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ON SOME CHARACTERIZATION OF INNER PRODUCT SPACES

BY

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Let X denote a normed space over the real field $\mathbb{R}$. In this paper we shall give some characterizations of inner product spaces by using (or perusing) the properties of norm derivatives. Similar results can be stated for complex spaces.

Set (for x, y in X) $\tau(x, y) := \lim\limits_{t \to 0+} (\|x+ty\| - \|x\|)/t$; note that the last expression is nondecreasing with t. For the sake of completeness, we list the properties of the functional τ that will be used in the sequel.

(i) $\tau(x, \lambda x + \mu y) = \lambda\|x\| + \mu\tau(x, y)$ for $\lambda \in \mathbb{R}$, $\mu \geq 0$;

(ii) $\tau(\lambda x, \mu y) = \mu\tau(x, y)$ for $\lambda, \mu > 0$;

(iii) $-\tau(x, -y) = \lim\limits_{t \to 0-} (\|x+ty\| - \|x\|)/t \leq \tau(x, y) \leq \|y\|$;

(iv) $\tau : X \times X \to \mathbb{R}$ is upper semicontinuous;

(v) $\tau(x, 0) = 0$, $\tau(0, y) = \|y\|$ for every x, y in X.

The space is smooth if and only if one of the following conditions holds:

(vi) $-\tau(x, -y) = \tau(x, y)$ for all x, y in X, $x \neq 0$;

(vii) $\tau(x, y+z) = \tau(x, y) + \tau(x, z)$ for all $x, y, z \in X$, $x \neq 0$.

Now, we state a well-known result that we shall use in the sequel.

Lemma 1 ([5], Theorem 6.4). *The normed space X is an inner product space if and only if*

$$\tau(x, y) = \tau(y, x) \text{ for all pairs } x, y \text{ in } X, \ \|x\| = \|y\| = 1. \tag{1}$$

For other characterizations of inner product spaces see [5—8] where further references are given.

1. The following result is a consequence of Lemma 1.

Theorem 1. *The normed linear space X is an inner product space if and only if there exists a mapping $T : X \to \mathbb{R}$ such that the following conditions hold*

$$\lim_{t \to 0+} T(ty)/t = 0 \text{ for all } y \text{ in } X; \tag{2}$$

$$\|x+y\|^2 \leq \|x\|^2 + 2\tau(x, y)\|y\| + T(y) \text{ for all } x, y \text{ in } X. \tag{3}$$

Proof. If $(X; (\,,\,))$ is an inner product space, then $T(y) := (y, y)$, $y \in X$ satisfies the assumptions (2) and (3).

Conversely, suppose that there exists a mapping T such that (2) and (3) hold. Then for every $t\in\mathbb{R}$ we have:

$$\|x+ty\|^2\leq\|x\|^2+2\tau(ty,x)|t|\|y\|+T(ty) \text{ for all } x,\ y \text{ in } X. \tag{4}$$

If we assume that $t>0$ then (4) becames

$$(\|x+ty\|^2-\|x\|^2)/2t\leq\tau(y,x)\|y\|+T(ty)/2t$$

what implies

$$\tau(x,y)\|x\|\leq\tau(x,y)\|y\| \text{ for all } x,y \text{ in } X. \tag{5}$$

If $\|x\|=\|y\|=1$, then (5) gives $\tau(x,y)=\tau(y,x)$ and from Lemma 1 we obtain the desired result.

Corollary. *A normed linear space X is an inner product space if and only if there exists a constant $k\geq 1$ such that*

$$\|x+y\|^2\leq\|x\|^2+2\tau(y,x)\|y\|+k^2\|y\|^2 \text{ for all } x,\ y \text{ in } X. \tag{6}$$

2. Now, we shall give the following definition.

Definition 1. *The mapping $(\,,\,)_T: X\times X\to\mathbb{R}$ given by*

$$(x,y)_T:=\tau(y,x)\|y\| \text{ for all } x,\ y \text{ in } X. \tag{7}$$

will be called semi-inner product in the sense of Tapia or T-semi-inner product, for short ([1], p. 392).

