By a standard principle of calculus of variations, $\delta\tau_M(f) = 0$ for all φ if and only if $p - H|f|^2 = 0$. We claim that this implies $|f| =$ constant. If we accept for the moment this fact, then the proof is achieved. In fact, if $|f|$ is a constant, then M is an immersed ovaloid. Therefore, by Hadamard's theorem on ovaloids [1, p.121], f must be a diffeomorphism onto the n-sphere of E^{n+1} with centre O and radius $|f|$.

To prove the claim, denote by Δ and Δ_1 respectively the Laplacian and the first Beltrami differntial parameter on M. If $f = (f^1, f^2, \dots, f^n)$, then $\Delta(f^i)^2 = 2f^i\Delta f^i + 2\Delta_1(f^i)$. Exploiting the well-known identity $\Delta f = nh$ [1, p.198] it follows, by a standard computation, that $\Delta |f|^2 = 2n(1 - Hp)$. Now, if $p - H|f|^2 = 0$, then $Hp \leq 1$, and hence $\Delta|f|^2 \geq 0$. Therefore, by Bochner's lemma [2, p.195], $|f|$ must be constant.

REFERENCES

1. M. Spivak — *A Comprehensive Introduction to Differential Geometry*, vol.IV, Publish or Perish, Inc., Berkeley, 1979.
2. T.J. Willmore — *Total Curvature in Riemannian Geometry*, Ellis Horwood Limited, Chichester, 1982.

Received 16.VII.1990

Dipartamento di Matematica e Applicationi "R. Caccioppoli"
Università degli Studi di Napoli
Via Mezzocannone, 8
80134 Napoli — Italia

ANALELE ŞTIINŢIFICE ALE UNIVERSITĂŢII "AL. I. CUZA" IAŞI
Tomul XXXVII, s. I. a. Matematică, 1991, f. 3

HOMOLOGY OF $P(w_0, w_1, w_2)$

BY

LOUIS FUNAR

Let $w = (w_0, \dots, w_n)$ be a set of positive integers and denote by S the polynomial ring $\mathbf{C}[x_0, \dots, x_n]$ graded by $\deg x_i = w_i \quad i = \overline{0,n}$. Then the projective variety $\operatorname{Proj}(S) = P(w)$ is called the weighted projective space of specified weights. For general w, $P(w)$ is a singular space. Its singularities are quotient ones, whose germs are isomorphic to $(\mathbf{C}^n/H, 0)$ where $H \subset GL(n, C)$ is a small abelian group, and whose corresponding links are generalized lens spaces [2, 3, 4]. It is known that the rational cohomology groups of $P(w)$ and $\mathbf{C}P^n$ agree [6] as graded abelian groups.

Our main result is that $H_\cdot(P(w))$ is torsion free for $n = 2$. This seems to have some significance when we to folow the lines developed in[1] in studying the quasi not-homogeneous singularities. We shall give a cell decomposition for $P(w)$ in same manner as Dold [5] has done for lens spaces. Then an elementary combinatorial computation will answer our question.

We restrict our attention to the case $n = 2$ and remember the an alternative way to describe $P(w)$ is as a quotient of CP^2 under the action of $G = Z/W_0Z \times \dots \times Z/W_2Z$

$$(k_0, k_1, k_2)(z_0, z_1, z_2) = (\xi_{w_0}^{k_0}, \xi_{w_1}^{k_1}, \xi_{w_2}^{k_2}),$$

where ξ_j is a primitive root of unity of order j. Now consider the Hopf fibration

$$\begin{tikzcd} S^1 \arrow[r] & S^5 \arrow[d] \arrow[dr, "\pi"] & \\ & CP^2 \arrow[r] & P(W) \end{tikzcd}$$

and set π for the composition of the canonical projections. Our aim is to give a G - equivariant cell decomposition of $\mathbf{C}P^2$. We shall define a decomposition of S^5 satisfying

i) the decomposition is G-equivariant;
ii) any set of the decomposition is stable with respect to the action of S^1 coming from the Hopf fibration;

iii) it induces a cell decomposition of CP^2, hence any set of the decomposition is topologically $S^1 \times$ cell.

