Corollary 4.6. *Let L be a weak SMP-space, N a Dedekind complete Riesz space, and $T \in \mathcal{L}_b(L, N)$ a positive map. Then*

$$\{f \in L : P_\alpha|f|\downarrow 0,\ P_\alpha \in \mathcal{P}(L) \text{ implies } TP_\alpha|f|\downarrow 0\}$$

is the largest ideal on which T is order continuous. □

Similarly we may generalize [11, 102.17] on locally convex solid Hausdorff weak SMP spaces.

Also the following formula for an order continuous component T_n of a positive linear map T can be established in a similar way using for example [8] and Lemma 4.4.

Proposition 4.6. *Let M be an order principal A-module, N a Dedekind complete Riesz space, and $T \in \mathcal{L}_b(M, N)$ a positive operator. Then for every $x \in M^+$,*

$$T_n x = \inf \sup\{Ta_\alpha x : a_\alpha \in A^+,\ a_\alpha x \uparrow x\}.$$

If A is zero-dimensional, the elements a_α can be taken from $\mathcal{B}_e$. □

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REFERENCES

1. Fremlin D. H. — *Topological Riesz spaces and measure theory*, Cambridge University Press, 1974.
2. Huijsmans C. B. and de Pagter B. — *Ideal theory in f-algebras*, Trans. Amer. Math. Soc., 269 (1982), 225 — 249.
3. Huijsmans C. B. and de Pagter B. — *Subalgebras and Riesz subspaces of an f-algebra*, Proc. London Math. Soc., (3) 48 (1984), 161 — 174.
4. Lavrič B. — *On Freudenthal's spectral theorem*, Indag. Math., 49 (196), 411 — 421.
5. Lavrič B. — *C-algebras and the Hahn-Banach extension theorem*, Proc. Roy. Soc. Edinburgh, 105 A (1987), 307 — 317.
6. Luxemburg W. A. J. and Zaanen A. C. — *Riesz Spaces I*, North-Holland, 1971.
7. Meyer M. — *Richesses du centre d'un espace vectoriel réticulé*, Math. Ann., 236 (1978), 147 — 169.
8. Schep A. R. — *Order continuous components of operators and measures*, Indag. Math., 40 (1978), 110 — 117.
9. Veksler A. I. — *Projection properties of vector lattices and Freudenthal's theorem*, Math. Nachr., 74 (1976), 7 — 25 (Russian).
10. Vuza D. — *Ideals and bands in principal modules*, Arch. Math., 45 (1985), 306 — 322.
11. Zaanen A. C. — *Riesz Spaces II*, North-Holland, 1983.

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A NOTE ON ADJOINT ABELIAN OPERATORS ON $C_0(X)$

BY

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1. Introduction. The concept of adjoint abelian operators on Banach spaces was introduced by Stampfli [12] and further developed by Fleming ([3], [4]) and Sen ([8] — [11]) by semiinner product techniques. In Fleming [4], a characterization of adjoint abelian operators on $C(X)$, X a compact metric space, is given. This note extends this result to $C_0(X)$, where X denotes a locally compact Hausdorff space (throughout this note) and $C_0(X)$ is the Banach space of all continuous complex functions on X that vanish at infinity. For literature on semiinner products (s.i.p.) we refer to Bonsall ([1], [2]).

2. Adjoint Abelian Operators. For our characterization of adjoint abelian operators on $C_0(X)$, The following result plays a central role.

Theorem 2.1. *Let $A \neq 0$ be an adjoint abelian operator on $C_0(X)$. Then there exists a positive constant λ such that $A^2 = \lambda I$.*

Proof. For the proof we need the following lemmas.

