$$-\frac{1}{2}|D_{q_j}f(P)|+\frac{1}{h_j}D_{r_{ij}}f(P)-\sum_{\substack{i=1\\ j\neq i}}^{n}\frac{1}{h_i}|D_{r_{ij}}f(P)|\geq 0,\ j=1,\ldots,n.$$

Remark 5. Suppose that the function f does not depend on r_{ij} for $i\neq j$. Consider the method (27) where $\Delta^{(2)}=(\Delta_{11},\ldots,\Delta_{nn})$. Assume that for $u:B[h]\to\mathbf{R}$ we have $Au^{(m)}(x)=u^{(m)}(x)$, $\Delta_i u^{(m)}(x)=h_i^{-1}[u^{(\bar{i}(m))}(x)-u^{(m)}(x)]$, $i=1,\ldots,n$, and Δ_{ii} are given by (26). Then the assumption (5) is equivalent with

$$D_{r_{ii}}f(P)\geq 0,\ h_iD_{q_i}f(P)+D_{r_{ii}}f(P)\geq 0,\ i=1,\ldots,n,\ P\in\Omega.$$

Remark 6. The results obtained in this paper can be extended to weakly coupled systems of differential-functional equations.

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INFINITELY MANY SOLUTIONS FOR A SINGULAR SEMILINEAR STURM-LIOUVILLE PROBLEM

BY

LIVIU I. NICOLAESCU

0. Introduction. This paper is devoted to the study of the following type of boundary-value problem

$$\frac{1}{\rho(x)}(\rho(x)u')'+k(x)f(u)=0,\qquad x\in(0,L),\qquad L>0,$$

(0.1)
$$u_x(0)=0,$$

$$au_x(L)+bu(L)=0,$$

where we allow functions $\frac{1}{\rho}$ and k to have a singularity in $x=0$. Using an idea of A. Castro-A. Kurepa [1], [2] and imposing certain conditions on f, ρ and k we prove the existence of infinitely many solutions to (0.1). Our paper partially extends the previous work of A. Castro-A. Kurepa [1] where they considered a similar problem for which $\rho(x)=x^{n-1}$, $n\in\mathrm{N}$, $n\geq 2$, $k(x)=1$. Their problem was motivated by the search of radially symmetric solutions for a semilinear elliptic problem in a ball.

Our result is aimed to deal with P.D.E. too but includes some degenerate elliptic semilinear problems as well.

The paper is divided into two sections. In section 1 we state and prove our main result namely Theorem 1. Section 2 is devoted to applications.

1. The Main Result. We consider the problem

$$\frac{1}{\rho(x)}(\rho(x)u')'+k(x)f(u)=0,\qquad x\in(0,L),\qquad L>0,$$

(1.1)
$$u_x(0)=0,$$

$$au_x(L)+bu(L)=0,\ a^2+b^2\neq 0.$$

Concerning (1.1) we make the following assumptions.

(1.2) $\rho\in C^1[0,L]$, $\rho(0)\geq 0$, $\rho'\geq 0$ on $(0,L)$,

(1.3) $k\in C^1[0,L]$, $\inf\{k(x)/x\in(0,L)\}=\alpha>0$, (1.4) $\exists \beta < 2$ such that $k(x) = O(x^{-\beta})$ as $x \to 0$,

(1.5) $\rho^2 k$ is increasing on $(0, L)$,

(1.6) $f: \mathbf{R} \to \mathbf{R}$ is locally Lipschitz, increasing and $f(0)=0$,

$$\lim_{|u| \to \infty} \frac{f(u)}{u} = +\infty, \tag{1.7}$$

$$\exists \lambda \in (0, 1) \text{ such that } \lim_{d \to \infty} F(\lambda d) \rho^2 \left(c \left(\frac{d}{f(d)} \right)^{\frac{1}{2-\beta}} \right) = +\infty, \quad \forall c > 0, \tag{1.8}$$

where $F(u) = \int_0^u f(s) ds$.

Theorem 1. *If conditions (1.2)–(1.8) are fulfilled then problem (1.1) has infinitely many solutions.*

Remark. Theorem 1 has a distinct feature. Namely, the assumptions about the behavoiour of f near $-\infty$ are very weak allowing nonlinearities of the type $-eu^{-3}$ as $u \to -\infty$.

