Theorem 4. *If the cluster set $C(f, \xi)$ of a continuous mapping $f: B \to X$ is degenerated at a point $\xi \in \partial B$, where $B = B(x_0, R)$, then, there exists an $n \in \in N$ such that all the corresponding sectional cluster sets are degenerated, and $C_{\Pi_n \cap B}(f, \xi) = \{\xi'\} \forall \Pi_n$ through ξ with $\Delta_n = \Pi_n \cap B \neq \varnothing$.*

Proof. Clearly, since $C(f, \xi) = \{\xi'\}$, then, all the finite-dimensional sections Π_n of B through ξ will have degenerated cluster sets, i.e. $C_{\Delta_n}(f, \xi) = \{\xi'\}$, as a consequence of the evident inclusion relation $C_{\Delta}(f, \xi) \subset C(f, \xi)$ and since $C_{\Delta_n}(f, \xi) \neq \varnothing$ $\forall \Delta_n \neq \varnothing$.

Now, let us consider the set $E^{x'}_{C,\varepsilon} = \{y \in C \cap D;\ d[f(y), x'] < \varepsilon\}$ and let us prove the following

Lemma. *If $f: D \to X$ is a continuous mapping, $\xi \in \partial D$ and $C \subset D$ is a cone with the vertex ξ, then, $C_c(f, \xi) = \{x' \in X;\ \xi \in \overline{E^{x'}_{C,\varepsilon}}\ \forall \varepsilon > 0\}$.*

Proof. Let us show first that $C_c(f, \xi) \subset \{x' \in X;\ \xi \in \overline{E^{x'}_{C,\varepsilon}}\ \forall \varepsilon > 0\}$. Let $\xi' \in C_c(f, \xi)$; then, on account of the definition of the cluster set, there is a sequence $\{x_n\} \subset C$ such that $f(x_n) \to \xi'$ as $x_n \to \xi$. But then, $\forall \varepsilon > 0$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ so that $\|f(x_{n_k}) - \xi'\| < \varepsilon$ $(k = 1, 2, \ldots)$, hence $\{x_{n_k}\} \subset \subset E^{\xi'}_{C,\varepsilon}$ and, since $x_{n_k} \to \xi$, it follows that $\xi \in \overline{E^{\xi'}_{C,\varepsilon}}$ and then, $\xi' \in \{x' \in X; \xi \in \in \overline{E^{x'}_{C,\varepsilon}}\ \forall \varepsilon > 0\}$.

But, also the converse is true. Indeed, let $\xi' \in \{x' \in X;\ \xi \in \overline{E^{x'}_{C,\varepsilon}}\ \forall \varepsilon > 0\}$. Then, $\forall \varepsilon > 0$, $\xi \in \overline{E^{\xi'}_{C,\varepsilon}}$ implying the existence of a sequence $\{x_n\} \subset C \cap D$ such that $x_n \to \xi$ and $\{x_n\} \subset E^{\xi'}_{C,\varepsilon}$, hence $\|f(x_n) - \xi'\| < \varepsilon$ $(n = 1, 2, \ldots)$. And now, let us consider a strictly decreasing sequence $\{\varepsilon_n\}$, $\varepsilon_n \to 0$ and $\forall n \in N$, let us denote by $\{x^m_n\}$ a sequence such that $x^m_n \to x_n$ and let us consider a sequence $\{y_n\}$ with $y_n = x^{m_n}_n$ chosen so that $y_n \to \xi$ and $\|f(y_n) - \xi'\| < \varepsilon_n$. Hence $f(y_n) \to \xi'$ and then $\xi' \in C_c(f, \xi)$, as desired.

Arguing as in the preceding theorem, we have also

Theorem 5. *If the cluster set $C_c(f, \xi)$ of a continuous mapping $f: B \to X$ at a point $\xi \in \partial B$ is degenerated for each cone $C \subset D$ with vertex ξ, i.e. $C_c(f, \xi) = \{\xi'\}$ $\forall C \subset D$, then, there exists an $n \in N$ such that the corresponding sectional cluster sets $C_{C \cap \Pi_n \cap D}(f, \xi) = \{\xi'\}$.*

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BALAYAGES AND MORPHISMS IN H-CONES

BY

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Introduction. The balayages were, together with the H-integrals, one of the fundamental models for the notion of a morphism of H-cones.

Taking as starting point the beautiful characterisation of the image of a balayage in standard H-cones, due to N. Boboc (unpublished), given here as th. 17, this note presents „nearly" a characterisation for the images of balayages in the general setting of H-cones: see th. 14,15 and 16.

The main step is th. 3, in which use is made of the simple remark that, since for any balayage B on a H-cone S, $B(S)$ is still a H-cone, we have another morphism $S \to B(S)$, which has many interesting properties. These properties may also serve as exemples for the general case of H-maps and adjoint morphisms.

The notions, notations and results about H-cones, used without reference, are in accordance with [3].

Let S be a H-cone $s \in S$ is called universally bounded (see also [2]) if, for any $\mu \in S^*$, $\mu(s) < +\infty$. $s \in S$ is called nearly continuous if, for any decreasing $F \subseteq S^*$ we have:

$$(\wedge F)(s) = \inf_{\mu \in F} \mu(s).$$

Let S, T be H-cones. We recall [4] that a morphism from S to T is a map $\varphi: \tilde{S} \to \tilde{T}$, which is additive, increasing and continuous (i.e. for any increasing $F \subseteq S: \varphi(\vee F) = \vee \varphi(F)$). We denote: $D(\varphi) = \{s \in S \mid \varphi(s) \in T\}$. φ is said to be semifinite if $D(\varphi)$ is dense in S (i.e. $\forall s \in S$ $\exists F \subseteq D(\varphi)$ increasing, such that $s = \vee F$).

