An. şt. Univ. „Al. I. Cuza" Iaşi
Matematica, 36 (1990)

SEMI-CONTINUITY PROPERTIES OF FUNCTIONS AND MULTIFUNCTIONS

BY

CĂTĂLIN STĂRICĂ

§ 0. Introduction. The development of optimization theory calls for an extension of the concept of semi-continuity to vector-valued functions (see [5], [6]). In this note, we introduce a new notion (see Definition 1.6), which plays a prominent part in the net characterization of quasi-lower semi-continuity (see Proposition 1.9) and in establishing a connection between lower semicontinuous functions and quasi-lower semi-continuous functions (see Corollary 1.11). In the second part of this note, we point out extensions for two theorems given by Berge [2], theorems concerning the semi-continuity of the value function and the solution multifunction when the objective function is valued in ordered topological spaces (see Propositions 2.4, 2.5).

§ 1. Quasi-inferior limit set. In what follows, $X=(X, \tau_X)$ denotes a topological space and $Y=(Y, \tau_Y)$ denotes a preordered topological space (i.e., a topological space endowed with a reflexive and transitive relation).

Let us first recall

Definition 1.1. *Let $Z\subset Y$. The decreasing hull and the increasing hull of Z are*

$$d(Z)=\{y\in Y/\exists z\in Z,\ y\leq z\},\qquad i(Z)=\{y\in Y/\exists z\in Z,\ z\leq y\}.$$

Definition 1.2. [5]. *A mapping $f: X\to Y$ is said to be:*

i) *lower semi-continuous (l.s.c.) at $x\in X$ if, for any neighborhood V of $f(x)$ in Y, there exists a neighhorhood U of x in X with $f(U)\subset i(V)$.*

ii) *quasi-lower semi-continuous (q.l.s.c.) at $x\in X$ if, for every b such that $f(x)\in Y-d(b)$, there exists a neighborhood U of x in X such that $f(a)\in Y-d(b)$ for every $a\in U$.*

Proposition 1.3. [5]. *If the order in Y is semi-closed, (i.e. the sets $i(y)$ and $d(y)$ are closed seets for every $y\in Y$) then every l.s.c. mapping at x is also q.l.s.c. at x.*

Definition 1.4. [4]. *Given a filter basis $\mathcal{B}$ in Y, the inferior limit set $\underline{L}(\mathcal{B})$ is the set of those $y\in Y$ such that, for any neighborhood U of y in Y, there exists $B\in\mathcal{B}$ with $B\subset i(U)$. (if N is a net we use $\underline{L}(N)$ for $\underline{L}(\mathcal{B}_N)$ where $\mathcal{B}_N$ is the filter basis generated by N).*

Propostion 1.5. *A function $f: X\to Y$ is l.s.c. at x iff, for every net $N: I\to X$ converging to x, we have $f(x)\in\underline{L}(f(N))$.*

We introduce now the notion announced previously, which is closely tied to the preorder relation.

Definition 1.6. *Given a filter basis $\mathcal{B}$ in Y, the quasi-inferior limit set $\underline{C}(\mathcal{B})$ is the set of $y\in Y$ such that, for any b with $y\in Y-d(b)$, there exists $B\in\mathcal{B}$ with $B\subseteq Y-d(b)$.*

The quasi-superior limit set $\overline{C}(\mathcal{B})$ can be defined analogously.

From the definitions follows

Proposition 1.7. i) *if $y\in\underline{C}(\mathcal{B})$ and $y'\leq y$, then $y'\in\underline{C}(\mathcal{B})$*; ii) *$\underline{C}(\mathcal{B})=\underline{C}(\mathcal{F}_B)$, where $\mathcal{F}_B$ is the filter generated by filter basis $\mathcal{B}$*;

iii) *if $\mathcal{B}_1\subseteq\mathcal{B}_2$, then $\underline{C}(\mathcal{B}_1)\subseteq\underline{C}(\mathcal{B}_2)$*;

iv) *$\underline{L}(\mathcal{B})\subseteq\underline{C}(\mathcal{B})$ whenever the preorder on Y is demi-closed.*

Now we will make a usual construction [4]: If $N: I\to Y$ is a net in Y, we can introduce the space $I'=I\cup\{w\}$ obtained by adding a point w to I endowed with discret topology. A basis of neigborhoods of w is $\{I_i\cup\{w\}/i\in I\}$, where $I_i=\{j\in I/j\geq i\}$. Now we recall that any filter basis $\mathcal{B}$ generates a net $N_B: I_B\to Y$, where $I_B=\{(y, B)/B\in\mathcal{B}, y\in B\}$ and N_B is the restriction of the first projection, I_B being preordered by $(x, B)\geq(y, C)$ if $B\subseteq C$.