Proof. Note that (1.1) can be rewritten as

$$u'' + \frac{\rho'}{\rho} u' + k(x) f(u) = 0, \qquad u'(0) = au'(L) + bu(L) = 0. \tag{1.9}$$

We shall use a shooting method. Namely we firstly interpret (1.9) as an initial value problem

$$u'' + \frac{\rho'}{\rho} u' + k(x) f(u) = 0, \tag{1.10}$$

$$u(0) = d, \ u'(0) = 0. \tag{1.11}$$

We denote the solution of (1.10), (1.11) with $u(x, d)$.

Then using a device due to A. Castro-A. Kurepa we shall prove the existence of infinitely many d such that

$$au'_x(L, d) + bu(L, d) = 0.$$

On to the proof.

Let us first observe that the solution of (1.10)–(1.11) are also solutions of the following integral equation

$$u(x) = d - \int_0^x \frac{dt}{\rho(t)} \int_0^t \rho(s) k(s) f(u(s)) ds. \tag{1.12}$$

The right-hand side integral operator defines a contraction mapping of a small ball of $C[0, \varepsilon]$.

Indeed let $\varepsilon, \delta > 0$ be small enough and define

$$B_{\varepsilon,\delta} = \{u \in C[0, \varepsilon] \mid |u(x) - d| \le \delta, \quad \forall x \in [0, \varepsilon]\}.$$

Denote the above integral operator by T. For every $u, v \in B_{\varepsilon,\delta}$ we have

$$|(Tu)(x) - (Tv)(x)| \le \int_0^x dt \int_0^t \frac{\rho(s) k(s)}{\rho(t)} |f(u(s)) - f(v(s))| ds. \tag{1.13}$$

Since f is locally Lipschitz there exists $L_\delta > 0$ such that whenever $|u-d|$, $|v-d| \le \delta$ we infer

$$|f(u) - f(v)| \le L_\delta |u - v|.$$

We infer from (1.13) that

$$|(Tu)(x) - (Tv)(x)| \le L_\delta \left(\int_0^x dt \frac{\rho(s) k(s)}{\rho(t)} ds \right) \|u - v\|_\infty . \tag{1.14}$$

Let us estimate the integral in (1.14). According to (1.2) we get that $\rho(s) \le \rho(t)$, $\forall s \in [0, t)$ and therefore the above integral is majorised by

$$\int_0^x dt \int_0^t k(s) ds.$$

Since $k(s) = O(s^{-\beta})$, we get that

$$K(t) = \int_0^t k(s) ds = O(t^{-\beta+1}),$$

and since $\beta < 2$ we see that $K \in L^1[0, L]$. Now let $\varepsilon > 0$ be so small such that

$$\int_0^\varepsilon K(t) dt < \frac{1}{2L_\delta} .$$

Therefore according (1.14) we get

$$|(Tu)(x) - (Tv)(x)| \le \frac{1}{2} \|u - v\|_\infty \ \forall x \in (0, \varepsilon), \tag{1.15}$$

or equivalently

$$\|Tu - Tv\|_\infty \le \frac{1}{2} \|u - v\|_\infty . \tag{1.15'}$$

The relation (1.15)′ says that T is a contradiction and $T(B_{\varepsilon,\delta}) \subset B_{\varepsilon,\delta}$.

Using Banach's contraction principle we infer local existence and unicity for problem (1.10)–(1.11).

A natural problem arises concerning the global existence for problem (1.10)—(1.11). This is precisely our next step.

Let us consider the energy

$$E(x, d)=\frac{1}{2}u'(x, d)^2+k(x)F(u(x, d)). \tag{1.16}$$

We shall reason by reductio ad absurdum. So let us suppose there exists a solution $u(x, d)$ of (1.10)—(1.11) which is not defined on the whole interval $[0, L]$.

Hence there exists $\bar{x}\in(0, L)$ and a sequence $(x_n)\subset(0, L]$ such that $x_n\to\bar{x}$ and

$$u^2(x_n, d)+(u'(x_n, d))^2\to\infty \text{ as } n\to\infty.$$

Using the mean value theorem we can choose the sequence x_n so that even $(u'(x_n, d))^2\to\infty$ as $n\to\infty$.