If S, T are H-cones of functions on the sets X, Y, then $\varphi: Y \to X$ is called H-map if for any bounded $s \in S$, $s \circ \varphi \in T$. Clearly, any H-map induces a morphism.

The semifinite morphism φ^* from T^* to S^* is called an adjoint of φ if: $\varphi^*(\mu)(s) = \mu(\varphi s)$, $\forall \mu \in T^*$, $s \in S$. We remark that not every morphism has an adjoint.

A Morphism Associated with a Balayage. To any balayage B on a H-cone S, can be associated a morphism $\hat{B}: S \to B(S)$, which has many interesting properties. Reciprocally, to an ordered convex subcone $S' \subseteq S$, satisfying some properties, can be associated a function $\hat{B}$.

Proposition 1. *Let S be a H-cone, $B: S\to S$ be a balayage on S. (i) $B(S)$ is an ordered, convex subcone in S, specifically solid and such that: $F\subseteq B(S)$ increasing and dominated in $S \Rightarrow \bigvee F\in B(S)$.*

(ii) *Let $\varepsilon: B(S)\to S$ be the natural inclusion. There exists a unique function $\hat B: S\to B(S)$ such that: $\varepsilon\circ\hat B = B$. Moreover $\hat B\circ\varepsilon = 1_{B(S)}$.*

Proof. (i) Clearly $B(S)$ is a convex subcone in S. If $Bs=t_1+t_2$ then: $Bt_1+Bt_2\le t_1+t_2=Bs=B(Bs)=B(t_1+t_2)$ show that $t_1=Bt_1$. Hence $B(S)$ is specifically solid in S. Let $F\subseteq B(S)$ be increasing and dominated in S. Then: $\forall t\in F: Bt\le \bigvee B(F)$. hence: $\forall t\in F: Bt\le B(\bigvee B(F))$. Let $v\in S$ be such that $Bt\le Bv$, $\forall t\in F$. Then: $\bigvee B(F)\le Bv$, hence $B(\bigvee B(F))\le Bv$ which shows that: $\bigvee F=B(\bigvee F)\in B(S)$.

(ii) $\hat B$ is simply the co-restriction of B.

The next result is a universality property. We keep the same notations as in prop. 1.

Proposition 2. *Let T be a H-cone; $u: S\to T$, $v: T\to S$ be finite morphisms, such that: u is surjective and $vu=B$.*

Then, there exists a unique isomorphism $\varphi: S'\to T$ such that: $\varphi\circ u=\hat B$ and $\varepsilon\circ\varphi=v$.

Proof. Let us define $\varphi: S'\to T$ by $\varphi=u\circ\varepsilon$. Then φ is a morphism and satisfies $\varphi\circ\hat B=u$. For each $t\in T$, let $s\in S$ be an element such that $t=u(s)$. We define $\psi(t)=\hat B(s)$. This definition is correct: if $s'\in S$ is such that $t=u(s')$, then: $v(us)=v(us')$ hence $Bs=Bs'$ and $\hat Bs=\hat Bs'$ also. Since $\hat Bs\in B(S)$ it follows that ψ is a well defined map from T to S'. By definition: $\psi\circ u=\hat B$. Moreover: $\varepsilon(\psi(t))=\varepsilon(\hat Bs)=Bs=v(us)=v(t)$ and: $v(\varphi s)=v(u(\varepsilon s))=B(\varepsilon s)=$ $=\varepsilon(\hat B(\varepsilon s))=\varepsilon(s)$. Also, the following relations hold: $\psi\circ\varphi=\psi\circ u\circ\varepsilon=\hat B\circ\varepsilon=1_{B(S)}$ and: $(\varphi\circ\psi)(t)=\varphi(\hat Bs)=u(s)=t$, $\forall t\in T$. It remains to show that ψ is a morphism. If $t_1=u(s_1)$ and $t_2=u(s_2)$, then $t_1+t_2=u(s_1+s_2)$ hence: $\psi(t_1+t_2)=$ $=\hat B(s_1+s_2)=\psi(t_1)+\psi(t_2)$. If $t_1\le t_2$ then $u(t_1)\le u(t_2)$. It follows that $v(us_1)\le$ $\le v(us_2)$, hence $Bs_1\le Bs_2$ and ψ results increasing. Finally, if $F\subseteq S$ is increasing and $t=\bigvee F$, then, from $t=u(s)$ and $v(us)=\bigvee(v\,u)(F)$ we get: $Bs=\bigvee B(F)$ and $\psi(t)=\bigvee\psi(F)$.

Next we consider a more general situation. Let S be a H-cone, and $S'\subseteq S$ an ordered convex subcone, specifically solid and such that: $F\subseteq S'$ increasing and dominated in $S\Rightarrow \bigvee F\in S'$.

$\varepsilon: S'\to S$ will denote the inclusion and $\hat B: S\to S'$ will be the map defined by

$$\hat Bs=\bigvee\{t\in S'\mid \varepsilon(t)\le s\}$$

Prop. 1 shows that such a situation occurs when $S'=B(S)$ with B balayage on S. However, only a partial converse is true. The next result represents the first step toward a characterisation of balayages by means of their image.

Theorem 3. (i) *S' is an H-cone (with respect to the induced operations and order). In particular, if B is a balayage on S, then $B(S)$ is a H-cone.*

We shall distinguish the various operations in S' by a'.

