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HOLOMORPHS OF CONJUGACY CLOSED LOOPS

BY

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1. Introduction. The study of holomorphy theory has received some considerations in the works of Bruck [1], Bruck and Paige [3], Robinson [7,8], and Huthance, Jr. [6]. In the present paper, we shall extend the study to G–loops.

Definition 1.1. *If $(G,.)$ is a loop and $a \in G$, the permutations $R(a)$ and $L(a)$, called the right and the left multiplications, respectively, are defined by $xR(a) = x.a$, and $xL(a) = a.x$, for all $x \in G$.*

Definition 1.2. *A loop $(G,.)$ satisfies "the right inverse property if there exists a function $\rho : G \to G$ such that $(x.y).y\rho = x$ for all $x, y \in G$. If there exists a function $\lambda : G \to G$ such that $y\lambda.(y.x) = x$ for all $x, y \in G$, then $(G,.)$ is said to satisfy the left inverse property."*

If $(G,.)$ is an inverse property loop, then for each $x \in G$, there exists an element $x^{-1} \in G$ such that $x.x^{-1} = x^{-1}.x = 1$, $R(a)^{-1} = R(a^{-1})$ and $L(a)^{-1} = L(a^{-1})$.

Definition 1.3. *Let $(G,.)$ be a loop, the triple $\langle U, V, W\rangle$ of permutations of G is called an autotopism of G if and only if $xU.yV = (x.y)W$ for all $x, y \in G$.*

It is well–known that the set of all autotopisms of a loop forms a group under componentwise multiplication.

Definition 1.4. *Let $(G,.)$ and $(K, \circ)$ be two loops. Then $(K, \circ)$ is said to be an isotope of $(G,.)$ if and only if there exist a triple $\langle U, V, W\rangle$ of one-to-one mappings of G onto K such that $xU \circ yV = (x.y)W$ for all $x, y \in G$. If K consists of the same elements as G and W is the identity mapping of G, then $(G, \circ)$ is called a principal isotope of $(G,.)$.*

Bruck [2] has shown that every isotope of a loop is isomorphic to a principal isotope of the loop, and every principal isotope $(G, \circ)$ of a loop $(G,.)$ is defined by $x \circ y = xR(b)^{-1}.yL(a)^{-1}$ for all $x, y \in G$, and some $a, b \in G$.

Definition 1.5. *A loop $(G,.)$ is called a G–loop if and only if it is isomorphic to all its isotopes.*

Definition 1.6. *A loop $(G,.)$ is conjugacy closed if and only if $R(x)^{-1}R(y)R(x) \in R(G,.)$ and $L(x)^{-1}L(y)L(x) \in L(G,.)$ for all $x,y \in G$, where $R(G,.) = \{R(a) \mid a \in G\}$ and $L(G,.) = \{L(a) \mid a \in G\}$.*

Goodaire and Robinson [5] have undertaken a study of conjugacy closed loops and have proved that they are G–loops.

We shall employ the following results about G–loops as established by Chiboka and Solarin [4], and in particular, about conjugacy closed loops as established by Goodaire and Robinson [5].

Lemma 1.1. *A loop $(G,.)$ is a G–loop if and only if there exists a permutation θ of G such that $\langle \theta R(b)^{-1}, \theta L(a)^{-1}, \theta\rangle$ is an autotopism of $(G,.)$ for $a,b \in G$.*

Lemma 1.2. *A loop $(G,.)$ is conjugacy closed if and only if the triples $\langle R(a), R(a)L(a)^{-1}, R(a)\rangle$ and $\langle L(a)R(a)^{-1}, L(a), L(a)\rangle$ are autotopisms of $(G,.)$ for all $a \in G$.*

Definition 1.7. *Let $(G,.)$ be a loop and let A be any group of automorphisms of G. Then the holomorph $A(G)$ of G by A, or the A–holomorph of G, is the set $H = A \times G$ with a binary operation defined by*

$$(1) \qquad (\alpha, x) \circ (\beta, y) = (\alpha\beta, x\beta.y)$$

for all $x,y \in G$ and $\alpha,\beta \in A$.

2. Main result. Lemma 2.1. *Let $(G,.)$ be a loop, A a group of automorphisms of G, and H the A-holomorph of G. Then $(H,\circ)$ is a conjugacy closed loop if and only if the triples*

$$(2) \qquad (i) \quad \langle L(x\alpha)R(x)^{-1}, L(x), L(x\alpha)\rangle$$

and

$$(3) \qquad (ii) \quad \langle R(x\alpha), R(x)L(x\alpha)^{-1}, R(x)\rangle$$

are autotopisms of G for all $x \in G$ and all $\alpha \in A$.