Therefore

$$\lim_{x_n\nearrow\bar{x}} E(x_n, \delta)=\infty. \tag{1.17}$$

Clearly

$$E'(x,d)=u'u''+k(x)f(u)u'+k'(x)F(u)=k'(x)F(u)-\frac{\rho'}{\rho}(u')^2\le k'(x)F(u)\le$$

$$\le|k'(x)|E(x, d).$$

Therefore

$$E'(x, d)\le|k'(x)|E(x, d) \tag{1.18}$$

$u(x, d)$ is defined on a small interval $[0, \varepsilon)$. Let $x_0>0$. Since $k\in C^1[x_0, L]$ we get from (1.18)

$$E'(x, d)\le CE(x, d),\ \forall x\in[x_0, x) \text{ for some } C>0. \tag{1.19}$$

Integrating (1.19) on $[x_0, x]$ we get

$$E(x, d)\le E(x_0, d)\exp C(x-x_0).$$

which contradicts (1.17).

Thus global existence is established.

The next step in the proof is the deriving of some energy estimates that we shall use later.

Lemma 2. *If f satisfy hypothesis (1.6)—(1.8) then*

$$\lim_{d\to\infty} E(x, d)=\infty \text{ uniformly with respect to } x\in[0, L]. \tag{1.20}$$

Proof. By (1.12)

$$u'(x)=-\frac{1}{\rho(x)}\int_0^x\rho(s)k(s)f(u(s))ds.$$

Let $\lambda\in(0, 1)$ be given by (1.8). We infer that $\forall d>0$ there exists at most one $x_0=x_0(d)\in[0, L]$

$$\lambda du(x, d)\le d,\ \forall x\in[0,x_0),\ u(x_0, d)=\lambda d, \tag{1.21}$$

$$u \text{ is decreasing on } (0,x_0) \tag{1.22}$$

We first estimate $x_0(d)$ assuming its existence.

Since f is increasing $f(u(s))$ is decreasing on $[0, x_0]$ and therefore

$$u'(x, d)=-\frac{1}{\rho(x)}\int_0^x\rho(s)k(s)f(u(s))ds\ge-\frac{f(d)}{\rho(x)}\int_0^x\rho(s)k(s)ds.$$

Integrating on $[0, x_0]$ we get

$$(1-\lambda)d\le f(d)\int_0^{x_0}\frac{dt}{\rho(t)}\int_0^t\rho(s)k(s)ds\le f(d)\int_0^{x_0}dt\int_0^t k(s)ds\le Cf(d)x_0^{2-\beta}$$

and therefore $\exists C>0$ such that

$$x_0(d)\ge C(d/f(d))^{1/(2-\beta)} \tag{1.23}$$

Let $d>0$ and $\bar{x}\in[0, L]$. If $\bar{x}\le x_0(d)$ then

$$E(\bar{x}, d)\ge k(\bar{x})F(\lambda d)\ge\alpha F(\lambda d) \text{ (by (1.3)).} \tag{1.24}$$

Let us suppose $\bar{x}\ge x_0(d)$. We multiply equation (1.10) with ρ^2u'. We get

$$(\rho u')'\rho u'+\rho^2k(x)f(u)u'=0. \tag{1.24}$$

Let us observe that since $\rho(x)=O(x)$ as $x\to0$ and $k(x)=O(x^{-\beta})$ as $x\to0$ then

$$\rho^2(x)k(x)=O(x^{2-\beta}) \text{ as } x\to0.$$

Since $\beta<2$ we see that ρ^2k has no singularity at $x=0$. (1.24) may be written as

$$\frac{1}{2}[(\rho u')^2]'+[\rho^2kF(u)]'-(\rho^2k)'F(u)=0.$$

We integrate on $[0, \bar{x}]$ and we get

$$\rho^2(\bar{x})E(\bar{x}, d)=\int_0^{\bar{x}}(\rho^2k)'F(u)ds=\int_0^{x_0(d)}(\rho^2k)'F(u)dx+\int_{\bar{x}}^{x_0(d)}(\rho^2k)'F(u)dx\ge\int_0^{x_0(d)}(\rho^2k)'F(u)ds$$

since ρ^2k is increasing. Therefore integrating by parts

$$\rho^2(\bar{x})\, E(\bar{x}, d) \geq \rho^2 k F(u) \Big|_0^{x_0(d)} - \int_0^{x_0} \rho^2 k f(u) u' dx \text{ on } [0, x_0] \geq$$

$$\rho^2(x_0) k(x_0) F(\lambda d) \text{ (since } u' \leq 0 \text{ on } [0, x_0]) \geq$$

$$\alpha \rho^2(x_0) F(\lambda d) \text{ (by (1.3))} \geq \alpha \rho^2 (C(d/f(d))^{\frac{1}{(2-\beta)}}) F(\lambda d) \text{ (by (1.23))}.$$