(ii) *ε is a bimonotone morphism of H-cones, and is also a morphism for the specific orders. If $t_1, t_2\in S'$ then: $\varepsilon(t_1\bigvee' t_2)=\varepsilon(t_1)\bigvee\varepsilon(t_2)$.*

(iii) *$\hat B$ is additive and increasing. $\forall F\subseteq S$, $\hat B(\bigvee F)=\bigvee'\hat B(F)$; $\forall s,t\in S$: $\hat B(R(s-t))=R'(\hat Bs-\hat Bt)$; $\hat B(s\bigvee t)=\hat Bs\bigvee'\hat Bt$. Let $F\subseteq S$ be specifically increasing and dominated in S. Then: $\hat B(\gamma F)=\gamma'\hat B(F)$, hence $\hat B$ is a specific morphism.*

(iv) *$\hat B\circ\varepsilon=1_{S'}$; $\varepsilon\circ\hat B$ is an idempotent contraction (i.e. a pseudo-balayage).*

Proof. The fact that S' is specifically solid implies the Riesz' decomposition property and the additivity of $\hat B$. By construction: $\forall t\in S'$: $\hat B(\varepsilon t)=t$. Hence $(\varepsilon\circ\hat B)^2=\varepsilon\circ\hat B$. For any $F\subseteq S$ and $s\in F$ we have: $\varepsilon(\bigwedge'\hat B(F))\le\varepsilon(\hat Bs)\le s$, hence $\varepsilon(\bigvee'\hat B(F))\le\bigvee F$. This proves that $\hat B(\bigwedge F)\ge\bigwedge'\hat B(F)$. The converse inequality is obvious. Now, for arbitrary $s\in S'$ we have:

$$s+\bigwedge'F=s+\hat B(\bigwedge\varepsilon(F))=\hat B[\varepsilon(s)+\bigwedge\varepsilon(F)]=\hat B[\bigwedge\varepsilon(s+F)]=\bigwedge'(s+F)$$

Let $t_1, t_2\in S'$. Then: $\varepsilon(t_1)\bigvee\varepsilon(t_2)$ is specifically dominated by $\varepsilon(t_1+t_2)$, hence belongs to S'. It results that: $\varepsilon(t_1)\bigvee\varepsilon(t_2)=\varepsilon(t_1\bigvee' t_2)$.

Let $s, t\in S$. Since $s\bigwedge t=R(s+t-s\bigvee t)$, we get:

$$\hat B(s\bigvee t)=\hat B(R(s+t-s\bigwedge t))=R'(\hat Bs+\hat Bt-\hat Bs\bigwedge\hat Bt)=\hat Bs\bigvee'\hat Bt.$$

Let $F\subseteq S$ be specifically increasing and dominated. For each $t\in S'$ such that $\varepsilon(t)\le\gamma F$, we have: $\gamma_{s\in F}(t\curlywedge s)=t\curlywedge\gamma F=t$. Since $t\curlywedge s\in S'$, $\forall s\in F$, we get: $t\le\gamma'\hat B(F)$, hence: $\hat B(\gamma F)\le\gamma'\hat B(F)$. The other facts are obvious.

Remarks. In general, $\hat B$ is not a morphism. However, it is clear that: ($\hat B$ morphism)$\Leftrightarrow$($\forall F\subseteq S$ increasing and dominated: $\hat B(\bigvee F)=\bigvee'\hat B(F)$) $\Leftrightarrow$ ($\varepsilon\circ\hat B$ is a balayage on S).

If S' is naturally solid in S, then $\hat B$ is morphism. More generally, $\hat B$ is a morphism iff: $\forall F\subseteq S$ naturally solid, increasing and dominated and such that $\bigvee F\in S'$, we have: $\bigvee(F\cap S')=\bigwedge F$.

Proof. If $\hat B$ is a morphism, then: $F\cap S'=\hat B(F)$ hence $\bigvee\hat B(F)=\bigvee F$, since $\bigwedge F\in S'=\hat B(S)$. Conversely, let us prove the continuity of $\hat B$. Let $F\subseteq S$ be increasing and dominated. If we denote

$$F_1=\{s\in S\mid \exists t\in F \text{ such that } s\le t\bigwedge\hat B(\bigvee F)\}$$

then F_1 is solid in S and $\bigvee F_1=\hat B(\bigvee F)\in S'$. Using the hypotheses, we get: $\hat B(\bigvee F)=\bigvee F_1=\bigvee(F_1\cap S')\le\bigvee\hat B(F)$

Also, for $F \subseteq S$ increasing and dominated, and such that $\bigvee F$ is u-continuous, or $s = \bigvee F$ has the property : $\bigwedge_{t \in F} R(s-t) = 0$ then $\hat{B}(\bigvee F) = \bigvee' B(F)$ (cf. [3], th. 2.2.10. b)). Particularly, if S is associated with a Dirichlet space, then $\hat{B}$ is always a morphism.

Exemple. Let S' be the convex cone of positive harmonic functions S being the cone of positive superharmonic functions on a harmonic space. Then S' satisfies the above conditions. However, $\hat{B}$ is only a specific morphism, but not a natural one.

The Adjoint of the Morphism Associated with a Balayage. We give next conditions for the existence of the adjoints for the morphisms ε and $\hat{B}$, together with their properties.

We consider the same situation as in §1, namely S is a H-cone, $S' \subseteq S$ is an ordered convex subcone.

Proposition 4. 1) *Each of the following conditions :*

(i) *The universally bounded elements form a dense part in S^*.*

(ii) *There exists a weak unit in S'.* *implies that ε^* exists and is semi-finite.*

2) *If : the universally bounded elements form a dense part in S' and $s \in S'$ universally bounded $\Rightarrow \varepsilon(s) \in S$ universally bounded, then ε^* exists and is finite.*

3) *If $\hat{B}$ is a morphism, then ε^* exists and is finite.*

Proof. 1) (i) If $\mu \in S^*$ is universally bounded, then $\varepsilon^*(\mu) \in S'^*$. (ii) Let $u \in S'$ be a weak unit. The set : $\{\mu \in S^* \mid \mu(\varepsilon(u)) < \infty\}$ is dense in S^* ; and for any such $\mu : \varepsilon^*(\mu) \in S'$.