Now below P^j consists of sets of dimension $j+1$. Since the action of S^1 is fixed point free P^{-1} must be void. Otherwise

(1) $P^0 = \{e_j^0, j \in \{0,1,2\}\}$ where $e_j^0 = \{z_i = 0 \text{ for } i \neq j\}$

(2) $P^1 = \{e_r^1(j), j \in \{0,1,2\}, r \in \{0,1,\ldots A-1\}\}$ where $A = \ell\mathrm{cm}(w_0, w_1, w_2) i.e.$ A is the least common multiple of the weights, and $e_r^1(2) = \{\arg z_0 - \arg z_1 = 2\pi r/A, z_2 = 0\}$, the others being obtained by cyclic permutations

(3) $P^2 = \{b_{rpq}, c_r(j); r,p,q \in \{0,1,\ldots A-1\}, r+p+q = 0(\bmod A), j \in \{0,1,2\}\}$ where $b_{rpq} = \{\arg z_0 - \arg z_1 = 2\pi r/A, \arg z_1 - \arg z_2 = 2\pi p/A, \arg z_2 - \arg z_0 = 2\pi q/A\}$ $c_r(2) = \{\arg z_0 - \arg z_1 \in (2\pi r/A, 2\pi(r+1)/A), z_2 = 0\}$, and the others are obtained by cyclic permutations

(4) $P^3 = \{e_{rpq}^3(j), r,p,q \in \{0,1,\ldots,A-1\}, r+p+q+1 = 0(\bmod A), j \in \{0,1,2,\}\}$, where $e_{rpq}^3 = \{\arg z_0 - \arg z_1 = 2\pi r/A, \arg z_1 - \arg z_2 \in (2\pi p/A, 2\pi(p+1)/A), \arg z_2 - \arg z_0 \in (2\pi q/A, 2\pi(q+1)/A)\}$ and the others are obtained by cyclic permutations

(5) $P^4 = \{e_{rpq}^4, r,p,q \in \{0,1,\ldots,A-1\}, r+p+q(\bmod A) \in \{-1,-2\}\}$ where
$e_{rpq}^4 = \{\arg z_0 - \arg z_1 \in (2\pi r/A, 2\pi(r+1)/A),$
$\arg z_1 - \arg z_2 \in (2\pi p/A, 2\pi(p+1)/A),$
$\arg z_2 - \arg z_0 \in (2\pi q/A, 2\pi(q+1)/A)\}.$

Let now make some notations

$$A/w_i = A_i, i \in \{0,1,2\} \quad d_i = \ell\mathrm{cm}(A_{i+1}, A_{i+2})$$

(cyclic numerotated),

$$d = \ell\mathrm{cm}(a_0, A_1, A_2),$$

$$N = A^3/Bd, B = w_0 w_1 w_2,$$

$$P = A^2/w_0 w_1 d_2 + A^2/w_1 w_2 d_0 + A^2/w_2 w_0 d_1.$$

We consider the following actions of G on $\mathbf{Z}/A\mathbf{Z}$

(i) $(k,1,m)(r) = r + kA_0 - 1A_1,$

(ii) $(k,1,m)(p) = p + 1A_1 - mA_2,$

(iii) $(k,1,m)(q) = q + mA_2 - kA_0.$

The orbits of theses actions are denoted by $(r)_2$, $(p)_1$, $(q)_0$ eventually the indices omitted, if are understood.

Let consider the action of G on $(\mathbf{Z}/A\mathbf{Z})^3$ obtained by summing the preceding ones i. e.

(iv) $(k,1,m)(r,p,q) = (r + kA_0 - 1A_1, p + 1A_1 - mA_2, q + mA_2 - kA_0)$ whose orbits we denote by (r,p,q). If $S_m = \{(r,p,q); r+p+q = -m(\bmod A)\}$, then S_m are G-invarinat. If we denote by K^i the projection under π of P^i we obtain a cell decomposition of $P(w)$, namely

$$K^0 = \{e_i^0, i \in 0,1,2\}, \text{ where } e_i^0 = \pi(\tilde{e}_i^0);$$

$$K^1 = \{e_{(r)_i}^1(i), i \in \{0,1,2\}\}, \text{ where } e_{(r)_i}^1(i) = \pi(\tilde{e}_r(i));$$

$$K^2 = \{c_{(r)_0}(i), b_{(rpq)}, i \in \{0,1,2\}, (r,p,q) \in S_0\}, \text{ where}$$
$$c_{(r)_i}(i) = \pi(c_r(i)) \text{ and } b_{(rpq)} = \pi(b_{rpq});$$

$$K^3 = \{e_{(rpq)}^3(i), i \in \{0,1,2\}, (r,p,q) \in \}S_1\}, \text{ where } e_{(rpq)}^3(i) = \pi(e_{rpq}^3(i));$$

$$K^4 = \{e_{(rpq)}^4, (r,p,q)\ inS_1 \cup S_2\}, \text{ where } e_{(rpq)}^4 = \pi(e_{rpq}^4).$$

