tr}\, L^2, \ldots, \mathrm{Tr}\, L^\lambda) = \mathrm{const.}$, gives

$$\sum_{\sigma=1}^{\lambda} \frac{\partial F}{\partial\, \mathrm{Tr}\, L^\sigma}\,\mathrm{grad}_2\mathrm{Tr}\, L^\sigma = 0$$

and

$$0 = \sum_{\sigma=1}^{\lambda} <\mathrm{grad}_2 \frac{\partial F}{\partial\, \mathrm{Tr}\, L^\sigma}, \mathrm{grad}_2\,\mathrm{Tr}\, L^\sigma>_2 + \sum_{\sigma=1}^{\lambda} \frac{\partial F}{\partial\, \mathrm{Tr}\, L^\sigma}\,\overset{2}{\Delta}\mathrm{Tr}\, L^\sigma \geqslant 0, \text{ at } P_0.$$

Hence, by assumption for F, each term must be 0. So there exists $\beta \geqslant 2$ such that

$$\overset{2}{\Delta}\mathrm{Tr}\, L^\beta(P_0) = \mathrm{div}\, L^{-1}\,\mathrm{grad}\,\mathrm{Tr}\, L^\beta(P_0) = 0.$$

From (23) and (19) we have

$$\sum_{i<j} (\lambda_i - \lambda_j)(\lambda_i^\beta - \lambda_j^\beta)K_{ij}(P_0) = 0.$$

Hence $\lambda_1(P_0) = \lambda_2(P_0) = \ldots = \lambda_n(P_0)$. Using the inequality

$$\mathrm{Tr}\, L^\lambda \geqslant n(\det L)^{\lambda/n},\quad \lambda = 1, 2, \ldots,$$

with equality only when $\lambda_1 = \ldots = \lambda_n$, we have for any $P \in M$

$$F(\mathrm{Tr}\, L(P), \mathrm{Tr}\, L^2(P), \ldots, \mathrm{Tr}\, L^\lambda(P)) = F(\mathrm{Tr}\, L(P_0), \mathrm{Tr}\, L^2(P_0), \ldots, \mathrm{Tr}\, L^\lambda(P_0)) =$$

$$= F(n(\det L)^{1/n}(P_0), n(\det L)^{2/n}(P_0), \ldots, n(\det L)^{\lambda/n}(P_0)) \leqslant$$

$$\leqslant F(n(\det L)^{1/n}(P), n(\det L)^{2/n}(P), \ldots, n(\det L)^{\lambda/n}(P)) \leqslant$$

$$\leqslant F(\mathrm{Tr}\, L(P), \mathrm{Tr}\, L^2(P), \ldots, \mathrm{Tr}\, L^\lambda(P)).$$

Hence we get

$$\mathrm{Tr}\, L^\beta(P) = n(\det L)^{\beta/n}(P)$$

which is possible if $\lambda_1(P) = \lambda_2(P) = \ldots = \lambda_n(P)$. The proof for the case $\mathrm{Tr}\, L(P_0) = \min_{P\in M} \mathrm{Tr}\, L(P)$ follows similarly from (21).

Theorem 6. *Let M, L be as in Theorem 5. Let $F: R_+^\lambda \to R_+$ and $f: R_+^2 \to R_+$ be C^2 functions satisfying (a_1, b_1) and (a) respectively. If*

$f(\det L, F(\operatorname{Tr} L, \operatorname{Tr} L^2, \ldots, \operatorname{Tr} L^\lambda)) = \text{const.}$ *(or* $f(\operatorname{Tr} L, F(\operatorname{Tr} L, \operatorname{Tr} L^2, \ldots, \operatorname{Tr} L^\lambda)) = \text{const.}$*) and* $\det L$ *(or* $\operatorname{Tr} L$*) attains its minimum at some interior point* $P_0 \in M$, *then* $\lambda_1 = \lambda_2 = \ldots = \lambda_n$ *on* M.