2) For any $\mu \in S^*$, $\mu \circ \varepsilon$ is a H-integral on S'. Indeed, if $t \in S'$ is universally bounded, then $\varepsilon(t) \in S$ is also universally bounded, *hence* $\varepsilon^*(\mu)(t) < \infty$ *on a dense part of* S'.

3) *Let again $\mu \in S^*$. Now $\mu \circ \varepsilon$ is a H-integral on S', since for each $t \in S'$ there exists an increasing family $F \subseteq S$ such that $\mu(s) < \infty$, $\forall s \in F$ and $t = \bigvee F$. It follows that $\hat{B}(F)$ is increasing in S', $t = \hat{B}t = \bigvee \hat{B}(F)$ and $\mu(\hat{B}s) < \infty$, $\forall s \in F$.*

Remark. If ε^* exists, then $\varepsilon^*(\mu)$ represents the restriction of the H-integral μ to S'.

The next result is connected to the property of ε^* to be a H-map.

Proposition 5. *If ε^* exists and is semi-finite, then :*

$$\forall \mu, \quad \nu \in D(\varepsilon^*) : \varepsilon^*(\mu \wedge \nu) = \varepsilon^*(\mu) \wedge \varepsilon^*(\nu)$$

Proof. Let us denote $\alpha = \varepsilon^*(\mu) \wedge \varepsilon^*(\nu) \in S'^*$. For any $t \in D(\alpha)$ we have $t \in D(\varepsilon^*(\mu \wedge \nu))$ and : $\varepsilon^*(\mu \wedge \nu)(t) = (\mu \wedge \nu)(\varepsilon t)$. Hence, by [3], there exists $s_1, s_2 \in S$ such that : $\varepsilon(t) = s_1 + s_2$ and $(\mu \wedge \nu)(\varepsilon(t)) = \mu(s_1) + \nu(s_2)$. It follows that $s_1, s_2 \in S'$ hence : $\mu(s_1) + \nu(s_2) \geq \alpha(s_1) + \alpha(s_2) = \alpha(s)$; $\varepsilon^*(\mu \wedge \nu) \geq \varepsilon^*(\mu) \wedge \varepsilon^*(\nu)$. The converse inequality is obvious.

Remark. If ε^* exists and is surjective, then, for any weak unit $\mu \in S^*$, $\varepsilon^*(\mu)$ is a weak unit. Hence, if S^* and S'^* are represented as H-cones of functions on the sets X^*_μ, $X'^*_{\varepsilon*(\mu)}$ then ε^* becomes a H-map.

In order to discuss the existence of $\hat{B}^*$, we must suppose of course that $\hat{B}$ is a morphism. Hence, for the last part of this paragraph, we suppose that

S is a H-cone, $B : S \to S$ is a balayage, $S' = B(S)$, $\varepsilon : S' \to S$ and $\hat{B} : S \to S'$. The same proof as in prop. 4.1) gives us

Proposition 6. *Suppose that one of the following hypotheses is fulfilled :*

(i) *The universally bounded elements form a dense part in S'^*.*

(ii) *There exists a weak unit in S.*

Then $\hat{B}^*$ exists and is semi-finite.

Proposition 7. *If $\hat{B}^*$ exists and is semi-finite, then :*

a) *$\hat{B}^*$ is injective and has a specifically solid image in S^*.*

b) *ε^* exists and is surjective.*

Proof. The existence of ε^* is already proved. From : $\hat{B} \circ \varepsilon = 1_{S'}$ we get $\varepsilon^* \circ \hat{B}^* = 1_{S'}$; hence ε^* is surjective and $\hat{B}^*$ results injective. Let $\nu \in D(\hat{B}^*)$, $\mu_1, \mu_2 \in S^*$ be such that : $\hat{B}^*(\nu) = \mu_1 + \mu_2$. It follows that : $\varepsilon^*(\mu_1) + \varepsilon^*(\mu_2) = \nu$ hence $\hat{B}^*(\varepsilon^*(\mu_1)) + \hat{B}^*(\varepsilon^*(\mu_2)) = \hat{B}^*(\nu) = \mu_1 + \mu_2$ hence : $\mu_1 = \hat{B}^*(\varepsilon^*(\mu_1))$ and $\hat{B}^*(S'^*)$ results specifically solid in S^*.

A partial converse also holds :

Proposition 8. *Suppose that ε^* exists and is surjective. Then $\hat{B}^*$ exists.*

Proof. Let $\nu \in S'^*$; we show that $\nu \circ \hat{B}$ is a H-integral. Since ε^* is surjective, there exists $\mu \in S^*$ such that $\nu = \varepsilon^*(\mu)$. Now $B^*\mu \in S^*$. Since: $B^*(\mu(s)) = \mu(Bs) = \mu((\varepsilon \circ \hat{B})(s)) = \varepsilon^*(\mu)(\hat{B}s) = \nu(Bs)$, $\forall s \in S$, the proof is ended.

Remark. Let us suppose that $\varepsilon : S' \to S$ and $\hat{B} : S \to S'$ have finite adjoints. Let us denote : $B = \varepsilon \circ \hat{B}$. B is a balayage, hence $B^* : S^* \to S^*$ exists and is also a balayage. Let us denote : $S^{*\prime} = B^*(S^*)$ and $i : S^{*\prime} \to S^*$ the canonical inclusion. Aplying the universality property from prop. 2 to the situation S replaced by S^* ; T replaced by S'^* ; $u = \varepsilon^*$; $v = \hat{B}^*$.

We get the existence of a canonical isomorphism between $S^{*\prime}$ and S'^*. Through this isomorphism, ε^* corresponds to B^* and $\hat{B}^*$ to i.

In view of the characterisation from the next paragraph, we consider next some connections of ε and $\hat{B}$ with the universally bounded and nearly continuous elements.

Proposition 9. *Let us suppose that the universally bounded elements form a dense part of S. Then : $\hat{B}^*$ exists and is a finite morphism iff : $\forall s \in S$ universally bounded, $\hat{B}s \in S'$ is universally bounded.*

In this case, the universally bounded elements form a dense part in S'.

