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ON OPIAL LIKE DISCRETE INEQUALITIES

BY

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1. Introduction. Discrete analogues of Opial's inequality [6] and its generalizations are established by W o n g [9], L e e [4] and B e e s a c k [1] (see, also [7, 8]). The discrete analogue of the Opial's inequality proved by W o n g (see, [5, Theorem 17, p. 161]) can be stated as follows.

Let $\{u_i\}$ be a nondecreasing sequence of nonnegative real numbers and let $u_0=0$. Then, for $p>1$, we have,

$$\sum_{i=1}^{n} u_i^p(u_i-u_{i-1}) \leq \frac{(n+1)^p}{p+1} \sum_{i=1}^{n} (u_i-u_{i-1})^{p+1}. \tag{1}$$

In fact, the inequality (1) is a discrete analogue of the variant of Opial's inequality given by H u a in [3]. In [4], Lee has given a discrete analogue of the variant of a further generalization of Opial's inequality given by Y a n g in [10] (see, also [5]). In 1967, C a l v e r t [2] found some interesting generalizations of the Opial's inequality and it appears that the discrete versions of Calvert's generalizations of Opial's inequality have been overlooked in the literature. This is apparently due in part, because the discrete versions of such inequalities are more difficult to establish and require rather somewhat different analysis in their proofs. Our objective here is to establish some new Opial like discrete inequalities which are closely related to the discrete analogues of the Calvert's generalizations of Opial's inequality. The analysis used for proving the inequalities in this paper is elementary and our results provide new estimates on these types of discrete inequalities.

2. Statement of results. In this section, we state our results to be proved in this paper. The difference operator ∇ used in our subsequent discussion is defined by $\nabla u_i = u_i - u_{i-1}$ for $i \in N$, where N is the set of natural numbers. We shall use, the usual convention that $\sum_{i=n_0}^{n_0-1} b_i = 0$ for $n_0 \in N$.

We first establish the following inequality which is closely related to the discrete analogue of the inequality given by Calvert in [2, Theorem 3].

Theorem 1. *Let $\{u_i\}$ and $\{v_i\}$ for $i \in N$ be sequences of real numbers. If $u_0=v_0=0$, then*

$$\sum_{i=1}^{n} [|u_i||\nabla v_i| + |v_i||\nabla u_i|] \leq \frac{n+1}{2} \sum_{i=1}^{n} [|\nabla u_i|^2 + |\nabla v_i|^2]. \tag{2}$$

Remark. 1. It is interesting to note that in the special case when $v_i=u_i$ inequality (2) reduces to

$$\sum_{i=1}^{n} |u_i||\nabla u_i| \le \frac{n+1}{2}\sum_{i=1}^{n} |\nabla u_i|^2, \tag{3}$$

which in turn is a discrete analogue of the Opial's inequality given in [5, Theorem 2' on p. 154]. When $n=1$, the constant in Wong's inequality (1) coincides with the constant obtained in inequality (3).

Or next two theorems deals with the inequalities which are closely related to the discrete analogues of the inequalities established by Calvert in [2, Theorems 1 and 2].

Theorem 2. *Let $\{u_i\}$ for $i\in N$ be a sequence of real numbers. Let $f(u_i)$ be a continuous function defined for all u_i in the range of $\{u_i\}$ and for all real u_i of the form $u_i=\sum_{k=1}^{i}\nabla u_k$, $i=1, \dots, n$. Suppose that $|f(u_i)|\le f(|u_i|)$ for all u_i and that $f(x)\le f(y)$ for $0\le x\le y$. Let $F(s)=\int_0^s f(x)dx$, $s>0$. If $u_0=0$, then*

$$\sum_{i=1}^{n} |f(u_i)\nabla u_i| \le F\Big(\sum_{i=1}^{n}|\nabla u_i|\Big)+\sum_{i=1}^{n}\Big[f\Big(\sum_{k=1}^{i}|\nabla u_k|\Big)-f\Big(\sum_{k=1}^{i-1}|\nabla u_k|\Big)\Big]|\nabla u_i|. \tag{4}$$

Remark 2. If we take $f(x)=x$, then we get $F(s)=\frac{1}{2}s^2$, and the inequality (4) reduces to the following inequality

$$\sum_{i=1}^{n} |u_i\nabla u_i| \le \left(\frac{n}{2}+1\right)\sum_{i=1}^{n} |\nabla u_i|^2. \tag{5}$$

Here we note that the constant appearing in (5) is larger than the constant obtained in inequality (3). The main reason for it is the wastage involved in proving the much more general result given in Theorem 2.