Proof. Since

$$f(\det L(P), F(\operatorname{Tr} L(P), \ldots, \operatorname{Tr} L^\lambda(P))) =$$

$$= f(\det L(P_0), F(\operatorname{Tr} L(P_0), \ldots, \operatorname{Tr} L^\lambda(P_0))) \leqslant f(\det L(P), F(\operatorname{Tr} L(P_0), \ldots, \operatorname{Tr} L^\lambda(P_0)))$$

we have

$$F(\operatorname{Tr} L(P), \ldots, \operatorname{Tr} L^\lambda(P)) \leqslant F(\operatorname{Tr} L(P_0), \ldots, \operatorname{Tr} L^\lambda(P_0)), \ \forall P \in M$$

that is

(24) $$F(\operatorname{Tr} L(P_0), \ldots, \operatorname{Tr} L^\lambda(P_0)) = \max_{P \in M} F(\operatorname{Tr} L(P), \ldots, \operatorname{Tr} L^\lambda(P)).$$

Hence

$$\overset{2}{\Delta} F(\operatorname{Tr} L(P), \ldots \operatorname{Tr} L^\lambda(P))|_{P_0} \leqslant 0.$$

From the proof of Theorem 5 we have seen that

$$\overset{2}{\Delta} F(P_0) = \sum_{\sigma=1}^{\lambda} \left\{ \left\langle \operatorname{grad}_2 \frac{\partial F}{\partial \operatorname{Tr} L^\sigma}, \operatorname{grad}_2 \operatorname{Tr} L^\sigma \right\rangle_2 (P_0) + \frac{\partial F}{\partial \operatorname{Tr} L^\sigma} \overset{2}{\Delta} \operatorname{Tr} L^\sigma(P_0) \right\} \geqslant 0.$$

Hence $\overset{2}{\Delta} F(P_0) = 0$ and so $\overset{2}{\Delta} \operatorname{Tr} L^\beta(P_0) = 0$. Therefore $\lambda_1(P_0) = \lambda(P_0) = \ldots = \lambda_n(P_0)$. Since

$$F(\operatorname{Tr} L(P_0), \operatorname{Tr} L^2(P_0), \ldots, \operatorname{Tr} L^\lambda(P_0)) = F(n(\det L)^{1/n}(P_0), n(\det L)^{2/n}(P_0) \ldots$$

$$n(\det L)^{\lambda/n}(P_0)) \leqslant F(n(\det L)^{1/n}(P), n(\det L)^{2/n}(P), \ldots, n(\det L)^{\lambda/n}(P)) \leqslant$$

$$\leqslant F(\operatorname{Tr} L(P), \operatorname{Tr} L^2(P), \ldots, \operatorname{Tr} L^\lambda(P)).$$

From (24) we take $\operatorname{Tr} L^\beta(P) = n(\det L)^\beta(P)$, which is possible if $\lambda_1(P) = \ldots = \lambda_n(P)$. The proof for the other case follows similarly from (21).

Remark. Theorems 5 and 6 are known for the case where $\operatorname{Tr} L$ attains its minimum [3].

Theorem 7. *Let M be an n-dimensional Riemannian manifold with positive sectional curvature. Let L be a positive definite Codazzi tensor, and $F: R_+^\lambda \to R_+$ be a C^2 function satisfying (a_1), (b_1). If*

$$F(\operatorname{Tr} L^{-1}, \operatorname{Tr} L^{-2}, \ldots, \operatorname{Tr} L^{-\lambda}) = \text{const.}$$

and $\operatorname{Tr} L$ *or* $\det L$ *attains its maximum at some interior point* $P_0 \in M$, *then* $\lambda_1 = \lambda_2 = \ldots \lambda_n$ *on* M.

Proof. Let $\operatorname{Tr} L(P_0) = \max_{P \in M} \operatorname{Tr} L(P)$. Then

$$\operatorname{grad} \operatorname{Tr} L(P_0) = 0 \text{ and } e_i e_i \operatorname{Tr} L(P_0) \leqslant 0.$$