In what follows, if N is a net, we use $\underline{C}(N)$ for $\underline{C}(\mathcal{B}_N)$, where $\mathcal{B}_N$ is the filter basis generated by N.

It is easy to see that the following characterization result holds.

Proposition 1.8. i) *Let $\mathcal{B}$ be a filter basis in Y. Then $y\in\underline{C}(\mathcal{B})$ iff y is a value one can take to extend N_B at w in such a way that the extended mapping $N_B: I'\to Y$ is q.l.s.c. at w.*

ii) *Let $N: I\to Y$ be a net. Then $y\in\underline{C}(N)$ iff $y\in\underline{C}(P)$ for any P a subnet of N.*

The notion we introduced helped us to give a characterization for q.l.s.c. mappings in the same manner as the already existing nets characterizations for continuous or semi-continuous mapping.

Proposition 1.9. *The mapping $f: X\to Y$ is q.l.s.c. at $x\in X$ iff, for every net $N: I\to X$ which converges to $x(N\to x)$, one has $f(x)\in\underline{C}(f(N))$.*

Proof. f q.l.s.c. at x and $N\to x$ entail that for every b with $f(x)\in Y-d(b)$ there exists $i\in I$ such that $f(N(I_i))\subseteq Y-d(b)$ and that means $f(x)(\in\underline{C}(f(N))$.

Conversely, suppose by reductio ad absurdum that f is not q.l.s.c. at x, i.e., there exists b with $f(x)\notin d(b)$ such that, for every V, neighborhood of x, there exists $x_V\in V$ with $f(x_V)\in d(b)$. That allows us to build a net $N: I\to X$, where $I=\mathcal{V}(x)$ ordered by inclusion and with $N(V)=x_V$. This net converges to x and $f(x)\notin\underline{C}(f(N))$, which is a contradiction.

We will establish a relation between l.s.c. mapping and q.l.s.c. mapping. We begin with a preliminary result.

Proposition 1.10. *Let $N: I\to Y$ be a net. If $y\in\underline{L}(N)$ and $y\in\underline{C}(\mathcal{V}(y))$ then $y\in\underline{C}(N)$.*

Proof. According to definition 1.2 it follows that for every $b\in Y-i(y)$, there exists $V\in\mathcal{V}(y)$ such that $b\in Y-i(V)$ and because $y\in\underline{L}(N)$ for this V there exists $i\in I$ such that $N(I_i)\subseteq V$ as claimed.

Corollary 1.11. *If $f: X\to Y$ is l.s.c. at x and if $f(x)\in\underline{C}(\mathcal{V}(f(x)))$ then f is q.l.s.c. at x.*

The proof is obvious using the net characterization for q.l.s.c. and l.s.c. mappings and the previous lemma.

In what follows, we prove that Corollary 1.11 is stronger than Proposition 1.3. In order to attain this purpose, we prove first a characterization result.

Proposition 1.12. *Let $y\in Y$. Then $y\in\underline{C}(\mathcal{V}(y))$ iff for every $b\in Y-i(y)$ there exist no nets $N: I\to Y$ with $N\to y$ and $b\geq N(i)$, for every $i\in I$.*

Proof. Let first suppose that $y\in\underline{C}(\mathcal{V}(y))$. By reductio ad absurdum, we assume that there exist $b\in Y-i(y)$ and $N\to y$ with $b\geq N(i)$, for any $i\in I$. Then, for this b, there exists a neighborhood V of y in Y such that $b\in Y-i(V)$ and for this V there exists $i\in I$ such that $N(I_i)\subset V$. This entails that $b\in i(V)$. Contradiction.

Conversely, we will use the same method of reductio ad absurdum. We assume that $y\in Y-\underline{C}(\mathcal{V}(y))$, i.e., there exists $b\in Y-i(y)$ such that, for every V, neighborhood of y, there exists $y_V\in V$ with $b\in i(y_V)$. Now we build the following net: $I=\mathcal{V}(y)$ preordered with inclusion and $N(V)=y_V$. This net converges to y and $b\geq N(i)$ for every $i\in I$. Contradiction.