Lemma 1. ([5], p. 218). *If C is compact, U is open, and $C \subset U$, then there exist sets C_0 and U_0 such that C_0 is a compact G_δ, U_0 is a σ-compact open set, and*

$$C \subset U_0 \subset C_0 \subset U.$$

Lemma 2. *If K_1 is a compact G_δ, K_2 is compact with $K_1 \cap K_2 = \emptyset$, and $0 < a < 1$, then there exists a function $h \in C_0(X)$ such that*

$$K_1 = \{t : h(t) = 1\},\ K_2 \subset \{t : h(t) = a\}$$

and $0 \leq h(t) \leq 1$.

Lemma 2 is an easy consequence Theorems B and C of Halmos ([5], p. 216).

Proof of Lemma 2. Since X is a locally compact Hausdorff space there exist compact neighborhoods V_1 and V_2 of K_1 and K_2 respectively (i.e. $K_i \subset \operatorname{int} V_i$, $i = 1, 2$), such that $V_1 \cap V_2 = \emptyset$. By Theorem C of Halmos ([5], p. 217) there exists a continuous function g such that $0 \leq g \leq 1$, $K_1 = \{x : g(x) = 1\}$. Again, by Theorem B of Halmos ([5], p. 216), there exists a continuous function f such that $0 \leq f \leq 1$, $f(x) = 1$ for $x \in K_1$ and $f(x) = 0$ for $x \in X \setminus V_1$. Let $h_1(x) = \min(f(x), g(x))$. Then $h_1 \in C_0(X)$ and $K_1 = \{x : h_1(x) = 1\}$. As K_2 is compact, by Theorem B of Halmos there exists a continuous function h_2 such

that $0 \le h_2 \le a$, $K_2 \subset \{x : h_2(x) = 1\}$, $h_2(x) = 0$ for $x \in X \setminus V_2$. Take $h(x) = \max(h_1(x), h_2(x))$. It follows that $h(x) = 0$ for $x \in X \setminus (V_1 \cup V_2)$, $h(x) = h_1(x) = 1$ for $x \in K_1$, $h(x) = a$ for $x \in K_2$. Hence $h \in C_0(X)$ has the required properties of Lemma 2.

Proof of our result. Let $Y = C_0(X)$. As A is adjoint abelian, there exists a s.i.p $[\ ,\ ]$ on Y compatible with the norm such that $[A g, f] = [g, A f]$, $\forall f, g \in Y$.

This s.i.p can be written as $[g, f] = \int_X g d\nu_f$, where ν_f is a regular complex Borel measure for each $f \in Y$, Rudin ([7], Theorem 6.19), and has the following properties

(i) $$\nu_f(P_f) = \|f\| \text{ for } f \ge 0$$

(ii) $$\operatorname{Supp} \nu_f \subset P_f \text{ (peak set of } f).$$

The proof of these properties is similar to that of Fleming [4]. As A_2 is Hermitian, by a result in Bonsall ([2], p. 92), $A^2 f = hf$, $\forall f \in Y$, where h is a real valued continuous function on X. In fact $h \ge 0$. If h is not constant, there exists a $t_0 \in X$ such that

$$0 \le h(t_0) < \|h\| = \operatorname{Sup}\{h(t) : t \in X\}.$$

Let $\varepsilon > 0$ be a number such that $h(t_0) + 3\varepsilon < \|h\|$. Then each of the three sets $C_1 = h^{-1}(h(t_0))$,