So putting $k+1 = -\lambda < 0$ in (21) we deduce that

$$\Delta \operatorname{Tr} L^{-\sigma}(P_0) \geqslant 0, \quad \sigma = 1, \ldots, \lambda.$$

fferentiation of the equality $F(\operatorname{Tr} L^{-1}, \ldots, \operatorname{Tr} L^{-\lambda}) = \text{const.}$, gives

$$\sum_{\sigma=1}^{\lambda} \frac{\partial F}{\partial \operatorname{Tr} L^{-\sigma}} \operatorname{grad} \operatorname{Tr} L^{-\sigma} = 0$$

$$0 = \sum_{\sigma=1}^{\lambda} \left\langle \operatorname{grad} \frac{\partial F}{\partial \operatorname{Tr} L^{-\sigma}}, \operatorname{grad} \operatorname{Tr} L^{-\sigma} \right\rangle + \sum_{\sigma=1}^{\lambda} \frac{\partial F}{\partial \operatorname{Tr} L^{-\sigma}} \Delta \operatorname{Tr} L^{-\sigma} \geqslant 0 \text{ at } P_0.$$

ence, there exists $\beta \geqslant 2$ such that

$$\Delta \operatorname{Tr} L^{-\beta}(P_0) = 0.$$

om (21) and (17) we have

$$\sum_{i<j} (\lambda_i - \lambda_j)(\lambda_i^{-\beta-1} - \lambda_j^{-\beta-1}) K_{ij}(P_0) = 0.$$

$$\lambda_1(P_0) = \lambda_2(P_0) = \ldots = \lambda_n(P_0).$$

sing the inequality

$$\frac{\operatorname{Tr} L^k}{n} \geqq \left(\frac{\operatorname{Tr} L}{n}\right)^k, \quad k = -1, -2, \ldots$$

th equality only when $\lambda_1 = \ldots = \lambda_n$, we have for any $P \in M$

$$F(\operatorname{Tr} L^{-1}(P), \operatorname{Tr} L^{-2}(P), \ldots, \operatorname{Tr} L^{-\lambda}(P)) = F(\operatorname{Tr} L^{-1}(P_0), \operatorname{Tr} L^{-2}(P_0), \ldots,$$

$$\operatorname{Tr} L^{-\lambda}(P_0)) = F\left(n\left(\frac{\operatorname{Tr} L}{n}\right)^{-1}(P_0), n\left(\frac{\operatorname{Tr} L}{n}\right)^{-2}(P_0), \ldots, n\left(\frac{\operatorname{Tr} L}{n}\right)^{-\lambda}(P_0)\right) \leqslant$$

$$\leqslant F\left(n\left(\frac{\operatorname{Tr} L}{n}\right)^{-1}(P), n\left(\frac{\operatorname{Tr} L}{n}\right)^{-2}(P), \ldots, n\left(\frac{\operatorname{Tr} L}{n}\right)^{-\lambda}(P)\right) \leqslant$$

$$\leqslant F(\operatorname{Tr} L^{-1}(P), \operatorname{Tr} L^{-2}(P), \ldots, \operatorname{Tr} L^{-\lambda}(P)).$$

ence

$$\frac{\operatorname{Tr} L^{-\beta}}{n}(P) = \left(\frac{\operatorname{Tr} L}{n}\right)^{-\beta}(P)$$

hich is possible if $\lambda_1(P) = \ldots = \lambda_n(P)$.

Now, let $\det L(P_0) = \max_{P \in M} \det L(P)$. Putting $k+1 = -\lambda < 0$ in (23) and orking as in Theorem 5 we finally get

$$\operatorname{div} L^{-1} \operatorname{grad} L^{-\beta}(P_0) = 0.$$

So, from (25) and (19), we take $\lambda_1(P_0) = \lambda_2(P_0) = \ldots = \lambda_n(P_0)$. Using the inequality

$$\operatorname{Tr} L^{-\lambda} \geqslant n(\det L)^{-\lambda/n}, \quad \lambda = 1, 2, \ldots,$$

with equality only when $\lambda_1 = \ldots = \lambda_n$, we have for any $P \in M$

$$F(\operatorname{Tr} L^{-1}(P), \ldots, \operatorname{Tr} L^{-\lambda}(P)) = F(\operatorname{Tr} L^{-1}(P_0), \ldots, \operatorname{Tr} L^{-\lambda}(P_0)) =$$

$$= F(n(\det L)^{-1/n}(P_0), \ldots, n(\det L)^{-\lambda/n}(P_0)) \leqslant$$

$$\leqslant F(n(\det L)^{-1/n}(P), \ldots, n(\det L)^{-\lambda/n}(P_0)) \leqslant F(\operatorname{Tr} L^{-1}(P), \ldots, \operatorname{Tr} L^{-\lambda}(P)).$$

Hence

$$\operatorname{Tr} L^{-3}(P) = n(\det L)^{3/n}(P)$$

which is possible if $\lambda_1(P) = \lambda_2(P) = \dots = \lambda_n(P)$.