Theorem 3. *Let $\{u_i\}$ and f be as in Theorem 2. Let $G(s)=\int_0^s \frac{1}{f(x)}\,dx, s>0.$ If $u_0=0$ then*

$$\sum_{i=1}^{n} \left|\frac{\nabla u_i}{f(u_i)}\right| \ge G\Big(\sum_{i=1}^{n}|\nabla u_i|\Big)-\sum_{i=1}^{n}\Big[1/f\Big(\sum_{k=1}^{i-1}|\nabla u_k|\Big)-1/f\Big(\sum_{k=1}^{i}|\nabla u_k|\Big)\Big]|\nabla u_i|. \tag{6}$$

Remark 3. We note that by making a suitable specialization of the inequality (6), it is easy to observe that the constant appearing therein is also not sharp. Because there is a significant wastage in the general case of the proof of Theorem 3.

3. Proofs of theorems 1—3. From the hypotheses of Theorem 1, it is easy to observe that the following identities hold.

$$u_i=\sum_{k=1}^{i}\nabla u_k, \quad i=1, \dots, n, \tag{7}$$

$$v_i=\sum_{k=1}^{i}\nabla v_k, \quad i=1, \dots, n. \tag{8}$$

Using (7), (8) and the elementary inequality $ab\le(a^2+b^2)/2$ (for a, b reals) we observe that.

$$\sum_{i=1}^{n}[|u_i||\nabla v_i|+|v_i||\nabla u_i|]\le\sum_{i=1}^{n}\Big[\Big(\sum_{k=1}^{i}|\nabla u_k|\Big)|\nabla v_i|+\Big(\sum_{k=1}^{i}|\nabla v_k|\Big)|\nabla u_i|\Big]=$$

$$=\sum_{i=1}^{n}\sum_{k=1}^{i}[|\nabla u_k||\nabla v_i|+|\nabla v_k||\nabla u_i|]\le\frac{1}{2}\sum_{i=1}^{n}\sum_{k=1}^{i}[|\nabla u_k|^2+|\nabla v_i|^2+|\nabla v_k|^2+|\nabla u_i|^2]=$$

$$=\frac{1}{2}\sum_{i=1}^{n}\sum_{k=1}^{i}[|\nabla u_i|^2+|\nabla u_k|^2]+\frac{1}{2}\sum_{i=1}^{n}\sum_{k=1}^{i}[|\nabla v_i|^2+|\nabla v_k|^2]=\frac{n+1}{2}\sum_{i=1}^{n}|\nabla u_i|^2+$$

$$+\frac{n+1}{2}\sum_{i=1}^{n}|\nabla v_i|^2=\frac{n+1}{2}\sum_{i=1}^{n}[|\nabla u_i|^2+|\nabla v_i|^2].$$

This completes the proof of Theorem 1.

From the hypotheses of Theorem 2, we see that the identity (7) holds. Hence we observe that .

$$\sum_{i=1}^{n}|f(u_i)\nabla u_i|=\sum_{i=1}^{n}\Big|f\Big(\sum_{k=1}^{i}\nabla u_k\Big)\nabla u_i\Big|\le\sum_{i=1}^{n}f\Big(\sum_{k=1}^{i}|\nabla u_k|\Big)|\nabla u_i|. \tag{9}$$

Now from the definition of F and the properties of the function f we obtain

$$F\Big(\sum_{k=1}^{i}|\nabla u_k|\Big)-F\Big(\sum_{k=1}^{i-1}|\nabla u_k|\Big)=\int_{\sum_{k=1}^{i-1}|\nabla v_k|}^{\sum_{k=1}^{i}|\nabla v_k|} f(x)dx\ge f\Big(\sum_{k=1}^{i-1}|\nabla u_k|\Big)|\nabla u_i|= \tag{10}$$

$$=f\Big(\sum_{k=1}^{i}|\nabla u_k|\Big)|\nabla u_i|+\Big[f\Big(\sum_{k=1}^{i-1}|\nabla u_k|\Big)-f\Big(\sum_{k=1}^{i}|\nabla u_k|\Big)\Big]|\nabla u_i|.$$

From (10) we observe that

$$f\Big(\sum_{k=1}^{i}|\nabla u_k|\Big)|\nabla u_i|\le F\Big(\sum_{k=1}^{i}|\nabla u_k|\Big)-F\Big(\sum_{k=1}^{i-1}|\nabla u_k|\Big)+\Big[f\Big(\sum_{k=1}^{i}|\nabla u_k|\Big)- \tag{11}$$

$$-f\Big(\sum_{k=1}^{i-1}|\nabla u_k|\Big)\Big]|\nabla u_i|.$$

Now substituting $i=1, \dots, n$ on both sides of the inequality (11) and summing up we obtain

$$\sum_{i=1}^{n}f\Big(\sum_{k=1}^{i}|\nabla u_k|\Big)|\nabla u_i|\le F\Big(\sum_{k=1}^{n}|\nabla u_k|\Big)+\sum_{i=1}^{n}\Big[f\Big(\sum_{k=1}^{i}|\nabla u_k|\Big)-f\Big(\sum_{k=1}^{i-1}|\nabla u_k|\Big)\Big]|\nabla u_i|. \tag{12}$$

Replacing k by i in the first term on the right side in (12) and using this bound in (9) we obtained the desired inequality in (4). This completes the proof of Theorem 2.

