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ABOUT CLUSTER SETS IN INFINITE-DIMENSIONAL SPACES

BY

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The theory of cluster sets, created in 1895 by P. Painleve [6] is a very important tool in the study of the boundary behaviour of the analytic and pseudo-analytic (quasiconformal) functions of a complex variable (cf. the monograph of K. Noshiro (1955) [5] and of E.F. Collingwood and A.J. Lohwater (1966) [3]). This concept was generalized in the n-dimensional case (cf. P.T. Church [2], R. Nakki [4] and M. Vuorinen [7]). In the present paper, we consider cluster sets in infinite-dimensional spaces.

Let X be a metric space $D \subset X$ a domain and $f: D \to X$ a continuous mapping. f is said to have a *sequential limit* $\alpha \in X$ at $\xi \in \partial D$ (where ∂D means the boundary of D) if there exists a sequence $\{x_n\} \subset D$ such that $x_n \to \xi$ and $f(x_n) \to \alpha$. Let γ be *an endcut of D from ξ*, i.e. $\gamma: [0, 1) \to X$ is such that $\gamma(t) \to \xi$ as $t \to 1$. f has *an asymptotic value* $\alpha \in X$ if there exists an endcut γ of D from ξ such that $f[\gamma(t)] \to \alpha$ as $t \to 1$. A sequence of points $\{x_n\} \subset D$ *converges in a cone to the boundary point* ξ if there exists a constant $a \in [1, \infty)$ such that $|x_n - \xi| \le a d(x_n, \partial D)$ $(n = 1, 2, \ldots)$. f is said to have an *angular limit* $\alpha \in X$ if, for each sequence $\{x_n\} \subset D$ converging in a cone to ξ, $f(x_n) \to \alpha$. f has a *limit* $\alpha \in X$ at ξ through a set E, if $\bar{E} \subset D \cup \{\xi\}$ and $f(x) \to f$ as $x \to \xi$ and $x \in E$.

A set $C(f, \xi)$ is said to be the *cluster set* of a continuous mapping $f: D \to X$ at $\xi \in \partial D$ if it is the set of all sequential limits of f at ξ.

Now let us give several other characterizations of the cluster sets.

Theorem 1. *A set $C(f, \xi)$ is a cluster set at $\xi \in \partial D$ of a continuous mapping $f: D \to X$ iff at least one of the following conditions holds:*

I. $C(f, \xi) = \cap \overline{f(U_\xi \cap D)}$, *where the intersection is taken over all the neighbourhoods U_ξ of ξ.*

II. $C(f, \xi) = \bigcap_{r>0} \overline{f[B(\xi, r) \cap D]}$, *where $B(\xi, r)$ is the ball of center ξ and radius r.*

III. $C(f, \xi) = \{x' \in X;\ \xi \in \overline{E_\varepsilon^{x'}}\ \forall \varepsilon > 0\}$, *where* $E_\varepsilon^{x'} = \{y \in D;\ d[f(y), x'] < \varepsilon\}$.

Proof. Since it is easy to see that conditions I and II represent equivalent characterizations for the cluster sets, we renounce to give a proof for them. In order to see that also the third condition is such an equivalent characterization, let us observe that

$$\{x' \in X;\ \xi \in \overline{E_\varepsilon^{x'}}\ \forall \varepsilon > 0\} = \{x' \in X;\ B(\xi, r) \cap E_\varepsilon^{x'} \ne \emptyset\ \forall r > 0,\ \forall \varepsilon > 0\} =$$

$$= \{x' \in X;\ \exists y_{r\varepsilon} \in D \cap B(\xi, r), \text{ where } d[f(y_{r\varepsilon}), x'] < \varepsilon,\ \forall r > 0,\ \forall \varepsilon > 0\} =$$

$= \{x' \in X; f[D \cap B(\xi, r)] \cap B(x', \varepsilon) \neq \varnothing \ \forall \varepsilon > 0, \forall r > 0\} = \bigcap_{r>0} \overline{f[D \cap B(\xi, r)]} = C(f, \xi)$,

where „$\exists$" means „there exists".

From the definition of a cluster set, it follows that $C(f, \xi) \subset \overline{f(D)}$ $\forall \xi \in \partial D$.

A domain $G \subset\subset D$ is called *normal* for the mapping $f: D \to X$, if $f(\partial G) = \partial f(G)$. Here $G \subset\subset D$ means that $\overline{G} \subset D$.