Theorem 8. *Let M be a compact, connected and orientable n-dimensional Riemannian manifold with positive sectional curvature. If L is a positive definite Codazzi tensor on M, then*

$$(25)\quad \int_M \sum_{\rho=0}^{k-1} \frac{1}{k-\rho} < L^\rho \operatorname{grad} \operatorname{Tr} L^{k-\rho}, \operatorname{grad} \operatorname{Tr} L > dM \geq 0, \quad k = 1, 2, \dots$$

$$(26)\quad \int_M \sum_{\rho=0}^{-k-2} \frac{1}{k+\rho+1} < L^{-\rho-1} \operatorname{grad} \operatorname{Tr} L^{k+\rho+1} + L^k \operatorname{grad} \log \det L, \operatorname{grad} \operatorname{Tr} L > dM \geq 0, \quad k = -1, -2, \dots \ (\rho \text{ non negative})$$

$$(27)\quad \int_M \sum_{\rho=0}^{k} \frac{1}{k-\rho+1} < L^\rho \operatorname{grad} \operatorname{Tr} L^{k-\rho+1}, L^{-1} \operatorname{grad} \log \det L > dM \geq 0, \quad k = 0, 1, 2, \dots$$

$$(28)\quad \int_M \sum_{\rho=0}^{-k-3} \frac{1}{k+\rho+2} < L^{-\rho-1} \operatorname{grad} \operatorname{Tr} L^{k+\rho+2} + L^{k+1} \operatorname{grad} \log \det L, L^{-1} \operatorname{grad} \log \det L > dM \geq 0, \quad k = -2, -3, \dots \ (\rho \text{ non negative})$$

Equality holds in any case, if $\lambda_1 = \dots = \lambda_n$ on M.

Proof. Let $e_i, \dots, e_j$ be a field of orthonormal frames in a neighbourhood U of P_0. Since

$$\sum_{i,j} < \nabla_{e_j}(\nabla_{e_i} L)e_i, L^k e_j > = \sum_{i,j} \{ < \nabla_{e_j}(L^k(\nabla_{e_i} L)e_i), e_j > - < (\nabla_{e_j} L^k)e_j, (\nabla_{e_i} L)e_i > \}$$

and

$$\sum_{i,j} \{ < \nabla_{e_j}((\nabla_{e_i} L)L^{-1}e_i, L^k e_j > - < L^{-1}(\nabla_{e_i} L(L^k e_i, (\nabla_{e_j} L)L^{-1}e_j > \} =$$

$$= \sum_{i,j} \{ < \nabla_{e_j}(L^k(\nabla_{e_i} L)L^{-1}e_i), e_j > - < L^{-1}(\nabla_{e_j} L^{k+1})e_j, (\nabla_{e_i} L)L^{-1}e_i >.,$$

using Lemma 2 and (9) we get from (5) and (6)

$$Q_k + P_k = \frac{1}{k+1} \Delta \operatorname{Tr} L^{k+1} - \operatorname{div} L^k \operatorname{grad} \operatorname{Tr} L +$$

$$(29)\quad + \begin{cases} \displaystyle\sum_{\rho=0}^{k-1} \frac{1}{k-\rho} < L^\rho \operatorname{grad} \operatorname{Tr} L^{k-\rho}, \operatorname{grad} \operatorname{Tr} L >, & k = 1, 2, \dots \\ \displaystyle -\sum_{\rho=0}^{-k-2} \frac{1}{k+\rho+1} < L^{-\rho-1} \operatorname{grad} \operatorname{Tr} L^{k+\rho+1} + L^k \operatorname{grad} \log \det L, \operatorname{grad} \operatorname{Tr} L >, & k = -2, -3, \dots \end{cases}$$

$$(30)\quad \Delta \log \det L - \operatorname{div} L^{-1} \operatorname{grad} \operatorname{Tr} L - < L^{-1} \operatorname{grad} \log \det L, \operatorname{grad} \operatorname{Tr} L > = Q_{-1} + P_{-1}$$