Proof. Let $s \in S$ be universally bounded. Then, for any $\mu \in S'^*$ we have $\hat{B}^*\mu \in S^*$, hence $\hat{B}^*\mu(s) = \mu(Bs) < \infty$ and $\hat{B}s$ results universally bounded. Conversely, if we define $\hat{B}^* : S \to \overline{R}_+$ by : $\hat{B}^*\mu(s) = \mu(Bs)$, then $\hat{B}^*\mu$ is finite on a dense part, hence is a H-integral. Finally, for each $t \in S'$ we can find an increasing family $F \subseteq S$ of universally bounded elements, such that $\bigvee F = \varepsilon(l)$. Now: $\bigvee \hat{B}(F) = l = \hat{B}(\varepsilon l)$ and each element from $\hat{B}(F)$ is universally bounded.

Proposition 10. *If $s \in S'$ is universally bounded, then $\varepsilon(s)$ is universally bounded in S.*

Proof. For any $\mu \in S^*$, we have $\varepsilon^*(\mu) = \mu \circ \varepsilon = \mu|_{S'} \in S'^*$, hence:

$$\mu(\varepsilon s) = \varepsilon^*(\mu)(s) = (\mu|_{S'})(s) < \infty.$$

Proposition. 11. *If $\hat{B}^*$ exists (as a semi-finite morphism) then: $s \in S'$ is nearly continuous $\Rightarrow \varepsilon(s)$ is nearly continuous in S.*

Proof. Let $F \subseteq S^*$ be a decreasing family. For each $\mu \in F$ we have:

$$\hat{B}^*[\bigwedge' \varepsilon^*(F)] \leq \hat{B}^* \varepsilon^*(\mu) \leq \mu$$

(since $\hat{B}^* \circ \varepsilon^* = (\varepsilon \circ \hat{B})^* = B^*$). We get next: $\hat{B}^*[\bigwedge' \varepsilon^*(F)] \leq \bigwedge F$

Since: $\varepsilon^* \circ \hat{B}^* = 1_{S^*}$: $\bigwedge' \varepsilon^*(F) \leq \varepsilon^*(\bigwedge F)$. The converse inequality is obvious. Let $s \in S'$ be nearly continuous. Using the equality proved above:

$$(\bigwedge F)(\varepsilon s) = \varepsilon^*(\bigwedge F)(s) = [\bigwedge' \varepsilon^*(F)(s)] = \inf_{\mu \in F} \varepsilon^*(\mu)(s) = \inf_{\mu \in F} \mu(\varepsilon s)$$

Let B be a balayage on the H-cone S. We call B nearly continuous if: $\forall F \subseteq S$ decreasing: $\bigwedge B(F) = B(\bigwedge F)$.

Proposition 12. *Let B be a balayage such that B^* is nearly continuous and $\hat{B}^*$ exists. Then: $s \in S$ nearly continuous $\Rightarrow Bs$ is nearly continuous in S'. Particularly, if the nearly continuous elements form a dense part in S, then the nearly continuous elements form a dense part in S'.*

Proof. Let $F \subseteq S'^*$ be a decreasing family. From the proof of prop. 11, we can write: $\bigwedge' F = \bigwedge'(\varepsilon^* \circ \hat{B}^*)(F) = \varepsilon^*(\bigwedge \hat{B}^*(F))$, hence:

$$\hat{B}^*(\bigwedge'(F)) = (\hat{B}^* \circ \varepsilon^*)(\bigwedge \hat{B}^*(F)) = B^*(\bigwedge \hat{B}^*(F))$$

Since B^* is supposed to be nearly continuous, we have further:

$$\hat{B}^*(\bigwedge' F) = \bigwedge B^* \circ \hat{B}^*(F) = \bigwedge \hat{B}^* \circ \varepsilon^* \circ \hat{B}^*(F) = \bigwedge \hat{B}^*(F)$$

Let now $s \in S$ be nearly continuous. Then:

$$(\bigwedge' F)(Bs) = \hat{B}^*(\bigwedge F)(s) = (\bigwedge \hat{B}^*(F)(s)$$

Since $\hat{B}^*(F)$ is a decreasing family from S^*, we have finally:

$$(\bigwedge' F)(\hat{B}s) = \inf_{\mu \in F} \hat{B}^*(\mu)(s) = \inf_{\mu \in F} \mu(\hat{B}s)$$

Characterisation of Balayages. The specific case can be easily characterised, using th. 3.

Theorem 13. *Let S be a H-cone, $S' \subseteq S$ be an ordered convex subcone. Then the following are equivalent:*

1) *There exists a specific balayage B on S such that $B(S) = S'$.*

2) *S' is specifically solid and for any specifically increasing and dominated $F \subseteq S'$, $\curlyvee F \in S'$.*

Proof. $1 \Rightarrow 2$ is easy and $2 \Rightarrow 1$ results from th. 3 (iii). The next theorem characterises the balayages among a class of morphisms.

Theorem 14. *Let S, S' be H-cones, $\hat{B}: S \to S'$ be a finite morphism. Then the following are equivalent:*

1) *$\hat{B}$ is surjective and for any $F \subseteq S$: $\hat{B}(\bigwedge F) = \bigwedge' \hat{B}(F)$.*

2) *There exists a balayage B on S and a canonical identification $\varepsilon: S' \to B(S)$ such that $B = i \circ \varepsilon \circ \hat{B}$, where $i: B(S) \to S$ is the canonical inclusion.*

Proof. $\underline{1 \Rightarrow 2}$. We begin by defining $\varepsilon: S' \to S$ by: $\forall l \in S'$ $\varepsilon(l) = \bigwedge \{s \in S \mid \hat{B}s = l\}$. The hypotheses show that the definition is valid and $\hat{B} \circ \varepsilon = 1_{S'}$. Hence ε is injective and if we denote $B: S \to S$, $B = \varepsilon \circ \hat{B}$, then: $B \circ B = (\varepsilon \circ \hat{B}) \circ (\varepsilon \circ \hat{B}) = \varepsilon \circ \hat{B} = B$. By definition, for any $s \in S$: $Bs = (\varepsilon \circ \hat{B})(s) \leq s$. It suffices to show that ε is a morphism of H-cones.