Proof. The loop $(H,\circ)$ is conjugacy closed if and only if the triples

$$(a) \quad \langle L_\circ(\alpha,x)R_\circ(\alpha,x)^{-1}, L_\circ(\alpha,x), L_\circ(\alpha,x)\rangle$$

and

$$(b) \quad \langle R_\circ(\alpha,x), R_\circ(\alpha,x)L_\circ(\alpha,x)^{-1}, R_\circ(\alpha,x)\rangle$$

are autotopisms of $(H,\circ)$ for all $(\alpha,x) \in H$. Condition (a) is equivalent to the identity

$$(4) \quad (\alpha,x)\circ[(\beta,y)\circ(\gamma,z)] = [(\alpha,x)\circ(\beta,y)]\,R_\circ(\alpha,x)^{-1}\circ[(\alpha,x)\circ(\gamma,z)]$$

while condition (b) is equivalent to the identity

$$(9) \quad [(\beta,y)\circ(\gamma,z)]\circ(\delta,t) = [(\beta,y)\circ(\delta,t)]\circ[(\gamma,z)\circ(\delta,t)]\,L_\circ(\delta,t)^{-1}$$

for all $t,x,y,z \in G$ and all $\alpha,\beta,\gamma,\delta \in A$, where $R_\circ(\alpha,x)$ is the right multiplication by (α,x) in $(H,\circ)$, and $L_\circ(\alpha,x)$ is the left multiplication by (α,x) in $(H,\circ)$. Equation (4) holds if and only if

$$(5) \qquad (\alpha\beta\gamma, x\beta\gamma.(y\gamma.z)) = (\alpha\beta, x\beta.y)R_\circ(\alpha,x)^{-1}\circ(\alpha\gamma, x\gamma.z)$$

Suppose $(\alpha\beta, x\beta.y)R_\circ(\alpha,x)^{-1} = (\delta,u)$ for some $u \in G$ and $\delta \in A$. Then,

$$(\alpha\beta, x\beta.y) = (\delta,u)(\alpha,x) \Longrightarrow (\alpha\beta, x\beta.y) = (\delta\alpha, u\alpha.x) \Longrightarrow$$
$$\Longrightarrow u = (x\beta.y)R(x)^{-1}\alpha^{-1} \quad \text{and} \quad \delta = \alpha\beta\alpha^{-1}.$$

Equation (5) becomes

$$(6) \qquad (\alpha\beta\gamma, x\beta\gamma.(y\gamma.z)) = (\alpha\beta\gamma, (x\beta.y)R(x)^{-1}\gamma.(x\gamma.z))$$

which is true if and only if

$$(7) \qquad (x\beta\gamma).(y\gamma.z) = (x\beta.y)R(x)^{-1}\gamma.(x\gamma.z)$$

if and only if

$$(8) \qquad x\beta.(y.z\gamma^{-1}) = (x\beta.y)R(x)^{-1}.(x.z\gamma^{-1})$$

Let $\overline{z} = z\gamma^{-1}$, then equation (8) holds if and only if $y.\overline{z}L(x\beta) = yL(x\beta)R(x)^{-1}.\overline{z}L(x) \Longleftrightarrow \langle L(x\beta)R(x)^{-1}, L(x), L(x\beta)\rangle$ is an autotopism of G for all $x \in G$ and all $\beta \in A$. Replacing β with α, we obtain (i) of the Lemma.

Let $(\beta,y),(\gamma,z),(\delta,t) \in (H,\circ)$. Then the second requirement for conjugacy closure is

$$(9) \quad [(\beta,y)\circ(\gamma,z)]\circ(\delta,t) = [(\beta,y)\circ(\delta,t)\circ(\gamma,z)\circ(\delta,t)]\,L_\circ(\delta,t)^{-1}$$

Equation (9) holds if and only if

$$(10) \qquad (\beta\gamma\delta, (y\gamma.z)\delta.t) = (\beta\delta, y\delta.t)\circ(\gamma\delta, z\delta.t)L_\circ(\delta,t)^{-1}$$

