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# THE METHOD OF LINES FOR PARABOLIC DIFFERENTIAL-FUNCTIONAL EQUATIONS WITH INITIAL-BOUNDARY CONDITIONS OF THE DIRICHLET TYPE

BY

ZDZISLAW KAMONT

**1. Introduction.** Let us denote by $\mathcal{F}(X, Y)$ the set of all functions defined on $X$ taking values in $Y$ : $X$, $Y$ being arbitrary sets. If $X$ and $Y$ are metric spaces then $C(X, Y)$ denotes the set of continuous functions with domain $X$ and range in $Y$. Let $E=(0, a]\times(-b, b)$, $E^{(0)}=([-\tau_0, a]\times[-c, c])\setminus E$, where $a>0$, $\tau_0\geq 0$, $b=(b_1, \ldots, b_n)$, $c=(c_1, \ldots, c_n)$ and $c_i\geq b_i>0$ for $i=1, \ldots, n$. If $z: E^{(0)}\cup E\to\mathbf{R}$ is a function of the variables $(x, y)=(x, y_1, \ldots, y_n)$ and there exist derivatives $D_{y_i}z$, $D_{y_i y_j}z$ for $i, j=1, \ldots, n$ then we denote $D_y z=(D_{y_1}z, \ldots, D_{y_n}z)$ and $D_{yy}z=[D_{y_i y_j}z]_{i,j=1,\ldots,n}$. Let $\Omega=\bar{E}\times\mathbf{R}\times C(E^{(0)}\cup E, \mathbf{R})\times\mathbf{R}^n\times\mathbf{R}^{n^2}$, ($\bar{E}$ is the closure of $E$), and assume that $f:\Omega\to\mathbf{R}$ and $\varphi: E^{(0)}\to\mathbf{R}$ are given functions. We consider the differential-functional problem

$$(1)\qquad \begin{aligned} &D_x z(x, y)=f(x, y, z(x, y), z(\cdot), D_y z(x, y), D_{yy}z(x, y)),\ (x, y)\in E,\\ &z(x, y)=\varphi(x, y)\ \text{for}\ (x, y)\in E^{(0)}. \end{aligned}$$

If $x\in[-\tau_0, a]$ then we denote

$$H_x=\{(\xi, \eta)=(\xi, \eta_1, \ldots, \eta_n)\in E^{(0)}\cup E : \xi\leq x\}.$$

If $z\in C(E^{(0)}\cup E, \mathbf{R})$ then $\|z\|_x=\max\{|z(\xi, \eta)| : (\xi, \eta)\in H_x\}$, $x\in[-\tau_0, a]$. Assume that the differential-functional problem (1) is the Volterra type i.e. if $x\in(0, a]$, $z, \bar{z}\in C(E^{(0)}\cup E, \mathbf{R})$ and $z(\xi, \eta)=\bar{z}(\xi, \eta)$ for $(\xi, \eta)\in H_x$ then $f(x, y, p, z, q, r)=f(x, y, p, \bar{z}, q, r)$ for $y\in[-b, b]$, $p\in\mathbf{R}$, $q=(q_1, \ldots, q_n)\in\mathbf{R}^n$, $r=[r_{ij}]_{i,j=1,\ldots,n}$, $r\in\mathbf{R}^{n^2}$. We shall be using vectorial inequalities, with the understanding that the same inequalities hold between their corresponding components.

We are interested in establishing a method of approximation of a solution of the problem (1) by means of solutions of an associated system of ordinary differential or differential-functional equations and in estimating of the difference between the exact and approximate solutions. The systems of ordinary differential (or differential-functional) equations mentioned above can be obtained by using a discretization in the spatial variable of the problem (1) and therefore they will be called differential-difference systems. We introduce a general class of differential-difference schemes which enable us to get approximate solutions of (1). The main problem in our investigations is to find such a differential-difference system which satisfies some consistency conditions with respect to the problem (1) and it is stable. We prove that if a differential-difference method satisfies a consistency condition and it is stable then it is convergent. We give an error estimate of the method, in particular as a function of power of the step $h$ indicating the order of the method. The basic tool in our investigations are theorems on differential or differential-functional inequalities.

There is an ample literature on the lines method. The monograph by Walter [17] contains a large bibliography. The lines method is also treated as a tool for proving of existence theorems for the first boundary value problem and for the Cauchy problem with respect to the parabolic differential equations [14], [15], [17], [18], (see also [5], [11]–[13], [16]). A simple example of the method of lines for the non-linear parabolic differential-functional equations was considered in [3]. The method of lines for first order hyperbolic systems in two independent variables was considered in [6]. The author gives in [6] a convergence theorem and an existence theorem based on the method of lines for the Cauchy problem with respect to the non-linear hyperbolic systems. For further bibliography see the references in the papers cited above, especially in [5], [11], [12] and in the monograph [17]. Our results are new also in the case when $f$ does not depend on the functional argument.

In the sequel we will use the following lemma on differential-functional inequalities of the form $D_-u(x)\le F(x, u(x), u)$ where $u=(u_1, \dots, u_k)$ and $D_-u(x)=(D_-u_1(x), \dots, D_-u_k(x))$ is the lefthand lower Dini derivative of $u$ at the point $x$.