For some usual properties of T-semi-inner products we send to [2–3] and to [1] pp. 392–400 where further references are given.

Definition 2. *The T-semi-inner product is said to be derivable on X if the following limit*

$$(y,x)_{TT}:=\lim_{t\to 0}[(y,x+ty)_T-(y,x)_T]/t \tag{8}$$

exists for all x, y in X.

It is easy to see that if $(\,,\,)_T$ is derivable, then $(\,,\,)_T$ is continuous, i. e.

$$(y,x)_T=\lim_{t\to 0}(y,x+ty)_T \text{ for all } x,\ y \text{ in } X. \tag{9}$$

The following proposition is a consequence of Giles' theorem [4].

Proposition 1. *A normed linear space is smooth if and only if $(\,,\,)_T$ is continuous.*

Now, we shall give some usual properties of T-semi-inner product derivative.

Proposition 2. *Let X be a smooth linear space endowed with a derivable T-semi-inner product. Then the following statements hold*

(i) $(y,y)_{TT}=\|y\|^2$ *for all* $y\in X$;
(ii) $(y,0)_{TT}=\|y\|^2$ *for all* $y\in X$;
(iii) $(\alpha y,x)_{TT}=\alpha^2(y,x)_{TT}$ *for all* $\alpha\in\mathbb{R}$ *and* $x,y\in X$;
(iv) $(y,\alpha x)_{TT}=(y,x)_{TT}$ *for all* $\alpha\in\mathbb{R}\setminus\{0\}$ *and* $x,y\in X$;
(v) $\|x\|^2(y,x)_{TT}\geq(y,x)_T^2$ *for all* $x,y\in X$.

Proof. The properties (i)–(iv) are obviously from the definition of $(\,,\,)_{TT}$. We only prove the relation (v).

It is easy to see that

$$(y,x+ty)_T-(y,x)_T\geq(y,x)_T(\|x+ty\|-\|x\|)/\|x\|$$

for all $x,y\in X$, $x\neq 0$. Then we obtain:

$$[(y,x+ty)_T-(y,x)_T]/t\geq(y,x)_T(\|x+ty\|-\|x\|)/(t\|x\|)$$

for all $t>0$, what implies

$$(y,x)_{TT}\geq(y,x)_T^2/\|x\|^2 \text{ for all } x,\ y\in X,\ x\neq 0,$$

and the proposition is proven.

Proposition 3. *Let X be a smooth normed linear space endowed with a derivable T-semi-inner product and x, y two given elements in X. Then the mapping*

$$\varphi_{x,y}:\mathbb{R}\to\mathbb{R},\quad \varphi_{x,y}(t):=\|x+ty\|^2 \tag{10}$$

is derivable of two orders on $\mathbb{R}$ and the second derivative is nonnegative on $\mathbb{R}$. In addition,

$$\varphi'_{x,y}(t)=2(y,x+ty)_T \text{ and } \varphi''_{x,y}(t)=2(y,x+ty)_{TT} \text{ for all } t\in\mathbb{R}. \tag{11}$$

Proof. Let $t_0\in\mathbb{R}$. Then we have

$$\lim_{t\to t_0}[\varphi_{x,y}(t)-\varphi_{x,y}(t_0)]/(t-t_0)=\lim_{h\to 0}(\|x+t_0y+hy\|^2-\|x+t_0y\|^2)/(t-t_0)=$$

$$=2(y,x+t_0y)_T \text{ for all } x,\ y\in X,$$

and similarly

$$\lim_{t\to t_0}[\varphi'_{x,y}(t)-\varphi'_{x,y}(t_0)]/(t-t_0)=2(y,x+t_0y)_{TT}$$

for all $x,y\in X$.

The second derivative of $\varphi_{x,y}$ is nonnegative from the property (v) of the above proposition and the proof is finished.