Let $l_1, l_2 \in S'$; if $s_1, s_2 \in S$ are such that $\hat{B}s_k = l_k$, $k = 1, 2$ then $\hat{B}(s_1 + s_2) = l_1 + l_2$ hence: $\varepsilon(l_1 + l_2) \leq s_1 + s_2$. We get $\varepsilon(l_1 + l_2) \leq \varepsilon(l_1) + \varepsilon(l_2)$. From Riesz' decomposition property, there exists $u_1, u_2 \in S$ such that: $\varepsilon(l_1 + l_2) = u_1 + u_2$ and: $u_k \leq \varepsilon(l_k)$, $k = 1, 2$. It results that: $\hat{B}u_k \leq l_k$, $k = 1, 2$ and $l_1 + l_2 = \hat{B}u_1 + \hat{B}u_2 \leq l_1 + l_2$. Hence $\hat{B}u_k = l_k$, $k = 1, 2$ and $u_k = \varepsilon(l_k)$, $k = 1, 2$. Let now $l_1, l_2 \in S'$ be such that $l_1 \leq l_2$. Let us denote $u = \varepsilon(l_1) \wedge \varepsilon(l_2)$. We have: $\hat{B}u = \hat{B}(\varepsilon(l_1) \wedge \varepsilon(l_2)) = l_1 \wedge l_2 = l_1$. We get: $\varepsilon(l_1) \leq \varepsilon(l_1) \wedge \varepsilon(l_2)$, hence $\varepsilon(l_1) \leq \varepsilon(l_2)$. Let finally $F \subseteq S'$ be an increasing family. $\varepsilon(F)$ is an increasing family from S, hence denoting $u = \bigvee \varepsilon(F)$, we get $\hat{B}u = \hat{B}(\bigvee \varepsilon(F)) = \bigvee'(\hat{B} \circ \varepsilon)(F) = \bigvee' F$. Hence $\varepsilon(l) \leq u$ and $\varepsilon(l) \leq \bigvee \varepsilon(F)$. The converse inequality is obvious.

For a dual characterisation of balayages, we need a definition. Let S be a H-cone; $S' \subseteq S$ a subcone which is a H-cone with respect to the induced operations and order. We say that S' posseses the extension property if

(E) There exists a solid, dense part $A \subseteq S'^*$, such that: $\forall \mu \in A$, $\exists \bar{\mu} \in S^*$ such that $\bar{\mu}|S' = \mu$.

Theorem 15. *Let S, S' be H-cone, satisfying: $S = S^{**}$, $S' = S'^{**}$. Let $\varepsilon: S' \to S$ be a finite morphism. The following are equivalent:*

1) *ε is injective; $s \in S'$ nearly continuous $\Rightarrow \varepsilon(s) \in S$ is nearly continuous; the image $\varepsilon(S')$ is specifically solid and has the extension property; ε^* has an adjoint.*

2) *There exists a finite morphism $\hat{B}: S \to S'$ which has an adjoint and such that: $\varepsilon \circ \hat{B}$ is a balayage on S; $\hat{B} \circ \varepsilon = 1_{S'}$.*

Proof. $1 \Rightarrow 2$ $D(\varepsilon^*) \subseteq S^*$ and $A \subseteq S'^*$ are by hypotheses dense and solid, hence H-cones; $\varepsilon^*: D(\varepsilon^*) \to A$ becomes then a finite morphism. Moreover, since ε is injective and has a specifically solid image, then [5] shows that: $\varepsilon^*(\mu \wedge \nu) = \varepsilon^*(\mu) \wedge \varepsilon^*(\nu)$, $\forall \mu, \nu \in S'^*$. Next, since ε preserves the nearly continuous elements it follows that ε^* commutes with decreasing infima, hence with arbitrary infima. Now, th. 14 can be applied to ε^*. This gives the existence of a finite morphism $\varphi: A\to D(\varepsilon^*)$, such that: $\varphi\circ\varepsilon^*$ is a balayage on $D(\varepsilon^*)$ and $\varepsilon^*\circ\varphi=1_A$. By prop. 4.3) φ^* exists and is finite. Denoting $\hat{B}=\varphi^*$ we have obtained $\hat{B}: S^{**}\to S'^{**}$ with all the required properties.

2⇒1 $\hat{B}\circ\varepsilon=1_{S'}$, shows that ε is injective. Since $B=\varepsilon\circ\hat{B}$ is a balayage on S, it follows from prop. 4.3) that ε^* exists and is finite. Now $\varepsilon(S')=B(S)$ is specifically solid. Since $\hat{B}^*$ exists, from prop. 6 we get the extension property and the fact that ε preserves the nearly continuous elements.

The next theorem contains the main result of the paper, giving sufficient and „nearly" necessary conditions, in order that an ordered convex subcone S' of an H-cone S, be the image of a balayage.

Theorem 16. *Let S be a H-cone such that $S=S^{**}$. Let $S'\subseteq S$ be an ordered convex subcone having the following properties:*

1) *$\forall F\subseteq S'$ increasing and dominated, $\vee F$, calculated in S, belongs to S'.*
2) *S' is specifically solid in S.*
3) *S' has the extension property.*
4) *The set of nearly continuous elements is dense in S'.*
5) *$s\in S'$ nearly continuous $\Rightarrow\varepsilon(s)$ is nearly continuous in S.*
6) *One of the following properties holds:*
   (i) *There exists a weak unit in S'.*
   (ii) *The set of universally bounded elements is dense in S^*.*
   (iii) *The set of universally bounded elements is dense in S' and: $s\in S'$ universally bounded $\Rightarrow\varepsilon(s)$ is universally bounded in S.*

Then, there exists a balayage B on S, such that $B(S)$ is canonically identified with S'^{**}.