**Lemma 1.** *Suppose that*

1° *the function* $F=(F_1, \dots, F_k) : [0, a]\times \mathbf{R}^k\times C([-\tau_0, a], \mathbf{R}^k)\to \mathbf{R}^k$ *of the variables* $(x, \eta, \xi)$, $\eta=(\eta_1, \dots, \eta_k)$, $\xi=(\xi_1, \dots, \xi_k)$, *is non-decreasing with respect to the functional argument and satisfies the Volterra condition,*

2°) *$F$ possesses the following quasi-monotone property: for each $i$, $1\le i\le k$, $F_i$ is non-decreasing in $\eta_j$ for $j=1, \dots, k$, $j\ne i$,*

3°) *there exists a function* $\sigma=(\sigma_1, \dots, \sigma_k)\in C([0, a]\times \mathbf{R}^k_+, \mathbf{R}^k_+)$, $\mathbf{R}_+=[0, +\infty)$, *of the variables $(x, \eta)$ such that*

(i) *for each $i$, $1\le i\le k$, $\sigma_i$ is non-decreasing in $\eta_j$, $j=1, \dots, k$, $j\ne i$,*

(ii) *the maximum solution of the problem $u'(x)=\sigma(x, u(x))$, $u(0)=0$, is $u(x)=0$ for $x\in[0, a]$,*

4°) *if $x\in[0, a]$, $u, \bar u\in C([-\tau_0, a], \mathbf{R}^k)$ and $u(t)\le \bar u(t)$ for $t\in[-\tau_0, x]$ then* $F(x, u(x), u)-F(x, \bar u(x), \bar u)\ge -\sigma(x, [\bar u-u]_{[-\tau_0, x]})$, *where* $[\bar u-u]_{[-\tau_0, x]}=(\max\{[\bar u_1(t)-u_1(t)] : t\in[-\tau_0, x]\}, \dots, \max\{[\bar u_k(t)-u_k(t)] : t\in[-\tau_0, x]\})$,

5°) *the functions* $\alpha, \beta\in C([-\tau_0, a], \mathbf{R}^k)$ *satisfy the initial inequality* $\alpha(x)\le\beta(x)$ *for* $x\in[-\tau_0, 0]$ *and the differential-functional inequality*

$$D_-\alpha(x)-F(x, \alpha(x), \alpha)\le D_-\beta(x)-F(x, \beta(x), \beta), \quad x\in(0, a].$$

*This being assumed, we have* $\alpha(x)\le\beta(x)$ *for* $x\in[-\tau_0, a]$.

Lemma 1 can be proved by the methods used in [1], [2], [4], [7]–[10].

**II. A finite-difference approximation.** For $y=(y_1, \dots, y_n)$, $\bar y=(\bar y_1, \dots, \bar y_n)$, $y, \bar y\in \mathbf{R}^n$, we write $y^*\bar y=(y_1\bar y_1, \dots, y_n\bar y_n)\in\mathbf{R}^n$. We define a mesh on $[-c, c]$ in the following way. Let $d=(d_1, \dots, d_n)\in\mathbf{R}^n$ and $d_i>0$ for $i=1, \dots, n$. Assume



that for a $h=(h_1, \dots, h_n)\in(0, d]$ there exists $N=(N_1, \dots, N_n)$ where $N_i$ are natural numbers such that $N^*h=c$. Denote by $I_d\subset(0, d]$ the set of all $h$ having the above property. For $h=(h_1, \dots, h_n)\in I_d$ we define $|h|=h_1+\dots+h_n$. Suppose that natural numbers $\tilde N_j$ are defined by $\tilde N^*h<b\le(\tilde N+1)^*h$ where $\tilde N=(\tilde N_1, \dots, \tilde N_n)$ and $\tilde N+1=(\tilde N_1+1, \dots, \tilde N_n+1)$. For $h\in I_d$ we denote $y^{(m)}=m^*h$ and $y^{(m)}=(y_1^{(m_1)}, \dots, y_n^{(m_n)})$ where $m=(m_1, \dots, m_n)$ and $m_j$ are integers. Write $J=\{m : -N\le m\le N\}$ and $\tilde J=\{m: -\tilde N\le m\le\tilde N\}$. For $1\le j\le n$ we denote $j(m)=(m_1, \dots, m_{j-1}, m_j+1, m_{j+1}, \dots, m_n)$ and $-j(m)=(m_1, \dots, m_{j-1}, m_j-1, m_{j+1}, \dots, m_n)$. Let $E[h]=\{(x, y^{(m)}) : x\in(0, a], m\in\tilde J\}$, $B[h]=\{(x, y^{(m)}) : x\in[-\tau_0, a], m\in J\}$ and $E^{(0)}[h]=B[h]\setminus E[h]$. For a function $z : B[h]\to\mathbf{R}$ we write $z^{(m)}(x)=z(x, y^{(m)})$. In the next parts of the paper we adopt additional conditions concerning the mesh $B[h]$.

Let $S=\{s=(s_1, \dots, s_n) : s_j\in\{-1, 0, 1\}$ for $j=1, \dots, n\}$ and $S'=S\setminus\{\Theta\}$ where $\Theta=(0, \dots, 0)\in\mathbf{R}^n$. We define the following operators $A$, $\Delta=(\Delta_1, \dots, \Delta_n)$, $\Delta^{(2)}=[\Delta_{ij}]_{i,j=1,\dots,n}$. If $z : B[h]\to\mathbf{R}$ and $m\in\tilde J$, $x\in[0, a]$ then

$$
\begin{aligned}
&Az^{(m)}(x)=\sum_{s\in S} a_s z^{(m+s)}(x),\\
(2)\qquad &\Delta_j z^{(m)}(x)=\frac{1}{h_j}\sum_{s\in S} b_s^{(j)} z^{(m+s)}(x),\quad j=1, \dots, n,\\
&\Delta_{ij} z^{(m)}(x)=\frac{1}{h_i h_j}\sum_{s\in S} c_s^{(i,j)} z^{(m+s)}(x),\quad i, j=1, \dots, n,
\end{aligned}
$$

where $a_s$, $b_s^{(j)}$, $c_s^{(i,j)}\in\mathbf{R}$. For $z : B[h]\to\mathbf{R}$ we write