Corollary 1.13. *If $d(y)$ is closed in Y topology for every $y\in Y$ then $y\in\underline{C}(\mathcal{V}(y))$ for every $y\in Y$.*

Proof. By reductio ad absurdum we presume that there exists $y\in Y$ with $y\in Y-\underline{C}(\mathcal{V}(y))$. Using the former proposition that is equivalent to: there exist $b\in Y-i(y)$ and $N: I\to Y$ with $N\to y$ and $b\geq N(i)$ for every $i\in I$. In other words, $N(i)\in d(b)$, for every $i\in I$. Because $N\to x$ and $d(b)$ is closed, it follows that $x\in d(b)$. Contradiction.

It is obvious now that Proposition 1.3 follows from our results: Corollary 1.11 and Corollary 1.13.

Example. We shall see now what we mean by $y\in\underline{C}(\mathcal{V}(y))$ for y belonging to a pretty particular ordered normed vectorial space, i.e., R^n (with euclidean norm and usual order). This space has a semi-closed order and, following lemma 1.4, one has that $y\in\underline{C}(\mathcal{V}(y))$ for every $y\in R^n$. We can obtain this result directly: Let $y=(y_1, \ldots, y_n)\in R^n$ and $b=(b_1, \ldots, b_n)\in R^n-i(y)$. In other words, there exists $i\in\{1, \ldots, n\}$ such that $b_i<y_i$. Let us write $y_i-b_i=\varepsilon$. If $V=\{x\in R^n/\|x-y\|<\varepsilon/2\}$ this is the neighborhood we have looked for because $b\in R^n-i(V)$.

§2. **Generalizations of Berge's theorems.** In what follows, $X=(X, \tau_X)$ and $Y=(Y, \tau_Y)$ are two topological spaces and $Z=(Z, \tau_Z, \leq)$ is an ordered topological space (endowed with a reflexive, tranzitive and antisymmetrical relation).

Let first recall some definitions.

Definition 2.1. *Let $F: X\to Y$ be a multifunction. F is said to be:*

i) *upper semi-continuous (u.s.c.) at $x\in X$ if, for each open subset G of Y containing $F(x)$, there exists a neighborhood U of x in X with $F(a)\subset G$ for each $a\in U$.*

ii) *lower semi-continuous (l.s.c.) at $x\in X$ if, for each open subset G of Y with $G\cap F(x)\neq\emptyset$, there exists a neighborhood U of x in X with $G\cap F(a)\neq\emptyset$, for each $a\in U$.*

iii) *closed at $x\in X$, if, for each net $N: I\to X$ with $N\to x$ and for each net $P: I\to Y$ such that $P(i)\in F(N(i))$ and $P\to y$, one has $y\in F(x)$.*

iv) *uniformly compact near* $x \in X$ *if there is a neighborhood* U *of* x *such that the closure of the set* $\bigcup_{a \in U} F(a)$ *is compact.*

Before proving our results let give the following two definitions.

Definition 2.2. *Let* $A \subset Z$. *The topological infimum (we write* $\inf_t A$*) of the set* A *is any element* $z \in Z$ *such that* $z \le a$ *for every* $a \in A$ *and, for every neighborhood* V *of* z *in* Z, *there exists* $a_V \in A$ *with* $a_V \in V$.

Definition 2.3. Z *is said to be lower pseudo-separated if, for any two elements* $z_1 \in Z$ *and* $z_2 \in Z$ *with* $z_1 > z_2$, *there exist a neighborhood* U *of* z_1 *and a neighborhood* V *of* z_2 *such that* $U \cap d(V) = \emptyset$.

Remark. It is easy to see that R^n, with the usual norm and order, *is a pseudo-separated ordered topological space.*

Now we can state our first result.

Proposition 2.4. *Let* $\Gamma: X \to Y$ *be a multivalued function with non void values, l.s.c. at* a *and let* $f: X \times Y \to Z$ *be an u.s.c. mapping on* $\{a\} \times \Gamma a$. *Then, the mapping* $m: X \to Z$, *given by* $m(x) = \inf_t \{f(x, y) / y \in \Gamma x\}$, *is u.s.c. at* a, *whenever* m *is single valued.*

Proof. Let $m(a) \in Z$ and V be an open neighborhood of $m(a)$. $m(a)$ is a topological infimum and that is why there exists $b \in \Gamma a$ such that V is a neighborhood of $f(a, b)$. But f is u.s.c. and for this V there exist U, a neighborhood of a and W, a neighborhood of b, such that $f(x, y) \in d(V)$ for any $x \in U$ and $y \in W$.