Suppose $(\gamma\delta, z\delta.t)L_\circ(\delta,t)^{-1} = (\sigma,s)$ for some $s \in G$ and some $\sigma \in A$, then

$$(11) \qquad (\gamma\delta, z\delta.t) = (\delta,t)\circ(\sigma,s) = (\delta\sigma, t\sigma.s)$$

if and only if $\sigma = \delta^{-1}\gamma\delta$ and $s = (z\delta.t)L(t\delta^{-1}\gamma\delta)^{-1}$. So equation (10) becomes

$$(\beta\gamma\delta, (y\gamma.z)\delta.t) = (\beta\gamma\delta, (y\delta.t)\delta^{-1}\gamma\delta.(z\delta.t)L(t\delta^{-1}\gamma\delta)^{-1})$$

which is true if and only if

$$(12) \qquad (y\gamma\delta.z\delta).t = (y\gamma\delta.t\delta^{-1}\gamma\delta).(z\delta.t)L(t\delta^{-1}\gamma\delta)^{-1}$$

Let $\overline{y} = y\gamma\delta, \overline{z} = z\delta$, and $\alpha = \delta^{-1}\gamma\delta$, then

$$(\overline{y}.\overline{z}).t = (\overline{y}.t\alpha).(\overline{z}.t)L(t\alpha)^{-1} \Longleftrightarrow$$
$$(\overline{y}.\overline{z})R(t) = \overline{y}R(t\alpha).\overline{z}R(t)L(t\alpha)^{-1} \Longleftrightarrow$$
$$\langle R(t\alpha), R(t)L(t\alpha)^{-1}, R(t)\rangle$$

is an autotopism of G for all $t \in G$ and all $\alpha \in A$. Replacing t with x yields (ii).

Lemma 2.2. *Let A be a group of automorphisms of a conjugacy loop $(G,.)$. The holomorph $(H,\circ)$ of $(G,.)$ is conjugacy closed if and only if*
(i) $\langle R(x)^{-1}R(x\alpha), L(x)L(x\alpha)^{-1}, I\rangle$

and
(ii) $\langle L(x\alpha)L(x)^{-1}, I, L(x\alpha)L(x)^{-1}\rangle$
are autotopisms of G for all $x\in G$ and all $\alpha\in A$.

Proof. By Lemma 2.1, the holomorph of any loop is conjugacy if and only if the triples $\langle L(x\alpha)R(x)^{-1}, L(x), L(x\alpha)\rangle$ and $\langle R(x\alpha), R(x)L(x\alpha)^{-1}, R(x)\rangle$ are autotopisms of G for all $x\in G$ and all $\alpha\in A$. Since $(G,.)$ is conjugacy closed, we have that $\langle R(x), R(x)L(x)^{-1}, R(x)\rangle$ and $\langle L(x)R(x)^{-1}, L(x), L(x)\rangle$ are autotopisms of G for all $x\in G$. Let

(13) $$B=\langle L(x\alpha)R(x)^{-1}, L(x), L(x\alpha)\rangle,$$

(14) $$C=\langle R(x\alpha), R(x)L(x\alpha)^{-1}, R(x)\rangle,$$

(15) $$S=\langle L(x)R(x)^{-1}, L(x), L(x)\rangle,$$

(16) $$T=\langle R(x), R(x)L(x)^{-1}, R(x)\rangle.$$

Since the set of autotopisms forms a group,

$$S^{-1}=\langle R(x)L(x)^{-1}, L(x)^{-1}, L(x)^{-1}\rangle$$

and

$$T^{-1}=\langle R(x)^{-1}, L(x)R(x)^{-1}, R(x)^{-1}\rangle$$

are also autotopisms of G for all $x\in G$. So $(H,\circ)$ is conjugacy closed if and only if

(17) $$T^{-1}C=\langle R(x)^{-1}R(x\alpha); L(x)L(x\alpha)^{-1}, I\rangle$$

and

(18) $$BS^{-1}=\langle L(x\alpha)L(x)^{-1}, I, L(x\alpha)L(x)^{-1}\rangle$$

are autotopisms of G for all $x\in G$ and all $\alpha\in A$.

Theorem 2.1. *Let A be a group of automorphisms of a loop $(G,.)$ which satisfies the inverse property. Then the holomorph $(H,\circ)$ of $(G,.)$ is conjugacy closed if and only if*
(i) $(G,.)$ is conjugacy closed

and
(ii) *every automorphism of G is nuclear in the sense that $x^{-1}.x\alpha\in N$ for all $x\in G$ and all $\alpha\in A$, where N is the nucleus of G.*

Proof. Suppose $(H,\circ)$ is conjugacy closed, then by Lemmas 2.1 and 2.2, we have the following autotopisms of G:

$$B=\langle L(x\alpha)R(x)^{-1}, L(x), L(x\alpha)\rangle,\qquad C=\langle R(x\alpha), R(x)L(x\alpha)^{-1}, R(x)\rangle,$$

$$D=\langle R(x)^{-1}R(x\alpha), L(x)L(x\alpha)^{-1}, I\rangle,\qquad E=\langle L(x\alpha)L(x)^{-1}, I, L(x\alpha)L(x)^{-1}\rangle.$$

So,

(19) $$E^{-1}B=\langle L(x)R(x)^{-1}, L(x), L(x)\rangle$$

and

(20) $$CD^{-1}=\langle R(x), R(x)L(x)^{-1}, R(x)\rangle$$

are autotopisms of G for all $x\in G$, hence $(G,.)$ is conjugacy closed.