$$
\begin{aligned}
(3)\qquad &\Delta z^{(m)}(x)=(\Delta_1 z^{(m)}(x), \dots, \Delta_n z^{(m)}(x)),\\
&\Delta^{(2)} z^{(m)}(x)=[\Delta_{ij} z^{(m)}(x)]_{i,j=1,\dots,n}
\end{aligned}
$$

where $x\in[0, a]$, $m\in\tilde J$. We will approximate $z^{(m)}(x)$, $D_y z^{(m)}(x)$ and $D_{yy} z^{(m)}(x)$ by means of $Az^{(m)}(x)$, $\Delta z^{(m)}(x)$, $\Delta^{(2)}z^{(m)}(x)$ respectively. Let $\mathcal{F}_c(B[h], \mathbf{R})$ be a class of all functions $w$ defined on $B[h]$ taking values in $\mathbf{R}$ such that $w(\cdot, y^{(m)})\in C([-\tau_0, a], \mathbf{R})$ for $m\in J$. We denote $\tilde E[h]=\{(x, y^{(m)}) : x\in[0, a], m\in\tilde J\}$ and $\Sigma_h=\tilde E[h]\times\mathbf{R}\times\mathcal{F}_c(B[h], \mathbf{R})\times\mathbf{R}^n\times\mathbf{R}^{n^2}$. Assume that for each $h\in I_d$ we have $\Phi_h : \Sigma_h\to\mathbf{R}$. We consider the following method of lines for the problem (1)

$$
\begin{aligned}
(4)\qquad &D_x w^{(m)}(x)=\Phi_h(x, y^{(m)}, Aw^{(m)}(x), w(\cdot), \Delta w^{(m)}(x), \Delta^{(2)}w^{(m)}(x))\\
&x\in[0, a],\ m\in\tilde J,\\
&w^{(m)}(x)=\varphi(x, y^{(m)})\ \text{ for } (x, y^{(m)})\in E^{(0)}[h].
\end{aligned}
$$

If $x\in[-\tau_0, a]$ then we denote $H_x[h]=\{(\xi, y^{(m)})\in B[h] : \xi\le x\}$. The function $\Phi_h : \Sigma_h\to\mathbf{R}$ is said to satisfy the Volterra condition if for $(x, y)\in\tilde E[h]$, $w, \bar w\in\mathcal{F}_c(B[h], \mathbf{R})$ such that $w|_{H_x[h]}=\bar w|_{H_x[h]}$ we have $\Phi_h(x, y, p, w, q, r)=\Phi_h(x, y, p, \bar w, q, r)$ where $p\in\mathbf{R}$, $q\in\mathbf{R}^n$ $r\in\mathbf{R}^{n^2}$. If $w\in\mathcal{F}_c(B[h], \mathbf{R})$ and $x\in[-\tau_0, a]$ then we define $\|w\|_{h,x}=\max\{|w(\xi, y^{(m)})| : (\xi, y^{(m)})\in H_x[h]\}$.

We assume that there exists a solution of (1). We investigate the question of under what conditions the solutions of (4) tend to a solution of (1) when the step size tends to zero.

**III. The convergence of the method of lines.** Our basic assumptions are the following.

*Assumption* $H_1$. Suppose that

1°) the function $\Phi_h : \Sigma_h \to \mathbf{R}$, $h \in I_d$, of the variables $(x, y, p, w, q, r)$ is non-decreasing with respect to the functional argument and satisfies the Volterra condition.

2°) for each $h \in I_d$, $m \in \tilde{J}$, the function $\Phi_h(\cdot, y^{(m)}, \cdot)$: $[0, a] \times \mathbf{R} \times \mathcal{F}_c(B[h], \mathbf{R}) \times \mathbf{R}^n \times \mathbf{R}^{n^2} \to \mathbf{R}$ is continuous.

3°) the derivatives $D_p\Phi_h$, $D_q\Phi_h = (D_{q_1}\Phi_h, \ldots, D_{q_n}\Phi_h)$, $D_r\Phi_h = [D_{r_{ij}}\Phi_h]_{i, j=1, \ldots, n}$, $h \in I_d$, exist on $\Sigma_h$, the matrix $D_r\Phi_h$ is symmetric and for each $(x, y, w) \in \bar{E}[h] \times \mathcal{F}_c(B[h], \mathbf{R})$ the functions $D_p\Phi_h(x, y, \cdot, w, \cdot, \cdot)$, $D_q\Phi_h(x, y, \cdot, w, \cdot, \cdot)$, $D_r\Phi_h(x, y, \cdot, w, \cdot, \cdot)$ are continuous and bounded on $\mathbf{R} \times \mathbf{R}^n \times \mathbf{R}^{n^2}$.

4°) for $P = (x, y, p, w, q, r) \in \Sigma_h$, $h \in I_d$, and for $s \in S'$ we have

$$a_s D_p\Phi_h(P) + \sum_{i=1}^{n} \frac{1}{h_i} b_s^{(i)} D_{q_i}\Phi_h(P) + \sum_{i,j=1}^{n} \frac{1}{h_i h_j} c_s^{(i,j)} D_{r_{ij}}\Phi_h(P) \geq 0, \tag{5}$$

5°) there exists $L \in \mathbf{R}_+$ such that we have on $\Sigma_h$, $h \in I_d$,

$$\begin{aligned} |\Phi_h(x, y, p, w, q, r) - \Phi_h(x, y, p, \bar{w}, q, r)| &\leq L\|w - \bar{w}\|_{h, x}, \\ |D_p\Phi_h(x, y, p, w, q, r)| &\leq L, \end{aligned} \tag{6}$$

6°) for $s \in S$ we have $a_s \geq 0$ and

$$\sum_{s \in S} a_s = 1, \quad \sum_{s \in S} b_s^{(i)} = \sum_{s \in S} c_s^{(i,j)} = 0, \quad i, j = 1, \ldots, n.$$

Now we state a result on the stability of the method (4).