By the same argument as in the finite-dimensional case, we obtain

Theorem 2. *If $f: D \to X$ is a homeomorphism, $C(f, \xi) \subset \partial f(D)$.*

We observe that in the case of a compact space, $C(f, \xi) \neq \varnothing$, while in a non-compact space (as for instance a Hilbert or a Banach space), the cluster set may be empty since in such a space, the sets $\overline{f(U_\xi \cap D)}$ are, in general, no more compact, so that $\cap \overline{f(U_\xi \cap D)}$ could be the empty set. However, in all the cases, the cluster sets are closed. Hence, it follows that in a compact space, the cluster sets are compact. In order that $C(f, \xi) \neq \varnothing$, it is sufficient that there exist a point $\xi' \in \overline{f(U_\xi \cap D)}$ $\forall U_\xi$.

If a cluster set comes to a point, it is said to be *degenerated*, otherwise, it is called *non-degenerated*. If $C(f, \xi) = X$, the cluster set is called *total*, otherwise *subtotal*. It may seem natural to use the term „degenerated" for the empty cluster set, but, we adopted the terminology corresponding to the compactified complex space $\overline{C}$, where $C(f, \xi) \neq \varnothing$.

Let us define a *cluster set* $C(f, E)$ of a continuous mapping $f: D \to X$ relatively to a set $E \subset \partial D$ by $C(f, E) = \bigcup_{\xi \in E} C(f, \xi)$. In the particular case, $E = \partial D$, $C(f, \partial D) = C(f)$.

A cluster set $C_E(f, \xi)$ of f at ξ and with respect to a set $E \subset D$, where $\overline{E} \subset D \cup \{\xi\}$ is the set of the sequential limits corresponding to all the sequences $\{x_n\} \subset E$. Clearly, $C_E(f, \xi) = C(f_E, \xi)$. In particular, if $\gamma \subset D$ is an endcut of D from ξ and $E = \gamma$, we obtain the *cluster set* $C_\gamma(f, \xi)$ *of f at ξ with respect to the arc* γ. If $E = C$ (where C is a cone contained in D and with the vertex ξ), we get the *cluster set* $C_c(f, \xi)$ *of f at ξ with respect to the cone* C. (Usually, in the definition of a cone, the concept of angle is involved, restraining us at most to Hilbert spaces, but, this may be avoided as we did in the definition of the convergence in a cone.) Combining the definitions from above, we may have cluster sets of the form $C_\gamma(f, E)$ and $C_c(f, E)$ $\forall E \subset \partial D$ and, in particular, $C_\gamma(f)$ and $C_c(f)$.

We recall that a domain $D \subset X$ is said to be *locally connected at a boundary point* $\xi \in \partial D$ if there exist arbitrary small neighbourhoods U_ξ of ξ such that $U_\xi \cap D$ is connected.

Theorem 3. *If X is a metric compact arcwise connected space, D is locally connected at $\xi \in \partial D$ and $f: D \to X$ is continuous, then, $C(f, \xi)$ is a continuum or a point.*

Proof. If U_ξ is a neighbourhood such that $U_\xi \cap D$ is connected, then, on account of the continuity of f, $f(U_\xi \cap D)$ will be connected too and $\overline{f(U_\xi \cap D)}$ will be a continuum. Let us show that $C(f, \xi) = \cap \overline{f(U_\xi \cap D)}$ will be a continuum or a point. Suppose, to prove it is false, that $C(f, \xi) = F_1 \cup F_2$, where F_1, F_2 are closed and $F_1 \cap F_2 = \varnothing$. Hence F_1, F_2 will be compact and $d(F_1, F_2) > 0$. But then, the set $F_3 = \{x; d(x, F_1) = \varepsilon\}$ will separate the sets F_1 and F_2 if $0 < \varepsilon < d(F_1, F_2)$. And now, let us consider a minimizing sequence $\{U_n\}$ $(U_{n+1} \subset U_n)$ of neighbourhoods U_ξ for $C(f, \xi)$, i.e. such that $C(f, \xi) = \bigcap_{n=1}^{\infty} \overline{f(U_n \cap D)}$, and the corresponding sequence $\{F_3 \cap \overline{f(U_n \cap D)}\}$. Clearly, $F_3 \cap \overline{f(U_n \cap D)} \neq \varnothing$ since X and $\overline{f(U_n \cap D)}$ are arcwise connected, $F_1 \cup F_2 \subset \overline{f(U_n \cap D)}$ $\forall n \in N$ and F_3 separates F_1 from F_2. But then, since X is assumed to be compact,

$$F_3 \cap C(f, \xi) = F_3 \cap \left[\bigcap_{n=1}^{\infty} \overline{f(U_n \cap D)}\right] = \bigcap_{n=1}^{\infty} \left[F_3 \cap \overline{f(U_n \cap D)}\right] \neq \varnothing.$$

But, on the other side, $C(f, \xi) \cap F_3 = (F_1 \cup F_2) \cap F_3 = \varnothing$. This contradiction implies that $C(f, \xi)$ is connected, so that, since $C(f, \xi)$ is also closed, it follows it is a continuum or a point, as desired.