$$C_2 = h^{-1}[h(t_0) - \varepsilon,\ h(t_0) + \varepsilon]$$

and $C_3 = h^{-1}[\|h\| - \varepsilon,\ \|h\| + \varepsilon]$ is a closed G_δ. Therefore by Lemma 1, we get compact G_δ sets K_1 and K_2 such that $t_0 \in K_1 \subset \operatorname{Int} C_2 \subset C_2$ and $t_1 \in K_2 \subset \operatorname{Int} C_3 \subset C_3$ where $t_1 \in \operatorname{Int} C_3$. Then $K_3 = K_1 \cap C_1$ is a compact G_δ on which $h(t) = h(t_0)$ and $K_2 \cap K_3 = \varnothing$. So by Lemma 2, there exists a $g \in C_0(X)$ such that $K_3 = \{t : g(t) = 1\}$ and $g(t) = 1/2\{1 + h(t_0)/\|h\|\}$ for $t \in K_2$. For $t \in K_2$ $hg(t) \ge 1/2(\|h\| - \varepsilon)\{1 + h(t_0)/\|h\|\} > h(t_0) = hg(t_0)$ if $\varepsilon > 0$ is sufficiently small. So for $t \in K_3$, $t \notin P_{hg}$. As P_{hg} is compact (since $hg \in C_0(X)$) there exists an $f \in C_0(X)$ such that $f(t) = 1$ on K_3 and $f(t) = \|h\|/\|hg\|$ for $t \in P_{hg}$.

Now, $[A^2 f, g] = \int_X hf d\nu_g = \int_{K_3} hf d\nu_g = \int_{K_3} h d\nu_g = h(t_0)\nu_g(P_g) = h(t_0)\|g\| = h(t_0)$ and

$[f, A^2 g] = [f, hg] = \int_X f d\nu_{hg} = \int_{P_{hg}} f\, d\nu_{hg} = \|h\|/\|hg\| \cdot \nu_{hg}(P_{hg}) = \|h\| \ne h(t_0)$..

Hence the result.

As Banach-Stone Theorem is also true for locally compact Hausdorff spaces (see Bonsall ([2], p. 93) and Kalton [6]), application of Theorem 2.1 above to the proof of Theorem 2 of Fleming [4] yields the following desired result.

Theorem 2.2. *Let A be a non-zero operator on $C_0(X)$. Then A is adjoint abelian iff there exists a homeomorphism, Φ on X, a positive constant λ and a unimodular function $\alpha \in C(X)$ such that for every $f \in C_0(X)$, $Af(t) = \lambda\alpha(t) \cdot f \circ \Phi(t)$, $\forall t \in X$, where*

(i) $$\Phi \circ \Phi(t) = t, \quad \forall t \in X,$$

(ii) $$\alpha(t)\alpha(\Phi(t)) = 1, \quad \forall t \in X.$$

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REFERENCE

1. F. F. Bonsall and J. Duncan — *Numerical ranges of operators on normed spaces and of elements of normed algebras*, London. Math. Soc. Lecture Notes (Camb. Univ. Press, 1971).
2. — *Numerical Ranges — II*, London Math. Soc. Lecture Notes (Camb. Univ. Press, 1973).
3. R. J. Fleming and J. E. Jamison — *Hermitian and adjoint abelian operators on certain Banach spaces*, Pac. J. Math. 52 (1974), 67 — 84.
4. — *Adjoint abelian operators on L^p and $C(K)$*, Trans. Amer. Math. Soc. 217 (1976), 87 — 98.
5. P. R. Halmos — *Measure Theory*, D. Van Nostrand C. (1950).
6. N. J. Kalton and G. V. Wood — *Orthogonal systems in Banach spaces and their applications*, Math. Proc. Camb. Phil. Soc. 79 (1976) 493—510.
7. W. Rudin — *Real and complex analysis*, Mc Graw Hill Inc (1966).
8. D. K. Sen — *Sur l'adjoint généralisé d'un opérateur*, C. R. Acad. Sci. Paris t-287 (1978), 13 — 14.
9. — *On adjoint of an operator on reflexive Banach spaces*, Math. Vesnik, Vol 4 (17) (32) (1980), 369 — 372.
10. — *Generalized p-self adjoint operators on Banach spaces*, Math. Japonica, Vol. 27 No. 1 (1981), 151 — 158.
11. — *On Generalized p-orthogonal projections on Banach spaces*. Anal. Sc. Univ. Al. I. Cuza (Romania) Vol. 34 (1988), 227 — 237.
12. J. G. Stampfli — *Adjoint abelian operators on Banach spaces*, Canad. J. Math. 21. (1969), 505 — 512.

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