Letting $\sigma = \inf \left\{ \frac{1}{\rho(x)} / x \in (0, L) \right\} > 0$, we get

$$E(\bar{x}, d) \geq \frac{\alpha}{\sigma} \rho^2 \left(C \left(\frac{d}{f(d)} \right)^{\frac{1}{2-\beta}} \right) F(\lambda d) = \frac{\alpha}{\sigma} A(\lambda, d). \tag{1.25}$$

Therefore for each $x \in [0, L]$ we have the estimation

$$E(\bar{x}, d) \geq \min \{d, \alpha F(\lambda, d), \frac{\alpha}{\sigma} A(\lambda, d)\}. \tag{1.26}$$

Since $\lim_{d\to\infty} A(\lambda, d) = \infty$ by (1.8) we get the desired result.

We now come to the final step of the proof.

We can use the shooting method via a phase plane analysis.

Let us denote $r^2(x, d) = u^2(x, d) + (u'(x, d))^2$

By Lemma 2 we deduce

$$\lim_{d\to\infty} r^2(x, d) = \infty \text{ uniformly on each } [\varepsilon, L] \text{ and moreover}$$

$r^2(x, d) \neq 0$ $d > d_0$, $x \in [0, L]$, d_0 large enough. Hence for $d > d_0$ one can find a continuous argument $\theta : [0, L] \to \mathbb{R}$ $\theta(x) = \theta(x, d)$ defined as

$$u(x, d) = r(x, d) \cos \theta(x, d), \quad u'(x, d) = -r(x, d) \sin \theta(x, d), \quad \theta(0, d) = 0. \tag{1.27}$$

This follows from the fact that the mapping $\mathbb{R} \xrightarrow{p} S^1$ given by $r \xrightarrow{p} e^{ir}$ is a covering mapping [3].

The condition $au'_x(L, d) + bu(L, d) = 0$ is equivalent to saying that $\theta(L, d) = \theta_0 + k\pi$, $k \in \mathbb{Z}$ where tg θ_0 is the slope of the line $\{(s, t) / at + bs = 0\}$ in the st-plane.

We shall prove in the sequel that

$$\lim_{d\to\infty} \theta(L, d) = \infty. \tag{1.28}$$

Hence in view of the continuity of θ, we get infinitely many values d_n such that

$$\theta(L, d_n) = \theta_0 + k_n \pi, \quad k_n \in \mathbb{N}.$$

and therefore the theorem will be proved.

(1.28) will be obtained by studying an ODE satisfied by θ. We see that

$$u' = r' \cos \theta - r \sin \theta\, \theta', \qquad u'' = -r' \sin \theta - r \cos \theta\, \theta'.$$

Hence

$$u' \sin \theta + u'' \cos \theta = -r\theta', \quad -r \sin^2 \theta + u'' \cos \theta = -r\theta',$$

$$\theta' = \sin^2\theta - (u''/r) \cos \theta, \quad \theta' = \sin^2 \theta + (\rho'/\rho + kf(u)) \frac{\cos\theta}{r}. \tag{1.29}$$

We can draw a first qualitative information from (1.29).

Remark 3. There is $R > 0$ such that if $r(x, d) > R$ $x \in [\bar{x}, L \in (\bar{x} > 0)$ and $\theta(x, d) = k\pi/2$, then $\theta(x, d) > k\pi/2$ for every $x \in [\bar{x}, L]$.

If $u_0 > 0$ we denote $m(u_0) : = \min \{f(u)/u \,|\, |u| > u_0\}$.

By (1.7) we have $\lim_{u_0\to\infty} m(u_0) = +\infty$.

Let $h = \max \left\{ \frac{\rho'(x)}{\rho(x)} ; x \in [L/4, L] \right\}$.