The map π being cellular we can compute how acts on the generators of chain groups

$$\partial e_i^0 = 0, \tag{6}$$

$$\partial e^1(r)_i(i) = e_{i-1}^0 - e_{i+1}^0, \tag{7}$$

$$\partial c_{(r)_i}(i) = e_{(r)_i}^1(i) - e_{(r+1)_i}^1(i), \tag{8}$$

$$\partial b_{(rpq)} = e_{(r)_2}^1(2) + e_{(p)_0}^1(0) + e_{(q)_1}^1(1) \tag{9}$$

with the mention that the actions $(i - iv)$ are compatible i.e. if

$$(rpq) = (\tilde{r}\tilde{p}\tilde{q}) \text{ then } (r)_2 = (\tilde{r})_2, (p)_0 = (\tilde{p})_0, (q)_1 = (\tilde{q})_1 \text{ so } (9)$$

has sense,

$$\partial e_{(rpq)}^3(2) = c_{(p)_0}(0) - c_{(q)_1}(1) + b_{(rp+1q)} - b_{(rpq+1)} \tag{10}$$

and the others are obtatined by cyclic permutations,

$$\partial e_{(rpq)}^4 = e_{(rpq)}^3(1) + e_{(rpq)}^3(2) + e_{(rpq)}^3(0), \quad \text{if } (rpq) \in S_1 \tag{11}$$

$$\partial e_{(rpq)}^4 = e_{(r+1\,pq)}^3(2) + e_{(rp+1\,q)}^3(0) + e_{(rpq+1)}^3(1) \quad \text{if } (r,p,q) \in S_2. \tag{12}$$

Now let C_j be the group of Z - chains of dimension j determined by the simplicial complex $K = \bigoplus_0^3 K^i$. The graded differential complex $C = \bigoplus_0^3 C_j$ is isomorphic with the following one

$$0 \to Z^{2N} \xrightarrow{D} Z^{3N} \xrightarrow{A} Z^{N+P} \xrightarrow{B} Z^P \xrightarrow{C} Z^3 \to 0$$

where A, B, C, D are suitable matrices, which acts by left composition and correspond to ∂ in the standard basic of C. Now by general arguments $H_0(P(w), \mathbf{Z}) = \mathbf{Z}), \pi_1(P(w)) = 0$ so $H_1(P(w), \mathbf{Z}) = 0$, and $H_4(P(w), \mathbf{Z}$. We need only look at H_2 and H_3. For *map* $\times$ *q* matrix with integer coefficients we denote by $J(m) = \min\{a; a \in \mathbf{Z}^+$ and there exists a $k \times k$ submatrix W, where $k =$ rank m such that $\det W = a\}$.

Lemma. *We have the followings*

$$\operatorname{rank} B = P - 2,$$

$$\dim_{\mathbf{Q}} \ker (B + f) \otimes_{\mathbf{Z}} \mathbf{Q} = N + 1,$$

$$\operatorname{rank} A = N + 1,$$

$$J(A) = 1,$$

$$\operatorname{rank} D = 2N - 1,$$

$$j(D) = 1,$$

where $f\colon Z^{N+P} \to Z$ *is induced by*

$$f\left(\sum_{(rpq)} n_{(rpq)} b_{(rpq)} + \sum_{i,(r)_i} v_{(r)_i, i} c_{(r)_i}(i)\right) = \sum_{i,(r)_i} v_{(r)_i}.$$

Observe that if lemma is proved then $\operatorname{im} A \otimes \mathbf{ZQ} \subset \ker (B+f) \otimes \mathbf{ZQ}$ and because have the same dimension, theses spaces coincide. Also if A_0 is a $N+1 \times N+1$ minor of A such that $\det A_0 = J(A) = 1$, then A_0 is z-invertible and means of a_0^{-1} we obtain an isomorphism between $\ker (B+f)$ and $\operatorname{im} A$; but f factors to an isomorphism f': $\ker B / \operatorname{im} A \to \mathbf{Z}$, because $\operatorname{rank} (B+f) = \operatorname{rank} B + 1$. Thus

$$H_2(P(w), \mathbf{Z}) = \mathbf{Z}$$

and similar calculus show that

$$H_3(P(w), \mathbf{Z}) = 0.$$

Hence it follows

Theorem. *The weighted projective spaces of dimension 2 have the same integer homology to those of* $\mathbf{C}P^2 = P(1,1,1)$.

Now the proof of the lemma is a consequence of the combinatorial description of the matrices A, B, D. So we have

$$A = \begin{bmatrix} T_2 & -T_2 & 0 \\ T_1 & 0 & T_1 \\ 0 & T_3 & -T_3 \\ I_N - Q_1 & I_N - Q_2 & I_N - Q_3 \end{bmatrix}$$

where T_1 is the $\sharp(\mathbf{Z}/A\mathbf{Z})/G(i) \times N$ matrix whose entries are

$$t_{(r)_2,(uvw)} = \begin{cases} 1, & \text{if there exists } p,q,s \text{ with } (p)_2 = (r)_2 \text{ and } (pqs) = (uvw) \\ 0, & \text{elsewhere,} \end{cases}$$