**Theorem 1.** *Suppose that*

1°) *Assumption* $H_1$ *is satisfied and* $\varphi \in C(E^{(0)}, \mathbf{R})$,

2°) *the function* $v_h \in \mathcal{F}_c(B[h], \mathbf{R})$ *satisfies the initial-boundary condition*

$$v_h(x, y^{(m)}) = \varphi(x, y^{(m)}) \quad on \quad E^{(0)}[h] \tag{7}$$

*and there exists* $\gamma : I_d \to \mathbf{R}_+$ *such that*

$$\begin{aligned} |D_x v_h^{(m)}(x) - \Phi_h(x, y^{(m)}, A v_h^{(m)}(x), v_h, \Delta v_h^{(m)}(x), \Delta^{(2)} v_h^{(m)}(x))| &\leq \\ \leq \gamma(h), \quad x \in [0, a], \ m \in \tilde{J}&, \end{aligned} \tag{8}$$

3°) $\omega_h : [0, a] \to \mathbf{R}_+$ *is defined by*

$\omega_h(x) = \gamma(h)(2L)^{-1}) \ (e^{2Lx} - 1)$ *if* $L > 0$,

$\omega_h(x) = \gamma(h)x$ *if* $L = 0$.

*Under these assumptions there exists a solution* $w_h \in \mathcal{F}_c(B[h], \mathbf{R})$ *of the problem* (4) *and*

$$\|w_h - v_h\|_{h, x} \leq \omega_h(x), \quad x \in [0, a]. \tag{9}$$



*Proof.* It follows from the conditions 3°, 5° of Assumption $H_1$ that the problem (4) has the solution $w_h$ on $[-\tau_0, a]$. We prove that

$$w_h(x, y^{(m)}) \leq v_h(x, y^{(m)}) + \omega_h(x), \quad x \in [0, a], \ m \in \tilde{J}. \tag{10}$$

We apply Lemma 1 for the proof of (10). Let

$$\tilde{v}_h(x, y^{(m)}) = \begin{cases} v_h(x, y^{(m)}) + \omega_h(x) & \text{for } x \in [0, a], \ m \in J, \\ v_h(x, y^{(m)}) & \text{for } x \in [-\tau_0, 0], \ m \in J, \end{cases} \tag{11}$$

and

$$R_{h, 0}(x, y^{(m)}) = D_x v_h^{(m)}(x) - \Phi_h(x, y^{(m)}, A v_h^{(m)}(x), v_h, \Delta v_h^{(m)}(x), \Delta^{(2)} v_h^{(m)}(x)),$$

$$\begin{aligned} R_{h, 1}(x, y^{(m)}) = \Phi_h(x, y^{(m)}, A v_h^{(m)}(x), v_h, \Delta v_h^{(m)}(x), \Delta^{(2)} v_h^{(m)}(x)) - \\ - \Phi_h(x, y^{(m)}, A \tilde{v}_h^{(m)}(x), \tilde{v}_h, \Delta \tilde{v}_h^{(m)}(x), \Delta^{(2)} \tilde{v}_h^{(m)}(x)), \quad x \in [0, a], \ m \in \tilde{J}. \end{aligned}$$

Then we have

$$\begin{aligned} D_x \tilde{v}_h^{(m)}(x) = R_{h, 0}(x, y^{(m)}) + R_{h, 1}(x, y^{(m)}) + \omega_h'(x) + \\ + \Phi_h(x, y^{(m)}, A \tilde{v}_h^{(m)}(x), \tilde{v}_h, \Delta \tilde{v}_h^{(m)}(x), \Delta^{(2)} \tilde{v}_h^{(m)}(x)), \\ x \in [0, a], \ m \in \tilde{J}. \end{aligned} \tag{12}$$

Since $R_{h, 0}(x, y^{(m)}) \geq -\gamma(h)$ for $x \in [0, a]$, $m \in \tilde{J}$ and

$$R_{h, 1}(x, y^{(m)}) \geq -2L\omega_h(x), \quad x \in [0, a], \ m \in \tilde{J},$$

we get, using (12), the inequality

$$D_x \tilde{v}_h^{(m)}(x) \geq \Phi_h(x, y^{(m)}, A \tilde{v}_h^{(m)}(x), \tilde{v}_h, \Delta \tilde{v}_h^{(m)}(x), \Delta^{(2)} \tilde{v}_h^{(m)}(x)), \quad x \in [0, a], m \in \tilde{J}. \tag{13}$$

It follows from the condition 1° of Assumption $H_1$ that the righthand sides of (13) are non-decreasing with respect to the functional argument and satisfy the Volterra condition. Now we prove that they possess the quasi-monotone property. Suppose that $w^* \in \mathcal{F}_c(B[h], \mathbf{R})$, $w, \bar{w} \in \mathcal{F}(B[h], \mathbf{R})$. Assume that $x \in [0, a]$, $m \in \tilde{J}$, are fixed and $w^{(m)}(x) = \bar{w}^{(m)}(x)$, $w^{(\bar{m})}(x) \leq \bar{w}^{(\bar{m})}(x)$ for $\bar{m} \in J$. Then we have using the mean value theorem