and

$$\bar{Q}_k + \bar{P}_k = \frac{1}{k+1} \operatorname{div} (L^{-1} \operatorname{grad} \operatorname{Tr} L^{k+1}) - \operatorname{div} (L^k \operatorname{grad} \log \det L) +$$

$$(31)\quad + \begin{cases} \displaystyle\sum_{\rho=0}^{k} \frac{1}{k-\rho+1} < L^\rho \operatorname{grad} \operatorname{Tr} L^{k-\rho+1}, L^{-1} \operatorname{grad} \log \det L >, & k = 1, 2, \dots \\ \displaystyle -\sum_{\rho=0}^{-k-3} \frac{1}{k+\rho+2} < L^{-\rho-1} \operatorname{grad} \operatorname{Tr} L^{k+\rho+2} + L^{k+1} \operatorname{grad} \log \det L, L^{-1} \operatorname{grad} \log \det L >, & k = -2, -3, \dots \ (\rho \text{ non negative}) \end{cases}$$

Integrating (28)—(31) over M and using Stokes' divergence Theorem, we get the inequalities (25)—(28). When equality holds in (25)—(28), then we get $Q_k = P_k = \bar{Q}_k = \bar{P}_k = 0$ on M. From $Q_k = 0$ we take $\nabla L = 0$ and so the eigenvalues of L are constant. $Q_k = 0$, implies $\lambda_1 = \dots = \lambda_n$ on M.

3. Applications to compact hypersurfaces. Let N be an $(n+1)$-dimensional Riemannian manifold of constant sectional curvature c and M be a hypersurface of N. The Weingarten operator L is a Codazzi tensor on M of type (1, 1). If $e_1, \dots, e_n$ is a basis in $T_P M$ ($P \in M$) of eigenvectors of L and $\lambda_1, \dots, \lambda_n$ the coresponding eigenvalues, then the sectional curvature K_{ij} of M for the plane $e_i \wedge e_j$ is given by Gauss equation $K_{ij} = c + \lambda_i \lambda_j$. So supposing M being strongly convex we have $K_{ij} > 0$ (if $c \geq 0$). After these remarks the following Theorems are immediate consequences of the Theorems of the section 2.

Theorem 9. *Let N be an $(n+1)$-dimensional Riemannian manifold of constant sectional curvature c ($c \geq 0$). Let M be a locally strongly convex hypersurface in N. Then each of the following conditions, implies $\lambda_1(P_0) = \dots = \lambda_n(P_0)$*

(α) $\det L$ or $\operatorname{Tr} L$ takes relative minimum at $P_0 \in M$ and $\operatorname{Tr} L^k$ ($k \geq 1$) takes relative maximum at P_0.

(β) $\det L$ or $\operatorname{Tr} L$ takes relative maximum at P_0 and $\operatorname{Tr} L^k$ ($k \leq -1$) takes relative maximum at P_0.

Theorem 10. *Let N be an $(n+1)$-dimensional Riemannian manifold of constant sectional curvature c ($c \geq 0$). Let M be a compact and locally strongly convex hypersurface in N. If there exists a C^2 function $F: R_+^\lambda \to R_+$ satisfy (a_1), (b_1) and such that*

$$F(\operatorname{Tr} L, \operatorname{Tr} L^2, \dots, \operatorname{Tr} L^\lambda) = \text{const.}, \text{ or } F(\operatorname{Tr} L^{-1}, \operatorname{Tr} L^{-2}, \dots, \operatorname{Tr} L^{-\lambda}) = \text{const.},$$

then M is totally umbilical.

Theorem 11. *Let N, M be as in Theorem 10. If there exist C^2 functions $F: R_+^\lambda \to R_+$ and $f: R_+^2 \to R_+$ satisfying (a_1, b_1) and (a) respectively and such that*

$f(\operatorname{Tr} L, F(\operatorname{Tr} L, \operatorname{Tr} L^2, \dots, \operatorname{Tr} L^\lambda)) = \text{const.}$ or $f(\det L, F(\operatorname{Tr} L, \operatorname{Tr} L^2, \dots, \operatorname{Tr} L^\lambda)) = \text{const.}$ then M is totally umbilical.