Remark. If the preceding theorem still holds in non-compact spaces (which I do not think), then, it would be necessary to find another argument, because, in the preceding proof, in order to obtain a contradiction, implying the connectedness of $C(f, \xi)$, we used precisely the fact that the intersection of a decreasing sequence of closed sets is not empty.

Finally, if X is a Hilbert space with a given basis, we want to point out the importance of the cluster sets $C_E(f, \xi)$ in the particular case $D = B(x_0, R) \subset X$ and $E = \Pi_n \cap D$, (where Π_n is an n-dimensional plane) i.e. the cluster sets with respect to finite-dimensional sections of the domain of definition. The importance of these kind of cluster sets is justified also by the fact that the concepts of extremal length, module and conformal capacity have no natural extension to the infinite-dimensional case and then, may be the finite-dimensional section method is the only one allowing us to use them in the infinite-dimensional case too. Thus, e.g., if the intersection $\Pi_n \cap D$ of D with an n-dimensional plane Π_n containing a point $\xi \in \partial D$ consists of several connected components, then, clearly, only the components having ξ as boundary point have an effective part in the definition of $C_{\Pi_n \cap D}(f, \xi) = \{C_{\Delta_1}(f, \xi), C_{\Delta_2}(f, \xi), \ldots\}$; there correspond, in general, several (at most a countable number) finite-dimensional cluster sets (each of them corresponding to a component Δ_n of $D_n \Pi_n$, with ξ as a boundary point). We observe that it is possible that $D \cap \Pi_n$ contain no component Δ_n with ξ as a boundary point and this for each Π_n through ξ. If U_ξ is a neighbourhood of $\xi \in \partial D$ in X and ξ is an isolated boundary point, then, $U_\xi - \{\xi\} \subset D$ and U_ξ is said to be a neighbourhood of ξ, while if all the balls with center ξ contain a point of $\partial D - \{\xi\}$, then, the components of $B(\xi, r) \cap D$ containing ξ as a boundary point are called *relative neighbourhoods of* ξ. In a similar way, the the components of $U_\xi \cap D \cap \Pi_n$, having ξ as a boundary point are called *sectional neighbourhoods of* ξ *relative to* D. We point out that we consider only the sections for which $U_\xi \cap D \cap \Pi_n$ is an open n-dimensional set with ξ as a boundary point for at least one of its components.

An interesting problem is the connection between the cluster set $C(f, \xi)$ of a domain in $D \subset X$ and the corresponding sectional ones. An important part is played by the so called degenerated cluster sets $C(f, \xi) = \xi'$, because, e.g. a continuous function $f: D \to X$ with the property that $C(f, \xi)$ is degenerated $\forall \xi \in \partial D$ may be continuously continued to $\overline{D}$.

Theorem 4. *If the cluster set $C(f, \xi)$ of a continuous mapping $f: B \to X$ is degenerated at a point $\xi \in \partial B$, where $B = B(x_0, R)$, then, there exists an $n \in N$ such that all the corresponding sectional cluster sets are degenerated, and $C_{\Pi_n \cap B}(f, \xi) = \{\xi'\} \forall \Pi_n$ through ξ with $\Delta_n = \Pi_n \cap B \neq \varnothing$.*

Proof. Clearly, since $C(f, \xi) = \{\xi'\}$, then, all the finite-dimensional sections Π_n of B through ξ will have degenerated cluster sets, i.e. $C_{\Delta_n}(f, \xi) = \{\xi'\}$, as a consequence of the evident inclusion relation $C_E(f, \xi) \subset C(f, \xi)$ and since $C_{\Delta_n}(f, \xi) \neq \varnothing$ $\forall \Delta_n \neq \varnothing$.

Now, let us consider the set $E^{x'}_{C, \varepsilon} = \{y \in C \cap D;\ d[f(y), x'] < \varepsilon\}$ and let us prove the following

Lemma. *If $f: D \to X$ is a continuous mapping, $\xi \in \partial D$ and $C \subset D$ is a cone with the vertex ξ, then, $C_c(f, \xi) = \{x' \in X;\ \xi \in \overline{E^{x'}_{C, \varepsilon}}\ \forall \varepsilon > 0\}$.*

Proof. Let us show first that $C_c(f, \xi) \subset \{x' \in X;\ \xi \in \overline{E^{x'}_{C, \varepsilon}}\ \forall \varepsilon > 0\}$. Let $\xi' \in C_c(f, \xi)$; then, on account of the definition of the cluster set, there is a sequence $\{x_n\} \subset C$ such that $f(x_n) \to \xi'$ as $x_n \to \xi$. But then, $\forall \varepsilon > 0$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ so that $\|f(x_{n_k}) - \xi'\| < \varepsilon$ $(k = 1, 2, \ldots)$, hence $\{x_{n_k}\} \subset E^{\xi'}_{C, \varepsilon}$ and, since $x_{n_k} \to \xi$, it follows that $\xi \in \overline{E^{\xi'}_{C, \varepsilon}}$ and then, $\xi' \in \{x' \in X;\ \xi \in \overline{E^{x'}_{C, \varepsilon}}\ \forall \varepsilon > 0\}$.