$$\begin{aligned} &\Phi_h(x, y^{(m)}, A w^{(m)}(x), w^*, \Delta w^{(m)}(x), \Delta^{(2)} w^{(m)}(x)) - \\ &- \Phi_h(x, y^{(m)}, A \bar{w}^{(m)}(x), w^*, \Delta \bar{w}^{(m)}(x), \Delta^{(2)} \bar{w}^{(m)}(x)) = \\ &= D_p\Phi_h(Q)[A w^{(m)}(x) - A \bar{w}^{(m)}(x)] + \\ &+ \sum_{i=1}^{n} D_{q_i}\Phi_h(Q) \, [\Delta_i w^{(m)}(x) - \Delta_i \bar{w}^{(m)}(x)] + \\ &+ \sum_{i,j=1}^{n} D_{r_{ij}}\Phi_h(Q)[\Delta_{ij} w^{(m)}(x) - \Delta_{ij} \bar{w}^{(m)}(x)] = \\ &= \sum_{s \in S'} (w^{(m+s)}(x) - \bar{w}^{(m+s)}(x)) \Big[ a_s D_p\Phi_h(Q) + \\ &+ \sum_{i=1}^{n} \frac{1}{h_i} b_s^{(i)} D_{q_i}\Phi_h(Q) + \sum_{i,j=1}^{n} \frac{1}{h_i h_j} c_s^{(i,j)} D_{r_{ij}}\Phi_h(Q) \Big] \leq 0, \end{aligned}$$

where $Q$ is an intermediate point. This completes the proof of the quasi-monotone property of the right-hand sides of (13). Since $w_h$ is a solution of (4) then we have using (7), (11) and Lemma 1 the estimation (10). In a similar way we prove that

$$(14)\qquad v_h(x, y^{(m)}) - \omega_h(x) \leq w_h(x, y^{(m)}), \quad x \in [0, a], \quad m \in \tilde{J}.$$

Hence, by (10), (14) we have (9), which completes the proof.

*Remark* 1. Let us consider two problems, the problem (4) and the following one

$$(\tilde{4})\qquad \begin{aligned} D_x w^{(m)}(x) &= \tilde{\Phi}_h(x, y^{(m)}, \tilde{A}w^{(m)}(x), w(\cdot), \tilde{\Delta}w^{(m)}(x), \\ &\qquad \tilde{\Delta}^{(2)}w^{(m)}(x)), \quad x \in [0, a] \quad m \in \tilde{J}, \\ w^{(m)}(x) &= \varphi(x, y^{(m)}) \text{ on } E^{(0)}[h], \end{aligned}$$

where $\tilde{\Phi}_h : \Sigma_h \to \mathbf{R}$ and $\tilde{A}, \tilde{\Delta}, \tilde{\Delta}^{(2)}$ are operators given by (2), (3) with $\tilde{a}_s$, $\tilde{b}_s^{(i)}$, $\tilde{c}_s^{(i,j)}$ instead of $a_s$, $b_s^{(i)}$, $c_s^{(i,j)}$. Suppose that there is a solution $\tilde{w}_h \in \mathcal{F}_c(B[h], \mathbf{R})$ of ($\tilde{4}$) and that there exists $\gamma : I_d \to \mathbf{R}_+$ such that

$$|\Phi_h(x, y^{(m)}, A\tilde{w}_h^{(m)}(x), \tilde{w}_h, \Delta\tilde{w}_h^{(m)}(x), \tilde{\Delta}^{(2)}\tilde{w}_h^{(m)}(x)) -$$

$$-\tilde{\Phi}_h(x, y^{(m)}, \tilde{A}\tilde{w}_h^{(m)}(x), \tilde{w}_h, \tilde{\Delta}\tilde{w}_h^{(m)}(x), \tilde{\Delta}^{(2)}\tilde{w}_h^{(m)}(x))| \leq \gamma(h), \quad x \in [0, a], \quad m \in \tilde{J}.$$

Assume that conditions 1°, 3° of Theorem 1 hold. Then $\|w_h - \tilde{w}_h\|_{h,x} \leq \omega_h(x)$, $x \in [0, a]$. This estimation follows from

*Assumption* $H_2$. Suppose that

1°) $f \in C(\Omega, \mathbf{R})$, $\varphi \in C(E^{(0)}, \mathbf{R})$ and there exists a solution $v$ of (1) such that $v|_{\bar{E}}$ is of class $C^3$,

2°) there exists a function $\gamma_0 : I_d \to \mathbf{R}_+$ such that for $x \in [0, a]$, $m \in \tilde{J}$ we have

$$(15)\qquad \begin{aligned} &|\Phi_h(x, y^{(m)}, Av_h^{(m)}(x), v_h, \Delta v_h^{(m)}(x), \Delta^{(2)}v_h^{(m)}(x)) - \\ &- f(x, y^{(m)}, Av^{(m)}(x), v, \Delta v^{(m)}(x), \Delta^{(2)}v^{(m)}(x))| \leq \gamma_0(h), \end{aligned}$$

where $v_h = v|_{B[h]}$ and

$$(16)\qquad \lim_{|h| \to 0} \gamma_0(h) = 0.$$

*Assumption* $H_3$. Suppose that the operators $A$, $\Delta$, $\Delta^{(2)}$ satisfy the conditions

1° for $i, j = 1, \ldots, n$ we have

$$\sum_{s \in S} s_j b_s^{(i)} = \delta_{ij},$$

where $\delta_{ij}$ is the Kronecker symbol,

2°) for $i, j, l, \tau = 1, \ldots, n$ we have

$$\sum_{s \in S} s_l c_s^{(i,j)} = 0, \quad \sum_{s \in S} s_l s_\tau c_s^{(i,j)} = \delta_{il}\delta_{j\tau} \text{ for } i \neq j,$$

$$\sum_{s \in S} s_l s_\tau c_s^{(i,j)} = 2\delta_{il}\delta_{j\tau} \text{ for } i = j.$$



3°) there exists $c_0 > 0$ such that $h_j h_i^{-1} \leq c_0$ for $h \in I_d$, $i, j = 1, \ldots, n$.

Now we prove that if the method of lines is stable and satisfies the consistency condition described by (15), (16) then it is convergent.