Since $(H,\circ)$ is conjugacy closed, the autotopism D implies that

(21) $$aR(x)^{-1}R(x\alpha).bL(x)L(x\alpha)^{-1}=(a.b)I$$

for all $a,b\in G$. In particular, $b=1$ in (21) gives

(22) $$\big[(a.x^{-1}).x\alpha\big].\big[(x\alpha)^{-1}.x\big]=a\implies aR(x)^{-1}R(x\alpha)R(x^{-1}\alpha.x)=a\implies$$

$$\implies R(x)^{-1}R(x\alpha)=R(x^{-1}\alpha.x)^{-1}.$$

For $a=1$ in (21), we get

$$L(x)L(x\alpha)^{-1}=L(x^{-1}.x\alpha)^{-1}$$

or

(23) $$L(x\alpha)L(x)^{-1}=L(x^{-1}.x\alpha).$$

So,

$$E=\langle L(x^{-1}.x\alpha), I, L(x^{-1}.x\alpha)\quad\implies\quad aL(x^{-1}.x\alpha).a).b=(x^{-1}.x\alpha).(a.b)$$

for all $a, b \in G$, hence $x^{-1}.x\alpha \in N$. Since the nuclei of an inverse property loop coincide (see [2] p.111), we have $x^{-1}.x\alpha \in N$ for all $x \in G$, hence α is nuclear.

If $(G,.)$ is conjugacy closed, Goodaire and Robinson [5] proved that the identities

$$g.(x.y) = (g.x)R(g)^{-1}.(g.y) \quad \text{and} \quad (x.y).f = (x.f).(y.f)L(f)^{-1}$$

hold for all $x, y, f, g \in G$.
If $x^{-1}.x\alpha \in N$, then

$$\left[(x^{-1}.x\alpha).a\right].b = (x^{-1}.x\alpha).(a.b)$$

for all $a, b \in G$, so the triple

$$\phi = \langle L(x^{-1}.x\alpha).x\alpha), I, L(x^{-1}.x\alpha)\rangle \tag{24}$$

in an autotopism of G for all $x \in G$.

Using (23), we have

$$\phi = \langle L(x\alpha)L(x)^{-1}, I, L(x\alpha)L(x)^{-1}\rangle$$

which is (ii) of Lemma 2.2.
Now,

$$(a, b).(x^{-1}.x\alpha) = a.\left[b.(x^{-1}.x\alpha)\right] \implies$$

$$a.b = \left[a.\left(b.(x^{-1}.x\alpha)\right)\right]\left[(x^{-1}.x\alpha)^{-1}\right] \implies$$

$$a.b = \left[a.(x^{-1}.x\alpha)^{-1}\right].\left[(x^{-1}.x\alpha).\left[\left[b.(x^{-1}.x\alpha)\right].(x^{-1}.x\alpha)^{-1}\right]\right]$$

since $(G,.)$ is conjugacy closed and has the inverse property. So,

$$a.b = \left(a.(x^{-1}.x\alpha)^{-1}\right).\left((x^{-1}.x\alpha).b\right) \implies \delta = \langle R(x^{-1}.x\alpha)^{-1}, L(x^{-1}.x\alpha), I\rangle$$

is an autotopism of G for all $x \in G$. Also,

$$\delta^{-1} = \langle R(x^{-1}.x\alpha), L(x^{-1}.x\alpha)^{-1}, I\rangle$$

is an autotopism of G for all $x \in G$. Using (22) and (23), we obtain

$$\delta^{-1} = \langle R(x)^{-1}R(x\alpha), L(x)L(x\alpha)^{-1}, I\rangle$$

which is (i) of Lemma 2.2. Hence $(H, \circ)$ is conjugacy closed.