Theorem 12. *Let N be an $(n+1)$-dimensional Riemannian manifold of constant sectional c ($c \geq 0$) and M be a compact, oriented and strongly convex* *hypersurface in N. Then the inequalities* (25) – (28) *are valid. The equalities holds, if M is totally umbilical.*

Remark. i) If the sectional curvature of N is negative, then Theorems 9 – 12 remain true, if we suppose M has positive sectional curvature. ii) Special cases of Theorems 9 and 11 are corresponding Theorems of SINGLEY for ovaloids in E^{n+1}.

iii) The inequality (27) in Theorem 12 is known for $k=0$ [1].

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Received 25. IV. 1989

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Matematică, 36 (1990)

RECENZII

IVOR GRATTAN-GUINNESS: *Convolutions in French Mathematics 1800—1840*: vol. I: The Settings; vol. II: The Turns; vol. III: The Data. Science Networks. Historical Studies. Birkhäuser Verlag: Basel-Boston-Berlin, 1990.

Matematica franceză a cunoscut o perioadă de mare dezvoltare după re- voluţia din 1789. Rezultate însemnate au fost obţinute în analiza reală şi com- plexă, mecanică, aplicaţii ale matematicii în fizică şi inginerie. Figuri însemnate de matematicieni au dominat epoca: Ampère, Cauchy, Fourier, Lagrange, Laplace, Legendre, Monge, Poisson ş.a.

Lucrarea recenzată constituie primul studiu exhaustiv al perioadei 1800—1840 în matematica franceză. În peste 1 500 de pagini, împărţite în trei volume, se realizează o analiză detaliată a rezultatelor matematice, a legăturilor şi in- fluenţelor reciproce dintre matematicieni francezi ai epocii, a diverselor insti- tuţii legate de cercetarea şi predarea matematicii (Institutul, Şcoala Politehnică, Şcoala Normală Superioară), precum şi a principalelor reviste consacrate artico- lelor de matematică.

Bazată pe o documentaţie vastă, care include pe lîngă cărţile şi articolele esenţiale şi multe lucrări mai puţin cunoscute, această masivă lucrare va con- stitui referinţa de bază în istoria matematicii franceze din prima jumătate a secolului trecut.

Eugen Popa

MICHAEL KENWOOD, CYRIL MOSS: Mathematics; Harlow (Essex): Longman Group U.K. Ltd., 1990; 311 p.; Seria: A-Level and AS-Level; £ 7.95.

Lucrarea de faţă este un ghid pentru elevii din clasele terminale, în ve- derea pregătirii examenelor de sfîrşit de liceu. În introducere este expus modul în care este structurat învăţămîntul matematic în Anglia şi cerinţele examina- torilor în vederea verificării cunoştinţelor celor examinaţi.

Cartea abundă în exerciţii şi probleme, la începutul fiecărui capitol fiind prezentate elementele teoretice şi aspectele metodice ce trebuiesc cunoscute pentru însuşirea noţiunilor, precum şi modele de rezolvare a exerciţiilor şi problemelor. În prima parte a cărţii sînt tratate elementele de bază din algebră, analiză matematică, geometrie analitică, pentru ca, în partea a doua, acestea să fie utilizate ca instrumente de lucru în rezolvarea problemelor de mecanică, teoria probabilităţilor şi statistică.

C. Chirilă

ERDÖS, P. (editor): Collected Papers of Paul Turan, vol. 1, 2, 3, Akadémiai Kiadó, Budapest, 1990; XXXVIII + XII + XII + 2 665 p. ISBN 963-05-4298-6 (vols. 1—3), ISBN 963-05-4299-4 (vol. 1), ISBN 963-05-43000-1 (vol. 2), ISBN 963-05-4301-X (vol. 3).

Paul Turan (1910—1976) a publicat multe articole originale, făcînd munca de pionierat, în multe ramuri de matematică: teoria numerelor (metoda sume- lor de puteri în studiul distribuţiei numerelor prime), teoria funcţiilor, teoria interpolării şi aproximării, ecuaţii diferenţiale, analiză numerică, teoria grupu- rilor, teoria grafurilor. Cele trei volume, însumînd peste 2 600 de pagini, conţin 246 de articole publicate de Turan.