But, also the converse is true. Indeed, let $\xi' \in \{x' \in X;\ \xi \in \overline{E^{x'}_{C, \varepsilon}}\ \forall \varepsilon > 0\}$. Then, $\forall \varepsilon > 0$, $\xi \in \overline{E^{\xi'}_{C, \varepsilon}}$ implying the existence of a sequence $\{x_n\} \subset C \cap D$ such that $x_n \to \xi$ and $\{x_n\} \subset E^{\xi'}_{C, \varepsilon}$, hence $\|f(x_n) - \xi'\| < \varepsilon$ $(n = 1, 2, \ldots)$. And now, let us consider a strictly decreasing sequence $\{\varepsilon_n\}$, $\varepsilon_n \to 0$ and $\forall n \in N$, let us denote by $\{x^m_n\}$ a sequence such that $x^m_n \to x_n$ and let us consider a sequence $\{y_n\}$ with $y_n = x^{m_n}_n$ chosen so that $y_n \to \xi$ and $\|f(y_n) - \xi'\| < \varepsilon_n$. Hence $f(y_n) \to \xi'$ and then $\xi' \in C_c(f, \xi)$, as desired.

Arguing as in the preceding theorem, we have also

Theorem 5. *If the cluster set $C_c(f, \xi)$ of a continuous mapping $f: B \to X$ at a point $\xi \in \partial B$ is degenerated for each cone $C \subset D$ with vertex ξ, i.e. $C_c(f, \xi) = \{\xi'\}$ $\forall C \subset D$, then, there exists an $n \in N$ such that the corresponding sectional cluster sets $C_{C \cap \Pi_n \cap D}(f, \xi) = \{\xi'\}$.*

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BALAYAGES AND MORPHISMS IN H-CONES

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Introduction. The balayages were, toghether with the H-integrals, one of the fundamental models for the notion of a morphism of H-cones.

Taking as starting point the beautiful characterisation of the image of a balayage in standard H-cones, due to N. Boboc (unpublished), given here as th. 17, this note presents „nearly" a characterisation for the images of balayages in the general setting of H-cones: see th. 14, 15 and 16.

The main step is th. 3, in which use is made of the simple remark that, since for any balayage B on a H-cone S, $B(S)$ is still a H-cone, we have another morphism $S \to B(S)$, which has many interesting properties. These properties may also serve as exemples for the general case of H-maps and adjoint morphisms.

The notions, notations and results about H-cones, used without reference, are in accordance with [3].

Let S be a H-cone $s \in S$ is called universally bounded (see also [2]) if, for any $\mu \in S^*$, $\mu(s) < +\infty$. $s \in S$ is called nearly continuous if, for any decreasing $F \subseteq S^*$ we have:

$$(\wedge F)(s) = \inf_{\mu \in F} \mu(s).$$

Let S, T be H-cones. We recall [4] that a morphism from S to T is a map $\varphi: \tilde{S} \to \tilde{T}$, which is additive, increasing and continuous (i.e. for any increasing $F \subseteq S$: $\varphi(\bigvee F) = \bigvee \varphi(F)$). We denote: $D(\varphi) = \{s \in S \mid \varphi(s) \in T\}$. φ is said to be semifinite if $D(\varphi)$ is dense in S (i.e. $\forall s \in S$ $\exists F \subseteq D(\varphi)$ increasing, such that $s = \bigvee F$).

If S, T are H-cones of functions on the sets X, Y, then $\varphi: Y \to X$ is called H-map if for any bounded $s \in S$, $s \circ \varphi \in T$. Clearly, any H-map induces a morphism.

The semifinite morphism φ^* from T^* to S^* is called an adjoint of φ if: $\varphi^*(\mu)(s) = \mu(\varphi s)$, $\forall \mu \in T^*$, $s \in S$. We remark that not every morphism has an adjoint.

A Morphism Associated with a Balayage. To any balayage B on a H-cone S, can be associated a morphism $\hat{B}: S \to B(S)$, which has many interesting properties. Reciprocally, to an ordered convex subcone $S' \subseteq S$, satisfying some properties, can be associated a function $\hat{B}$.