Proof. The proof proceeds in the following steps, which are essentialy the same as in the preceeding theorem.

1) and 2) guarantees that S' is a H-cone and $\varepsilon: S'\to S$ is a morphism of H-cones. Each of the properties 6 (i), 6 (ii) assures the existence of ε^* as a semifinite morphism; moreover, 6 (iii) implies ε^* finite. The extension property shows now that $\varepsilon^*: D(\varepsilon^*)\to A$ is a finite, surjective morphism of H-cones. Finally 2), 4) and 5) show that $\varepsilon^*(\wedge F)=\wedge\varepsilon^*(F)$, for any $F\subseteq S'^*$. Hence, th. 14 can be applied to $\varepsilon^*: D(\varepsilon^*)\to A$ and the conclusion follows.

Comments. 1) and 2) hold when S' is the image of a balayage. The second part of 6 (iii) (the fact that ε preserves the universally bounded elements) holds for the images of balayages. The extension property 3) holds as soon as $\hat{B}$ has an adjoint. If $\hat{B}^*$ is finite, then $A=S'^*$. Conditions 4) and 5) are discussed in prop. 11 and 12.

Balayages on Standard H-cones. For standard H-cones, there exists a nice characterisation for the images of balayages (Boboc, unpublished).

Theorem 17. *Let S be a standard H-cone and $S'\subseteq S$. The following are equivalent:*

1) *S' is an ordered convex subcone, specifically solid; $F\subseteq S'$ increasing and dominated $\Rightarrow\vee F\in S'$; S' is a standard H-cone and any universally continuous element of S' is a universally continuous element in S.*

2) *There exists a balayage B on S, such that $B(S)=S'$.*

Proof. 1⇒2 Using th. 3, it is enough to show that, for any $F\subseteq S$ increasing and dominated, we have: $\hat{B}(\vee F)=\vee'\hat{B}(F)$. Let $(t_i)_{i\in I}$ be an increasing family of universally continuous elements from S' (hence also from S), such that: $\hat{B}(\vee F)=\bigvee_{i\in I}t_i$. For each $i\in I$ we have $t_i=\bigvee_{t\in F}(t_i\wedge t)$; hence, $u\in S'$ being a weak unit, we get:

$$\forall\varepsilon>0\ \exists t_\varepsilon\in F \text{ such that: } t_i\le t_\varepsilon\wedge t_i+\varepsilon u$$

Now: $t_i\le\hat{B}(t\wedge t_i)+\varepsilon\hat{B}u\le\hat{B}t_\varepsilon+\varepsilon\hat{B}u$

$$\hat{B}(\vee F)=\bigvee_{i\in I}t_i\le\hat{B}t_\varepsilon+\varepsilon\hat{B}u\le\vee\hat{B}(F)+\varepsilon\hat{B}u$$

Since $\varepsilon>0$ is arbitrary, it follows that: $\hat{B}(\vee F)\le\vee'B(F)$. The converse inequality is obvious.

2⇒1 From [3, prop. 5.2.6. c)], $B(S)$ is a standard H-cone. Let $s\in S'$ be a universally continuous element from S' and $u\in S'$ be a weak unit. For any family $(t_i)_{i\in I}$ from S, increasing to $s\in S'$, we have: $s=Bs=\vee Bt_i$. Now $Bu\in S'$ is a weak unit in S', since: $\forall t\in S': t=Bt=B(\bigvee_{n\in N}(t\wedge n.u))\le\bigvee_{n\in N}(Bt\wedge n.Bu)=\bigvee_{n\in N}t\wedge B(n.u)\le t$. Since s is universally continuous, there exists $i_0\in I$ such that: $s\le Bt_{i_0}+Bu$. Hence $s\le t_{i_0}+u$ and s is universally continuous in S.

Remarks. 1⇒2 holds, with the same proof, if S, S' are regular stonian cones [1] and $s\in S'$ regular $\Rightarrow s$ regular in S.

A different proof for 1⇒2 can be obtained, using th. 16. We must suppose that $S=S^{**}$. The conditions 1) and 2) from th. 16 are imposed also in 1); conditions 4), 5) and any one of 6 (i), (ii) or (iii) are clearly fulfilled here. It remains to verify the extension property. The part $A\subseteq S'^*$ can be taken as the set of universally bounded elements. For such $\mu\in A$, we define $\tilde{\mu}: S\to\overline{R}_+$ by: $\tilde{\mu}(s)=\sup\{\mu(t)\mid t\in S',\ t\le s\}$, $\tilde{\mu}$ is clearly well defined and increasing. The additivity of $\tilde{\mu}$ is also easily checked. For $s\in S_0$ and $(s_i)_{i\in I}$ increasing to s, we have: $\tilde{\mu}(s)=\sup_{i\in I}\tilde{\mu}(s_i)$. Let now $\overline{\mu}$ denote the unique extension of $\tilde{\mu}|_{S_0}$ to S. We get then the desired extension.

Now, th. 16 can be applied, giving the balayage B on S, such that $B(S)$ is isomorphic to S'^{**}. But, again from the extension property, it follows that $S'^{**}=S'$ and the proof is ended.

We end with a theorem containing results about balayages on a standard H-cone.