In order to show (1.28) we shall prove that given $J \in \mathbb{N}$ there is $d_0 > 0$ such that $d \geq d_0$ implies $\theta(L, d) \geq J\pi$. This thing will be achieved by proving that the growth of $\theta(x, d)$ on $[L/4, L]$ is large provided that d is large enough. Now let $\delta \in (0, \pi/4)$ small enough and $r_0 > 0$ large enough so that the following inequalities should be fulfilled

$$\cos^2 \delta - \frac{h}{2} \sin 2\delta > 1/2, \tag{1.30}$$

$$v(r_0, \delta) \geq h/r_0^2, \tag{1.31}$$

where $v(r_0, \delta) = \alpha m(r_0 \cos \delta) \sin^2 \delta$

$$3L/(4J) \geq 8\delta + 2(\pi - 2\delta)/v(r_0, \delta). \tag{1.32}$$

We distinguish two situations

A. $x \geq L/4$, $\theta(x, d) \in (J\pi/2) - \delta$, $(J\pi/2) + \delta)$.

From (1.29) we get

$$\theta'(x, d) \geq \sin^2 \theta + \frac{kf(u) - u}{r^2} - \frac{h|uu'|}{r^2}$$

$$\geq \cos^2 \delta - \frac{h}{2} |\sin 2\theta| \geq \cos^2 \delta - \frac{h}{2} \sin 2\delta.$$

By (1.30) we see that in case A

$$\theta'(x, d) \geq \frac{1}{4}. \tag{1.33}$$

B. $x > L/4$, $\theta(x, d) \in [(J\pi/2) + \delta, (J+2)\pi/2 - \delta]$

$$\theta'(x, d) \geq k \frac{f(u)}{r} \cos \theta - h/2r_0^2$$

$$k \frac{f(u)}{u} \cos^2 \theta - \frac{h}{2r_0^2} \geq v(r_0, \delta) - h/2r_0^2.$$

By (1.31) we see that in case B

$$\theta'(x, d) \geq v(r_0, \delta)/2. \tag{1.34}$$

From (1.33) and (1.34) one infers that $\theta(x, d)$ is increasing on $[L/4, L]$. We also see that θ goes through the interval $[J\pi/2-\delta, J\pi/2+\delta)$ of length 2δ in a time less than 8δ and an interval of type $[(J\pi/2)+\delta, (j+2)\pi/2-\delta]$ of length $\pi-2\delta$ in a time less than $2(\pi-2\delta)/v(r_0, \delta)$.

Hence θ varies in an interval of length π in a „time" less than $8\delta+$ $+2(\pi-2\delta)/v(r_0, \delta)$.

Our desire is to make the variation of θ greater than $J\pi$ while $x\in[L/4, L]$, hence in a „time" $L-L/4=3L/4$. This is what inequality (1.32) aims at.

Therefore as soon as (1.30)—(1.32) are satisfied then

$$\theta(L, d)-\theta(L/4, d)\geq J\pi.$$

By remark 3 we see that (1.28) is proved and hence our theorem too.

Q.E.D.

2. Applications. We shall present two applications of our result. The first one is a degenerate elliptic semilinear problem while the second one is an one dimensional Sturm-Liouville problem arising in genetics.

A. Consider the following boundary value problem in the ball $B_R(0)\subset$ $\subset\mathbf{R}^N$, $N>1$,

$$\begin{aligned} &-\operatorname{div}(k(|x|)\nabla u(x))=P(|x|)f(u), \quad x\in\mathbf{R}^N \\ &a\partial u/\partial\nu+bu=0, \quad a^2+b^2\neq 0 \text{ on } |x|=R. \end{aligned} \tag{2.1}$$

We look for classical solutions to (2.1). Here $f(0)=0$, f is locally Lipschitzian and increasing.

If we restrict our search to radially symmetric solutions then (2.1) reduces to a Sturm-Liouville problem of type (1.1).

Denote $r=|x|$, $x=(x_1, \ldots, x_N)^T$. Then $\dfrac{\partial}{\partial x_i}=\dfrac{\partial r}{\partial x_i}\dfrac{\partial}{\partial r}=\dfrac{x_i}{r}\dfrac{\partial}{\partial r}$.