**Theorem 2.** *Suppose that*

1°) *Assumptions* $H_1 - H_3$ *are satisfied.*

2°) $w_h \in \mathcal{F}_c(B[h], \mathbf{R})$ *denotes the solution of* (4).

*Then* $\lim_{|h| \to 0} \|w_h - v_h\|_{h,x} = 0$ *uniformly with respect to* $x \in [0, a]$.

*Proof.* It follows from condition 1° of Assumption $H_2$ that there exists $\bar{C} > 0$ such that

$$(17)\qquad |v(x, y)|, \ |D_{y_i}v(x, y)|, \ |D_{y_i y_j}v(x, y)|, \ |D_{y_i y_j y_l}v(x, y)| \leq \bar{C},$$

where $i, j, l = 1, \ldots, n$, $(x, y) \in \bar{E}$. Now, using Assumptions $H_1$, $H_3$ we obtain

$$(18)\qquad \begin{aligned} &|Av^{(m)}(x) - v^{(m)}(x)| \leq \bar{C}|h|, \\ &|\Delta_i v^{(m)}(x) - D_{y_i} v^{(m)}(x)| \leq |h| c_0 \frac{n}{2} \bar{C} \max_{1 \leq j \leq n} \sum_{s \in S} |b_s^{(j)}|, \\ &|\Delta_{ij} v^{(m)}(x) - D_{y_i y_j} v^{(m)}(x)| \leq |h| \frac{\bar{C}}{6} (c_0 n)^2 \Gamma, \\ &\Gamma = \max_{1 \leq \tau, \tau' \leq n} \sum_{s \in S} |c_s^{(\tau,\tau')}|, \end{aligned}$$

where $x \in [0, a]$, $m \in \tilde{J}$, $i, j = 1, \ldots, n$. We define

$$(19)\qquad \begin{aligned} \tilde{R}_{h,0}(x, y^{(m)}) &= f(x, y^{(m)}, Av^{(m)}(x), v, \Delta v^{(m)}(x), \Delta^{(2)} v^{(m)}(x)) - \\ &\quad - \Phi_h(x, y^{(m)}, Av_h^{(m)}(x), v_h, \Delta v_h^{(m)}(x), \Delta^{(2)}v_h^{(m)}(x)), \\ \tilde{R}_{h,1}(x, y^{(m)}) &= f(x, y^{(m)}, v^{(m)}(x), v, D_y v^{(m)}(x), D_{yy} v^{(m)}(x)) - \\ &\quad - f(x, y^{(m)}, Av^{(m)}(x), v, \Delta v^{(m)}(x), \Delta^{(2)} v^{(m)}(x)), \quad x \in [0, a], \quad m \in \tilde{J}. \end{aligned}$$

It follows that there exists a function $\gamma_1 : I_d \to \tilde{\mathbf{R}}_+$ such that $|\tilde{\mathbf{R}}_{h,1}(x, y^{(m)})| \leq \gamma_1(h)$, $x \in [0, a]$, $m \in \tilde{J}$, and $\lim_{|h| \to 0} \gamma_1(h) = 0$.

Let

$$(20)\qquad \gamma(h) = \gamma_0(h) + \gamma_1(h).$$

Then

$$|D_x v_h^{(m)}(x) - \Phi_h(x, y^{(m)}, Av_h^{(m)}(x), v_h, \Delta v_h^{(m)}(x), \Delta^{(2)}v_h^{(m)}(x))| \leq$$

$$\leq |R_{h,0}(x, y^{(m)})| + (|R_{h,1}(x, y^{(m)})| \leq \gamma(h), \quad x \in [0, a], \quad m \in \tilde{J},$$

and $\lim_{|h| \to 0} \gamma(h) = 0$. By Theorem 1 we obtain our assertion.

Theorem 2 guarantees that a consistent and stable method of lines is convergent. The next remark states that if some additional conditions hold then the error of the line method is of the $\delta$-th order.

*Remark* 2. Suppose that all the assumptions of Theorem 2 are satisfied and that there exists $\delta > 0$, $C_1 > 0$ such that $|\gamma(h)| \leq C_1(|h|)^\delta$, $h \in I_d$, where $\gamma$ is given by (20). Then there exists $C_0 \geq 0$ such that $\|w_h - v_h\|_{h,x} \leq C_0|h|^\delta$, $x \in [0, a]$.

*Remark* 3. We assume that the consistency condition (15), (16) is satisfied along a given solution $v$ of (1). It is obvious that Theorem 2 is true if we assume (15), (16) along an arbitrary function $v \in C(E^{(0)} \cup E, \mathbf{R})$ such that $v|_{\overline{E}}$ is of class $C^3$.

**IV. Examples.** For $h \in I_d$ we define $T_h : \mathcal{F}_c(B[h], \mathbf{R}) \to C(E^{(0)} \cup E, \mathbf{R})$ in the following way [1]. Suppose that $u \in \mathcal{F}_c(B[h], \mathbf{R})$ and $y \in [-c, c]$. Then there exists $m \in J$ such that $y^{(m)} \leq y \leq y^{(m+1)}$, $m+1 = (m_1+1, \ldots, m_n+1)$, and $m+1 \in J$. We define $S_0 = \{s = (s_1, \ldots, s_n) : s_i \in \{0, 1\},\ i = 1, \ldots, n\}$,

$$(T_h u)(x, y) = \sum_{s \in S_0} u^{(m+s)}(x) \left(\frac{y-y^{(m)}}{h}\right)^s \left(1 - \frac{y-y^{(m)}}{h}\right)^{1-s}$$

where $x \in [-\tau_0, 0]$ and

$$\text{(21)} \qquad \left(\frac{y-y^{(m)}}{h}\right)^s = \prod_{i=1}^{n} \left(\frac{y_i - y_i^{(m_i)}}{h_i}\right)^{s_i}, \qquad \left(1 - \frac{y-y^{(m)}}{h}\right)^{1-s} = \prod_{i=1}^{n} \left(1 - \frac{y_i - y_i^{(m_i)}}{h_i}\right)^{1-s_i},$$

and we take $0^0 = 1$ in (21). Thus we have $T_h u : E^{(0)} \cup E \to \mathbf{R}$. The operator $T_h$ has the following properties [1].