Now, we shall give the following characterization theorem of inner product spaces in the class of smooth normed linear spaces endowed with a derivable T-semi-inner product.

Theorem 2. *Let X be a smooth normed linear space as above. Then the following statements are equivalent*

(i) *X is an inner product space;*
(ii) *The mapping* $\psi_{x,y}:\mathbb{R}\to\mathbb{R}$, $\psi_{x,y}(t)=(y,tx)_{TT}$ *is continuous in 0 for all x, y in X;*
(iii) *For every x, y in X there exists a sequence* $\alpha_n\in\mathbb{R}\setminus\{0\}$, $\alpha_n\to 0$ *such that* $\lim_{n\to\infty}(y,\alpha_nx)_{TT}=(y,0)_{TT}$;
(iv) *For every x, y in X we have*

$$(y,x)_{TT}=\|y\|^2.$$

Proof. (i)$\Rightarrow$(ii). It's obvious observing that

$$(y, x)_{TT} = \|y\|^2 \text{ for all } x, y \in X.$$

(ii)$\Rightarrow$(iii). It's obvious.

(iii)$\Rightarrow$(iv). Let $x, y \in X$ and $\alpha_n \to 0$ $(\alpha_n \neq 0)$. Since $(y, x)_{TT} = (y, \alpha_n x)_{TT}$, we obtain $(y, x)_{TT} = \lim\limits_{n \to \infty}(y, \alpha_n x)_{TT} = (y, 0)_{TT} = \|y\|^2$.

(iv)$\Rightarrow$(i). By Taylor's formula for $\varphi_{x,y}$ we have

$$\|x + ty\|^2 = \|x\|^2 + 2(y, x)_T t + \|y\|^2 t^2$$

for all $t \in \mathbb{R}$, what implies the parallelogram identity:

$$\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2) \text{ for all } x, y \in X,$$

i.e., X is an inner product space.

The proof is finished.

Finally, we shall give some examples of smooth normed linear spaces endowed with derivable T-semi-inner products.

Examples. Let $(\Omega, \mathcal{A}, \mu)$ be a measure space consisting of a set Ω, a σ-algebra $\mathcal{A}$ of subsets of Ω and a countably additive and positive measure μ on $\mathcal{A}$ with values in $\mathbb{R} \cup \{\infty\}$. If we denote $L^p(\mu) = L^p(\Omega, \mathcal{A}, \mu)$ and $p > 1$, then it is well-known that (see for example [1], p. 312):

$$\lim_{t \to 0}(\|x + ty\|_p - \|x\|_p)/t = \|x\|_p^{1-p} \int_\Omega |x(s)|^{p-1} \operatorname{sgn} x(s) d\mu(s)$$

for all $x, y \in L^p(\mu)$, $x \neq 0$.

Suppose $p \geq 2$ and put $p = 2k+2$, $k \geq 0$. Then we have:

$$(y, x)_{Tp} := \lim_{t \to 0}(\|x + ty\|_p^2 - \|x\|_p^2)/2t = \|x\|_p^{-2k} \int_\Omega x^{2k+1}(s) y(s) d\mu(s),$$

for all $x, y \in L^p(\mu)$, $x \neq 0$ and $(y, 0)_{Tp} = 0$ if $y = L^p(\mu)$.

A simple calculus gives:

$$(y, x)_{TTp} = (2k+1)\|x\|_p^{-2k} \int_\Omega x^{2k}(s) y^2(s) d\mu(s) - 2k\|x\|_p^{2k-2} \left(\int_\Omega x^{2k+1}(s) y(s) d\mu(s)\right)^2$$

for all $x, y \in L^p(\mu)$, $x \neq 0$ and $(y, 0)_{TTp} = \|y\|^2$ if $y \in L^p(\mu)$.

Remark. The above example shows that there exists smooth normed linear spaces endowed with a derivable T-semi-inner product which are not inner product spaces. See for example $L^p(\mu)$ with $p > 2$.

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