Denote differentiation with respect to r by „$'$". Hence

$$\nabla u=\frac{x}{r}u'$$

$$-\operatorname{div}(k(r)\nabla u)=-\sum_{i=1}^{N}\frac{\partial}{\partial x_i}\left(x_i\frac{k(r)}{r}u'\right)\left(h(r)-\frac{k(r)}{r}\right)=$$

$$=-\sum_{i=1}^{N}\left[\left(h(r)+h'(r)\frac{x_i^2}{r}\right)u'+\frac{x_i^1}{r}h(r)u''\right]=$$

$$=-[rh(r)u''+(Nh(r)+h'(r)ru']=$$

$$=-\left[k(r)u''+\left(Nh(r)+\frac{k'(r)r-k(r)}{r}\right)u'\right]=$$

$$=-k(r)u''-\left[(N-1)\frac{k(r)}{r}+k'(r)\right]u'$$

$k=1$ corresponds to the usual laplacian operator $-\Delta$ and is treated in the papers [1], [2] of A. Castro-A. Kurepa.

A straightforward computation shows that

$$-k(r)u''-\left[(N-1)\frac{k(r)}{r}+k'(r)\right]u'=-\frac{1}{r^{N-1}}[k(r)r^{N-1}u']'.$$

Hence problem (2.1) reduces to

$$\begin{aligned} &-r^{-N+1}[k(r)r^{N-1}u']'=P(r)f(u) \quad r\in(0, R), \\ &u'(0)=0, \\ &au'(R)+bu(R)=0, \end{aligned} \tag{2.2}$$

or, equivalently to

$$\begin{aligned} &-1/(r^{N-1}K(r))[r^{N-1}k(r)u']'=\frac{P(r)}{k(r)}f(u), \quad r\in(0, R) \\ &u'(0)=au'(R)+bu(R)=0. \end{aligned} \tag{2.3}$$

This is a problem of type (1.1) with $\rho(r)=r^{N-1}k(r)$. Conditions (1.2)—(1.8) become

$$\rho(r)\in C^1(0, R), \quad \rho(0)=0, \quad \rho'\geq 0 \text{ on } (0, R). \tag{2.4}$$

$$\frac{P}{k}\in C^1(0, R], \quad \inf\left\{\frac{P(r)}{k(r)}/r\in(0, R)\right\}=\alpha>0. \tag{2.5}$$

$$\begin{aligned} &\exists\beta, \gamma\in\mathbf{R} \text{ such that } P(r)=O(r^{\beta}) \quad r\to 0, \\ &\frac{1}{k(r)}=O(r^{-\gamma}) \quad r\to 0 \text{ and } \gamma-\beta<2. \end{aligned} \tag{2.6}$$

$$r^{2N-2}k(r)P^2(r) \text{ is increasing on } (0, R). \tag{2.7}$$

$$f:\mathbf{R}\to\mathbf{R} \text{ is locally lipschitzian, increasing and } f(0)=0 \tag{2.8}$$

$$\lim_{|u|\to\infty}\frac{f(u)}{u}=+\infty \tag{2.9}$$

$$\exists\lambda\in(0, 1) \text{ such that } \lim_{d\to\infty}A_c(\lambda, d)=+\infty \quad \forall c>0, \tag{2.10}$$

where $A(\lambda, d)=F(d)(d/f(d))^{\frac{2N-2}{2-\gamma+\beta}}k^2(c(d/f(d))^{\frac{1}{2-\gamma+\beta}})$,

$$F(u)=\int_0^u f(s)ds.$$

As a strightforward consequence of Theorem 1 we get

Proposition 1. *If (2.4)—(2.10) are fulfilled then (2.1) has infinitely many radially symmetric solutions.*