Theorem 18. *Let S be a standard H-cone of functions on the nearly saturated space X. Let B be a balayage on S.*

1) *$B(S)$ is a standard H-cone of functions on the nearly saturated space $b(B)$. If X is (semi-) saturated, then $b(B)$ is also (semi-) saturated with respect to $B(S)$. There exists a canonical bijection between the balayages B_1 on $B(S)$ and those balayages $\tilde{B}_1$ on S, satisfying: $\tilde{B}_1\circ B=B\circ\tilde{B}_1=\tilde{B}_1$.*

2) *The natural and fine topologies on $b(B)$ (with respect to $B(S)$) coincide respectively, with the traces of the natural and fine topologies on X. The canonical inclusion $i: b(B)\to X$ is a H-map, the associated morphism being $\tilde{B}: S\to B(S)$. Hence, the H-map i is: naturally and finely continuous, naturally and finely open on the image; and i^{-1} preserves the polar and semi-polar sets.*

3) *Let Ω be a Green set for (S, u, u^*). Then the inclusion $b(B^*)\to\Omega$ induces the morphism $\varepsilon^*: S^*\to B(S)^*$. Since $b(B)$ and $b(B^*)$ differ by a semi-polar set in Ω, there exists a Green set Ω' for $(B(S), Bu, \varepsilon^*(u^*))$ and an inclusion $\Omega'\to\Omega$ which induces both morphism $\hat{B}$ and ε^*. Moreover, $G: X_u\times X^*_{\mu}\to \bar{\mathbb{R}}_+$* and $G': X'_{Bu}\times X'^*_{\varepsilon^*(u^*)}\to\bar{\mathbb{R}}_+$ being the Green functions, we have: $G'\circ(\hat{B}^*\times\varepsilon)=G$.

Proof. 1) Let $t\in B(S)$; there exists $s\in S$ such that $t=Bs$. The correspondence $t\to s|_{b(B)}$ is well defined and realises an isomorphism between $B(S)$ and $S|_{b(B)}$. Let B_1 be a balayage on $B(S)$. If we put $\tilde{B}_1=\varepsilon\circ\hat{B}_1\circ\hat{B}$, then $\tilde{B}_1$ is a balayage on S and $\tilde{B}_1\le B$, hence $b(\tilde{B}_1)\subseteq b(B)$. Reciprocally, if $\tilde{B}_1$ is a balayage on S satisfying the given conditions, then $B_1=\hat{B}\circ\tilde{B}_1\circ\varepsilon$ defines a balayage on $B(S)$. Now $B_1=B^{b(\tilde{B}_1)}$ shows that $b(B)$ is nearly saturated for $B(S)$. If X is supposed to be semi-saturated [2], then let μ be a H-measure on $b(B)$ and $\nu\le\mu$ in $B(S)^*$. We have: $B^*\nu\le B^*\mu$. Since $\hat{B}^*\mu(s)=\mu(\hat{B}s)=\int\hat{B}s\,d\mu$ on $b(B)$ and $s=Bs$ on $b(B)$, $\hat{B}^*\mu$ results a H-measure on X. Hence $\tilde{B}'\nu$ is also a H-measure on X. But $\varepsilon^*(\hat{B}^*\nu)=\nu$ and ν is carried by $b(B)$, since:

$$B^*\nu(s)=\int s\,d\mu=\int Bs\,d\mu=B^*(\nu)(Bs)$$

hence: $\int(s-Bs)d\mu=0$, $\forall s\in S$. It follows that ν is a H-measure on $b(B)$, thus proving that $b(B)$ is semi-saturated. Let now X be saturated and denote by X' the saturated space for $B(S)$ with respect to the weak unit B_u. X' can be identified with $b(B)$ in the following way: if $\mu\in X'$, then $B^*\mu\in b(B)$ since $Bs(B^*\mu)=s(B^*\mu)$, $\forall s\in S$.

2) Taking into account the above identification, i induces $\hat{B}$. i is finely continuous, since (Bs is universally continuous in $B(S)$)$\Rightarrow$(s is universally continuous in S), i results naturally open on its image and the restriction of the natural topology to $b(B)$ results finer than the natural topology of $B(S)$. Using [3], i follows naturally continuous, hence the coincidence of the natural topologies. Analogously, i results finely open on its image, hence the coincidence of the fine topologies. Let now $A\subseteq b(B)$ be a polar (resp. totally thin) set in X with respect to S. Then: $\bigwedge'\{u\in B(S)\mid u\ge 1 \text{ on } A\}\le\bigwedge'\{s|_{b(B)}\mid s\in S,\ s\ge 1 \text{ on } A\}$ (resp. if $x\in b(B)$ and $s\in S$ is such that $Bs(x)<s(x)$ then: $\bigwedge'\{u\in B(S)\mid u\ge s \text{ on } A\}\le\bigwedge'\{t|_{b(B)}\mid t\in S,\ s\le t \text{ on } A\}$. Now $i^{-1}(A)$ is polar (resp. totally thin) since $\bigwedge' B(F)=B(\bigwedge F)$.

3) The canonical inclusion $b(B^*)\to\Omega$ induces the morphism $\hat{B}^*$; the above discussion proves that $\hat{B}^*$ can be identified with ε^*. Now [3, prop. 5.5.8] proves the second assertion. Finally, identifying $X_u\subseteq S^*$, $X^{**}_u\subseteq S$, $X'_{B u}\subseteq B(S)^*$, $X'^*_{\varepsilon^*(u^*)}\subseteq B(S)$, then for any $x\in B(S)^*$, $y\in B(S)$ we have:

$$G(\hat{B}^*x, \varepsilon(y))=\hat{B}^*x(\varepsilon(y))=x(\hat{B}\varepsilon y)=x(y)=G'(x, y)$$

Remark. There exists a bijective correspondence between the morphisms $\varphi: B(S)\to B(S)$ and those morphisms $\tilde{\varphi}: S\to S$ which satisfy: $\tilde{\varphi}\circ B=B\circ\tilde{\varphi}=\tilde{\varphi}$ obtained in the same way as for balayages.

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