**Lemma 2.** *If* $w \in \mathcal{F}_c(B[h], \mathbf{R})$ *then* $T_h w \in C(E^{(0)} \cup E, \mathbf{R})$. *If*

1°) $w \in C(E^{(0)} \cup E, \mathbf{R})$, the derivatives $(D_{y_1} w, \ldots, D_{y_n} w) = D_y w$ *exist on* $E^{(0)} \cup E$ *and* $D_y w(x, \cdot) \in C([-b, b], \mathbf{R}^n)$ for $x \in [-\tau_0, a]$,

2°) *there exists* $C_0 \geq 0$ *such that* $|D_{y_i} w(x, y)| \leq C_0$, $(x, y) \in E^{(0)} \cup E$, $i = 1, \ldots, n$,

*then* $\|T_h w_h - w\|_x \leq C_0 |h|$, $x \in [-\tau_0, a]$, *where* $w_h = w|_{B[h]}$.

We will consider the problem (4) with $\Phi_h(x, y, p, w, q, r) = f(x, y, p, T_h w, q, r)$. We list below a few examples of the method of lines which can be derived from (4) by specializing the operators $A$, $\Delta$, $\Delta^{(2)}$. Let $M = \{(i, j) : i, j = 1, \ldots, n,\ i \neq j\}$. Suppose that we have defined $M^{(-)}$, $M^{(+)} \subset M$ such that $M^{(-)} \cup M^{(+)} = M$ and $M^{(-)} \cap M^{(+)} = \emptyset$ (in particular it may be $M^{(-)} = \emptyset$ or $M^{(+)} = \emptyset$). Assume that $(j, i) \in M^{(-)}$ *if* $(i, j) \in M^{(-)}$. Suppose that $A$, $\Delta$, $\Delta^{(2)}$ are operators defined in the following way. Let $u : B[h] \to \mathbf{R}$, $x \in [0, a]$, $m \in \tilde{J}$, then

$$\text{(22)} \qquad Au^{(m)}(x) = \frac{1}{2n} \sum_{i=1}^{n} [u^{(i(m))}(x) + u^{(-i(m))}(x)]$$

and

$$\text{(23)} \qquad \Delta_i u^{(m)}(x) = (2h_i)^{-1}[u^{(i(m))}(x) - u^{(-i(m))}(x)], \quad i = 1, \ldots, n.$$

For $(i, j) \in M^{(+)}$ we define

$$\text{(24)} \qquad \Delta_{ij} u^{(m)}(x) = (2h_i h_j)^{-1}[-u^{(i(m))}(x) - u^{(j(m))}(x) - u^{(-i(m))}(x) - u^{(-j(m))}(x) + 2u^{(m)}(x) + u^{(i(j(m)))}(x) + u^{(-i(-j(m)))}(x)].$$

If $(i, j) \in M^{(-)}$ then

$$\text{(25)} \qquad \Delta_{ij} u^{(m)}(x) = (2h_i h_j)^{-1}[u^{(i(m))}(x) + u^{(j(m))}(x) + u^{(-i(m))}(x) + u^{(-j(m))}(x) - 2u^{(m)}(x) - u^{(i(-j(m)))}(x) - u^{(-i(j(m)))}(x)],$$



and

$$\text{(26)} \qquad \Delta_{ii} u^{(m)}(x) = h_i^{-2}[u^{(i(m))}(x) - 2u^{(m)}(x) + u^{(-i(m))}(x)], \quad i = 1, \ldots, n.$$

Under these assumptions concerning the operators $A$, $\Delta$, $\Delta^2$ consider the method of lines

$$\text{(27)} \qquad \begin{aligned} &D_x w^{(m)}(x) = f(x, y^{(m)}, Aw^{(m)}(x), T_h w, \Delta w^{(m)}(x), \Delta^{(2)} w^{(m)}(x)), \\ &\qquad\qquad m \in \tilde{J},\ x \in [0, a], \\ &w^{(m)}(x) = \varphi(x, y^{(m)}) \text{ on } E^{(0)}[h]. \end{aligned}$$

*Assumption* $H_1'$. Suppose that

1°) the function $f \in C(\Omega, \mathbf{R})$ of the variables $(x, y, p, z, q, r)$ is non-decreasing with respect to the functional argument and satisfies the Volterra condition,

2°) the derivatives $D_p f$, $D_q f = (D_{q_1} f, \ldots, D_{q_n} f)$, $D_r f = [D_{r_{ij}} f]_{i,j=1,\ldots,n}$ exist on $\Omega$ and for each $(x, y, z) \in E \times C(E^{(0)} \cup E, \mathbf{R})$ the functions $D_p f(x, y, \cdot, z, \cdot, \cdot)$, $D_q f(x, y, \cdot, z, \cdot, \cdot)$, $D_r f(x, x, \cdot, z, \cdot, \cdot)$ are continuous and bounded on $\mathbf{R} \times \mathbf{R}^n \times \mathbf{R}^{n^2}$,

3°) the matrix $D_r f$ is symmetric on $\Omega$ and there exists $c_0 > 0$ such that $h_i h_j^{-1} \leq c_0$ for $i, j = 1, \ldots, n$, $h \in I_d$,