Theorem 2.2. *If $(G,.)$ is an inverse property conjugacy closed loop, then each inner mapping $R(x,y)$ is a nuclear automorphism of $(G,.)$.*

Proof. Let $a = b^{-1}.bR(x,y)$, then $a.b = [(b.x).y].(x.y)^{-1} \implies$ $\implies (a.b).(x.y) = (b.x).y$. Since $R(x,y)$ is an automorphism of $(G,.)$ we have $A = \langle R(x,y), R(x,y), R(x,y)\rangle$ is an autotopism of $G(,.)$ for all $x, y \in G$. Since $(G,.)$ is a Moufang loop, $R(x,y)$ is also a pseudo-automorphism with companion $[x,y]$ (Bruck [2] Ch VII, Lemma 2.2), so $B = \langle R(x,y), R(x,y)R([x,y]), R(x,y)R([x,y])\rangle$ is an autotopism of $(G,.)$ for all $x, y \in G$. Hence

$A^{-1}B = \langle I, R([x,y]), R([x,y])\rangle$ is an autotopism of $(G,.)$ for all $x, y \in G$. Thus,

$$aI.bR([x,y]) = (a.b)R([x,y]) \implies a.(b.[x,y]) = (a.b).[x,y]$$

for all $a, b, x, y \in G$. So all loop commutators of G are in the right nucleus of G. Since the nuclei coincide, all loop commutators are in the nucleus N of G, so the commutator subgroup G' of G is a subset of N, and N is normal in G, so G/N is commutative.

Now, G/N is a homomorphic image of G, hence G/N is conjugacy closed and also commutative. Since a commutative conjugacy closed loop is a group, it follows that G/N is a group.

Let θ be the natural homomorphism of G onto G/N. Then N is the kernel of θ, so,

$$[(a.b).(x.y)]\theta = [(b.x).y]\theta \implies a\theta.b\theta.x\theta.y\theta = b\theta.x\theta.y\theta \implies$$

$a\theta$ is the identity of G/N, hence $a \in \ker\theta$, and $a = b^{-1}.bR(x,y) \in N$. Hence, each $R(x,y)$ is a nuclear automorphism of G.

Since an inverse property conjugacy closed loop is a Moufang loop, and for any Moufang loop, $R(x,y) = L(x^{-1}, y^{-1})$ for all x, y in the loop (Bruck [2], Ch. VII, Lemma 2.2) the inner mappings $L(x,y)$ are also include in the above result.

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ON SUPER ORDER PRINCIPAL MODULES

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1. Introduction. The sequential analogue of the notion of order principal module introduced in [4] will be called super order principal module. In the present paper we shall characterize super order principal modules and extend an abstract Egoroff type theorem [5, Theorem 74.3] to this more general framework.

Let M be a lattice ordered module over an Archimedean unital f–algebra A. Then M has also a natural Riesz space structure. We assume throughout this paper that M is Archimedean. M is said to be super order principal A–module, if for every $x \in M^+$ the Riesz subspace $Ax = \{ax : a \in A\}$ of M is super order dense in the band $\{x\}^{dd}$ generated by x.

Let L be an Archimedean Riesz space and Orth(L) the f–algebra of all orthomorphisms on L. Denote by $\mathcal{P}(L)$ the Boolean algebra of all band projections on L and by $\mathcal{M}(L)$ the Archimedean f–algebra generated by $\mathcal{P}(L)$. Evidently L is an Archimedean lattice ordered $\mathcal{M}(L)$– module. With this notation we can formulate the well known Freudenthal's spectral theorem [5, Theorem 40.3(unbounded case)] simple by saying that a Riesz space with the principal projection property is a super order principal $\mathcal{M}(L)$–module. Therefore, our characterization of super order principal modules describes (as a special case) all Riesz spaces in which the unbounded Freudenthal's spectral theorem holds.

For the terminology concerning Riesz spaces we refer the reader to [5], and for elementary f–algebra theory we refer to [1] and [7].

2. Preliminaries. Let L be an Archimedean Riesz space. If a sequence $(u_n) \subset L$ is increasing we shall write $u_n \uparrow$ while $u_n \uparrow u$ (with $u \in L$) means that also $u = \sup_n u_n$. The meanings of $u_n \downarrow$ and $u_n \downarrow u$ are analogous. Let D be a nonempty subset of L. Then D^1 denotes the set

$$D^1 = \{u \in L : \text{ there exists a sequence } (u_n) \subset D \text{ with } u_n \uparrow u\}.$$

Given the element $u \in L^+$, $\mathcal{B}_u$ will denote the Boolean algebra of all components of u, that is

$$\mathcal{B}_u = \{v \in L : \quad v \wedge (u - v) = 0\}.$$

The Riesz subspace of L generated by $\mathcal{B}_u$ is denoted by $\mathcal{M}_u$. An order bounded band preserving linear endomorphism of L is called an orthomorphism.