$b \in \Gamma a \cap W$ and Γ l.s.c. entail that there exists U' a neighborhood of a such that $\Gamma x \cap W \neq \emptyset$ for every $x \in U'$. Now it is clear that for every $x \in U \cap U'$ we have $m(x) \in d(V)$.

Proposition 2.5. *Suppose that* Z *is lower pseudo-separated. Let* $\Gamma: X \to Y$ *be a multivalued mapping with non void values, closed at* a, *l.s.c. at* a, *uniformly compact near* a *and let* $f: X \times Y \to Z$ *be a continuous mapping on* $\{a\} \times \Gamma$. *If* $m: X \to Z$ *is defined then* $M: X \to Y$, *given by:*

$$M(x) = \{y \in \Gamma x / f(x, y) \le m(x)\},$$

is u.s.c. at a.

Proof. By reductio ad absurdum, we suppose that M is not u.s.c. at a, i.e., there exists a closed subset F of Y such that $F \cap Ma = \emptyset$ and for every neighborhood V of a there exists $x_V \in V$ with $F \cap Mx_V \neq \emptyset$. Now we build two nets $N_1: I \to X$ and $N_2: I \to Y$ with $I = \mathcal{V}(a)$ and $N_1(V) = x_V$, $N_2(V) = y_V$ where $y_V \in F \cap Mx_V$. It is clear that $N_1 \to a$ and N_2 has a subnet P, which converges to b (that is so because Γ is uniformly compact near a). Γ is closed at a and that allows us to say that $b \in \Gamma a$: it is true that $f(a, b) > m(a)$. Because $f(x_V, y_V) = m(x_V)$ and f is continuous, one has that $m(x_V) \to f(a, b)$. Now, using the u.s.c. of m at a and the lower pseudo-separability of Z, we obtain the contradiction.

Corollary 2.6. *Suppose that* Z *is lower pseudo-separated and* Y *is* T_2. *Let* $\Gamma: X \to Y$ *be a multivalued mapping with non void values, continuous at* a, *uniformly compact near* a, *with* Γa *a closed subset of* Y *and let* $f: X \times Y \to Z$ *be a continuous function on* $\{a\} \times \Gamma a$. *If* $m: X \to Z$ *is defined then* M *is u.s.c. at* a.

Proof. Berge shows in [2] that if Γ is u.s.c. at a and Γa is compact then Γ is closed at a. The proof of our corollary is visible now using the proposition above.

Proposition 2.7. *Suppose that* Z *is lower pseudo-separated. Let* $\Gamma: X \to Y$ *be a multivalued mapping with non void values, closed at* a, *l.s.c. at* a *and let* $f: X \times Y \to Z$ *be a continuous function on* $\{a\} \times \Gamma a$. *If* $m: X \to Z$ *is defined then* M *is closed at* a.

Proof. Supposing that $N: I \to X$ converges to a and $N_1: I \to Y$ with $N_1(i) \in M(N(i))$ converges to b, it follows that $b \in \Gamma a$ (because Γ is closed at a). From $m(N(i)) = f(N(i), N_1(i))$ it arises that $m \circ N$ converges to $f(a, b)$. If we assume that $f(a, b) > m(a)$, using the pseudo-separability of Z and the upper-semicontinuity of m, we obtain a contradiction.

Acknowledgement. I am gratefull to Professor Th. Precupanu for his guidance, for his confidence and last but not least for his observations which improved this note.

REFERENCES

1. Bednarczuk E. — *On upper-semicontinuity of global minima in constrained optimization problems.* J. Math. Anal. Appl. 86 (1982), 309 — 318.
2. Berge C. — *Espaces Topologiques*, Dunod, Paris, 1951.
3. Hogan W. W. — *Point-to-set maps in mathematical programming.* SIAM Rev. 15 (1973), 591 — 603.
4. Penot J. P. — *Compact nets, filters and relations.* J. Math. Anal. Appl. 93 (1983), 400 — 418.
5. Penot J. P., Thera M. — *Semi-continuité des applications et des multiapplications.* C. R. Acad. Sci., Série A (1979).
6. Penot J. P., Thera M. — *Semi-continuous mapping in general topology.* Arch. Math. 38 (1982), 158 — 166.

Received 1. VIII. 1988
Revised 12. III. 1990

Faculty of Mathematics
University of Iași
6600 — Iași, România