Let us investigate (2.10) in a special situation when $f(u)$ behaves as u^{δ} as $u\to\infty$ and $k(r)$ behaves as r^{γ} as $r\to 0$. Then $F(\lambda, d)$ behaves as $d^{\sigma+1}$, $\dfrac{d}{f(d)}$ as $u^{1-\sigma}$ and therefore $A(\lambda, d)$ behaves as d^{0} where

$$0=\sigma+1+\frac{(2N-2)(1-\sigma)}{2+\beta-\gamma}+\frac{2\gamma(1-\sigma)}{2+\beta-\gamma}=$$

$$=\frac{(\sigma+1)(2+\beta-\gamma)+(2N-2)(1-\sigma)+2\gamma(1-\sigma)}{2+\beta-\gamma}=$$

$$=\frac{\sigma(1+\beta-3\gamma-2N)+(2N+\gamma+\beta)}{2+\beta-\gamma}.$$

The relation (2.10) becomes $0>0$ or equivalently

$$2N+\gamma+\beta>\sigma(2N+3\gamma-\beta-4). \tag{2.11}$$

When $\gamma<1-N$ then $2N+3\gamma-\beta-4<0$ and therefore (2.11) implies

$$(2N+\gamma+\beta)/(2N+3\gamma-\beta-4)<\sigma, \tag{2.12}$$

and therefore (2.1) *has infinitely many solutions for arbitrarily large* σ. This is different from the usual case of Laplacian where $\gamma=0$ and our results give an answer only for exponents $\sigma<N/(N-2)$ (see also Rabinowitz [6] Sect. 10).

We see that the balance between the rate of degeneracy of ellipticity and the behaviour of f near ∞ plays an important role in existence results.

B. Our second application arises in the field of genetics, epidemiology (Fife [4], Henry [5]).

The problem we consider is the following

$$\begin{cases}(x(1-x)u')'+P(x)f(u)=0,\\ u'(0)=u'(1)=0.\end{cases} \tag{2.13}$$

Here P is C^1 on (0, 1) and symmetric with respect to $x=\frac{1}{2}$. We rewrite (2.13) as

$$\begin{cases}\dfrac{1}{x(1-x)}(x(1-x)u')'+\dfrac{P(x)}{x(1-x)}f(u)=0, \quad x\in(0,1),\\ u'(0)=u'(1)=0.\end{cases} \tag{2.14}$$

Denote $k(x)=\dfrac{P(x)}{x(1-x)}$.

At a first sight (2.14) deos not lie in the frame of Theorem 1 since it has two singularities in $x=0$ and $x=1$.

We consider the auxiliary problem

$$\begin{cases}\dfrac{1}{x(1-x)}(x(1-x)u')'+k(x)f(u)=0, \quad x\in(0,1/2),\\ u'(0)=u'(1/2)=0.\end{cases} \tag{2.15}$$

After we prove the existence of solutions to (2.15) we extend them by symmetry about $x=1/2$ and hence we obtain strong solutions to (2.14).

(2.15) lie in the frame of Theorem 1. Hypotheses (1.2)–(1.8) become

$$P\in C^1(0,\ 1/2),\quad \inf\left\{\frac{P(x)}{x}\,\middle|\,x\in(0,1/2)\right\}=\alpha>0, \tag{2.16}$$

$$\exists\beta<1 \text{ such that } \rho(x)=O(x^{-\beta}) \text{ as } x\to 0. \tag{2.17}$$

$$x(1-x)P(x) \text{ is increasing on } (0, L). \tag{2.18}$$

$$f:\mathbf{R}\to\mathbf{R} \text{ is locally lipschitzian, increasing and } f(0)=0. \tag{2.19}$$

$$\lim_{|u|\to\infty}\frac{f(u)}{u}=+\infty. \tag{2.20}$$

$$\exists\lambda\in(0,1) \text{ such that } \lim_{d\to\infty}A(\lambda, d)=+\infty, \tag{2.21}$$

where $A(\lambda, d)=F(\lambda, d)(d/f(d))^{2/(1-\beta)}$.

We get the following result

Proposition 2. *If* (2.16)–(2.21) *are fulfilled then* (2.13) *has infinitely many strong solutions.*

It would be interesting to make the same discussion as the one at the end of example A. Let us suppose f behaves as u and p as $x^{-\beta}$ when $u\to\infty$ and $x\to 0$ respectively.

Condition (2.21) becomes where

$$0=\sigma+1+\frac{2(1-\sigma)}{1-\beta}=\frac{(\sigma+1)(1-\beta)+2(1-\sigma)}{1-\beta}=$$

$$=\frac{-\sigma(1+\beta)+3-\beta}{1-\beta}$$

or equivalently

$$3-\beta>\sigma(1+\beta).$$

When $\beta\leq-1$ we see that f may have arbitrarily large polynomial growth.

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