From the hypotheses of Theorem 3, we see that the identity (7) holds. Hence we observe that

$$\sum_{i=1}^{n}\left|\frac{\nabla u_i}{f(u_i)}\right|=\sum_{i=1}^{n}|\nabla u_i|/\Big|f\Big(\sum_{k=1}^{i}\nabla u_k\Big)\Big|\ge\sum_{i=1}^{n}|\nabla u_i|/f\Big(\sum_{k=1}^{i}|\nabla u_k|\Big). \tag{13}$$

From the definition of G and the properties of f we obtain

$$G(\sum_{k=1}^{i}|\nabla u_k|)-G(\sum_{k=1}^{i-1}|\nabla u_k|)=\int\limits_{\sum_{k=1}^{i-1}|\nabla u_k|}^{\sum_{k=1}^{i}|\nabla u_k|}\frac{1}{f(x)}dx\leq|\nabla u_i|/f(\sum_{k=1}^{i-1}|\nabla u_k|)= \tag{14}$$

$$=|\nabla u_i|/f(\sum_{k=1}^{i-1}|\nabla u_k|)-|\nabla u_i|/f(\sum_{k=1}^{i}|\nabla u_k|)+|\Delta u_k|/f(\sum_{k=1}^{i}|\nabla u_k|).$$

From (14) we have

$$|\nabla u_i|/f(\sum_{k=1}^{i}|\nabla u_k|)\geq G(\sum_{k=1}^{i}|\nabla u_k|)-G(\sum_{k=1}^{i-1}|\nabla u_k|)-[1/f(\sum_{k=1}^{i}|\nabla u_k|)- \tag{15}$$

$$-1/f(\sum_{k=1}^{i}|\nabla u_k|)]|\nabla u_i|.$$

Now substituting $i=1,\ldots,n$ on both sides of the inequality (15) and summing up we obtain

$$\sum_{i=1}^{n}|\nabla u_i|/f(\sum_{k=1}^{i}|\nabla u_k|)\geq G(\sum_{k=1}^{n}|\nabla u_k|)-\sum_{i=1}^{n}[1/f(\sum_{k=1}^{i-1}|\nabla u_k|)-1/f(\sum_{k=1}^{i}|\nabla u_k|)]|\nabla u_i|. \tag{16}$$

Replacing k by i in the first term on the right side in (16) and using this bound in (13) we obtain the required inequality in (6). This completes the proof of Theorem 3.

Remark 4. The discrete analogues of Opial's inequality or its variants are established in [1, 4, 7—9] by using different techniques. As noted earlier in Remarks 2 and 3, the steps in the proof of Theorem 2 there is a wastage at (9) and (10) and in particular in specializing Theorem 2 as mentioned in Remark 2 is there a further wastage while using a Schwarz inequality in order to obtain a net constant in inequality (5). Similarly, we can see that, there is a wastage at (13) and (14), in the general case of the proof of Theorem 3.

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THE GAUGE-COVARIANT EXTENSION

$g_{pq}=\frac{1}{2}D_pD_qF^2$

OF FINSLERIAN METRIC TENSOR

BY

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The usual Finslerian metric tensor (7) and hence all the usual Finsler geometry are not covariant under the gauge transformations (1), for they are based on the condition $C_{i0j}=0$ which is not gauge-covariant. To remedy this, we make the note that the Miron's regularity condition $C_{00i}=0$ is gauge-covariant, which suggests us to propose the explicity gauge-covariant and attractive definition (9) of the metric tensor. Nearest implications of the definition are considered.

In the previous publications [1—5] the gauge transformations

$$y^q=y^q(x^i,\ y'^p) \tag{1}$$

were introduced to extend the Finslerian methods and to bring them near real applications in the modern theoretical physics. In (1), x^i, mean the local components of points of the base manifold, and y^q are the components of tangent vectors with respect to the local natural frame; $i, j, \cdots p, q, \cdots =1, 2, \cdots N$. The conditions rank $(\partial y^p/\partial y'^q)=N$ together with the positive first-degree homogeneity,

$$y^q(x,\ ky')=ky^q(x,\ y'),\quad k>0 \tag{2}$$

will be assumed. Sufficiently smoothness of all the objects under consideration will be tacitly implied. The transformations mean the local representation of restricted automorphisms of the tangent bundle, that is, the automorphisms which act as the identity on the base manifold. In other words, (1) represent arbitrary, nonlinear in general, one-to-one mappings of tangent bundle onto tangent bundle subject to the condition that the mappings do not shift the points of support x^i. It can also be said that (1) represent the one-to-one mappings of tangent spaces onto tangent spaces provided that *the mapping depend on the background points in a sufficiently smooth way*. The last circumstance makes it legitimate to apply to (1) the epithet 'gauge' in accordance with the terminology adopted in the theoretical physics (see, for example, [6—8]). Since y^q are vectors under the general local coordinate transformations

$$x^i=x^i(\bar{x}^j), \tag{3}$$