4°) there exists $L \in R$ such that

$$|f(x, y, p, z, q, r) - f(x, y, p, \bar{z}, q, r)| \leq L\|z - \bar{z}\|_x,$$

$$|D_p f(x, y, p, z, q, r)| \leq L \text{ on } \Omega.$$

**Theorem 3.** *Suppose that*

1°) *Assumption* $H_1$ *is satisfied and for* $P = (x, y, p, z, q, r) \in \Omega$ *we have*

$$\text{(28)} \qquad \frac{1}{n} D_p f(P) - h_j^{-1}|D_{q_j} f(P)| + 2h_j^{-2} D_{r_{jj}} f(P) - 2 \sum_{\substack{i=1 \\ i \neq j}}^{n} (h_i h_j)^{-1} |D_{r_{ij}} f(P)| \geq 0, \quad j = 1, \ldots, n,$$

*and*

$$\text{(29)} \qquad D_{r_{ij}} f(P) \geq 0 \textit{ for } (i, j) \in M^{(+)}, \quad D_{r_{ij}} f(P) \leq 0 \textit{ for } (i, j) \in M^{(-)}.$$

2°) $w_h$ *denotes a solution of* (27) *and there exists a solution* $v$ *of* (1) *such that* $v$ *is of class* $C^1$ *on* $E^{(0)} \cup E$ *and the function* $v|_{\overline{E}}$ *is of class* $C^3$.

*Let* $v_h = v|_{B[h]}$. *Under these assumptions we have* $\lim_{h \to 0} \|w_h - v_h\|_{h,x} = 0$ *uniformly with respect to* $x \in [0, a]$.

*Proof*. The method (22)–(27) satisfies all the assumptions of Theorem 2. The conditions (28), (29) for (22)–(27) are equivalent with (5). The consistency condition (15), (16) follows from Lemma 2 and from (22)–(26). This completes the proof.

*Remark* 4. Consider the method of lines

$$D_x w^{(m)}(x) = f(x, y^{(m)}, w^{(m)}(x), T_h w, \Delta w^{(m)}(x), \Delta^{(2)} w^{(m)}(x)), \quad x \in [0, a],\ m \in \tilde{J},$$

$$w^{(m)}(x) = \varphi(x, y^{(m)}) \text{ on } E^{(0)}h.$$

where $\Delta$ and $\Delta^{(2)}$ are given by (23)–(26). Then the assumption (5) is equivalent with (29) and

$$-\frac{1}{2}|D_{q_j}f(P)|+\frac{1}{h_j}D_{r_{jj}}f(P)-\sum_{\substack{i=1\\ j\neq i}}^{n}\frac{1}{h_i}|D_{r_{ij}}f(P)|\geq 0,\ j=1,\ldots,n.$$

*Remark* 5. Suppose that the function $f$ does not depend on $r_{ij}$ for $i\neq j$. Consider the method (27) where $\Delta^{(2)}=(\Delta_{11},\ldots,\Delta_{nn})$. Assume that for $u: B[h]\to\mathbf{R}$ we have $Au^{(m)}(x)=u^{(m)}(x)$, $\Delta_i u^{(m)}(x)=h_i^{-1}[u^{(i(m))}(x)-u^{(m)}(x)]$, $i=1,\ldots,n$, and $\Delta_{ii}$ are given by (26). Then the assumption (5) is equivalent with

$$D_{r_{ii}}f(P)\geq 0,\ h_iD_{q_i}f(P)+D_{r_{ii}}f(P)\geq 0,\ i=1,\ldots,n,\ P\in\Omega.$$

*Remark* 6. The results obtained in this paper can be extended to weakly coupled systems of differential-functional equations.

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*Institute of Mathematics,*
*University of Gdansk*
*57 Wit Stwosz Street Poland*

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# INFINITELY MANY SOLUTIONS FOR A SINGULAR SEMILINEAR STURM-LIOUVILLE PROBLEM

BY

LIVIU I. NICOLAESCU

**0. Introduction.** This paper is devoted to the study of the following type of boundary-value problem

$$\text{(0.1)}\qquad \begin{gathered}\frac{1}{\rho(x)}(\rho(x)u')'+k(x)f(u)=0,\qquad x\in(0,L),\qquad L>0,\\ u_x(0)=0,\\ au_x(L)+bu(L)=0,\end{gathered}$$

where we allow functions $\frac{1}{\rho}$ and $k$ to have a singularity in $x=0$. Using an idea of A. Castro-A. Kurepa [1], [2] and imposing certain conditions on $f$, $\rho$ and $k$ we prove the existence of infinitely many solutions to (0.1). Our paper partially extends the previous work of A. Castro-A. Kurepa [1] where they considered a similar problem for which $\rho(x)=x^{n-1}$, $n\in\mathbf{N}$, $n\geq 2$, $k(x)=1$. Their problem was motivated by the search of radially symmetric solutions for a semilinear elliptic problem in a ball.

Our result is aimed to deal with P.D.E. too but includes some degenerate elliptic semilinear problems as well.

The paper is divided into two sections. In section 1 we state and prove our main result namely Theorem 1. Section 2 is devoted to applications.

**1. The Main Result.** We consider the problem

$$\text{(1.1)}\qquad \begin{gathered}\frac{1}{\rho(x)}(\rho(x)u')'+k(x)f(u)=0,\qquad x\in(0,L),\qquad L>0,\\ u_x(0)=0,\\ au_x(L)+bu(L)=0,\ a^2+b^2\neq 0.\end{gathered}$$

Concerning (1.1) we make the following assumptions.

(1.2) $\qquad \rho\in C^1[0,L],\ \rho(0)\geq 0,\ \rho'\geq 0$ on $(0,L)$,

(1.3) $\qquad k\in C^1[0,L],\ \inf\{k(x)/x\in(0,L)\}=\alpha>0,$