and T_2, T_3 are deduced by analogy from this, I_N is the $N \times N$ matrix and Q_1, Q_2, Q_3 are circulant matrices corresponding to the permutations of $1, 2, \ldots, N$ induced by

$$(r+1pq) \xrightarrow{q_1} (rp+1q), (rp+1q) \xrightarrow{q_2} (rpq+1), (rpq+1) \xrightarrow{q_3} (r+1pq),$$

which can be written as $(Q_1)_{uv} = \partial_{u,q_1}(v)$. From arithmetic considerations order $(q_i) = N$ and by bordering the minor $I_N - Q_2$ by one stratum is obtained a $N+1 \times N+1$ matrix with unit determinant so $\operatorname{rank} A \geq N+1$. Looking to B is easy to see that

$$B = \begin{bmatrix} T_1 & K_1 & 0 & 0 \\ T_2 & 0 & K_2 & 0 \\ T_3 & 0 & 0 & K_3 \end{bmatrix}$$

where K_1 is a $\sharp(\mathbf{Z}/A\mathbf{Z})/G(i)$ square matrix

$$K_1 = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & \ldots & 0 & 0 \\ . & . & . & . & . & \ldots & . & . \\ -1 & 0 & 0 & 0 & 0 & \ldots & 0 & 1 \end{bmatrix}$$

and similarly for K_2, K_3. Then $\operatorname{rank} B = P - 2$ and because $H_2(P(w), \mathbf{Q} = \mathbf{Q}$ we have $\operatorname{im} A \otimes_{\mathbf{Z}} \mathbf{Q} \subset \ker (B+f) \otimes_{\mathbf{Z}} \mathbf{Q}$ so $\operatorname{rank} A = N+1$ and $J(A) = 1$.

Also D as the form

$$D = \begin{bmatrix} I_N & -R_1 \\ I_N & -R_2 \\ I_N & -R_3 \end{bmatrix}$$

where R_1, R_2, R_3 are the square matrices corresponding to the following permutations of $1, 2, \ldots, N$

$$S_2/G \ni (rpq) \xrightarrow{t_1} (r-1pq) \in S_1/G$$

and the analoug ones. We have order $(t_i) = N$ and then the $2N-1 \times 2N-1$ principal minor of D is of determinant 1. Because $H_3(P(w), \mathbf{Q}) = 0$ so $\operatorname{im} D \otimes_{\mathbf{Z}} \mathbf{Q} \subset \ker A \otimes_{\mathbf{Z}} \mathbf{Q}$ then $J(D) = 1$ which finishes the proof of the lemma.

Remark. Using other methods the author has proved that in fact theorem holds in any dimension. This will be explained in a further paper.

Acknowledgements. I am indebted to A. Dimca for proposing this problem and to S. Papadima and P. Ionescu for valuable suggestions. Many thanks to my mother Maria for typing this work.

REFERENCES

1. Brieskorn, E — *Die Monodromie der Isolierten Singularitaten von Hyperflachen.* Manuscr. Math. 2, 103–61 (1970).
2. Dimca, A. — *On analytic abelian coverings.* Math. Ann.279, 501–515 (1988).
3. Dimca, A. — *On the Milnor fibrations of weighted homogeneous polynomials*, Preprint INCREST, 1989.
4. Dimca, A. — *On the homology and cohomology of complete intersections with isolated singularities.* Compositio Math. 58, 321–339 (1986).
5. Dold, A. — *Lectures on Algebraic Topology.* Springer–Verlag, Berlin–Heidelberg–New York, 1972.
6. Dolgachev, I. — *Weighted Projective Varities in Group Actions and Vector Fields.* Lectures Notes. Math. no. 956, Springer–Verlag, Berlin–Heidelberg–New York, 1982.

Received 3.IV.1990
Revised 18.VI.1990

University of Bucharest
Department of Mathematics
str. Academiei 14,
70109 Bucharest,
România

ANALELE ŞTIINŢIFICE ALE UNIVERSITĂŢII "AL. I. CUZA" IAŞI
Tomul XXXVII, s. I. a. Matematică, 1991, f. 3

CONE MAXIMAL POINTS IN TOPOLOGICAL LINEAR SPACES

BY

MIHAI TURINICI

§1. **Preliminaries.** Let E be a Hausdorff (real) topological linear space. By a *cone* in E we shall mean any part C of E with

(1.1) $$C+C \subseteq C \quad ; \quad \lambda C \subseteq C \quad , \quad 0<\lambda \in R \quad ; \quad 0 \in C.$$

Given such an object, denote

$$\mathrm{lin}(C) = C \wedge (-C), \quad \mathrm{pt}(C) = C \wedge (\mathrm{lin}(C))^c.$$

(Here, for any subset X of E, X^c stands for the absolute complement of X). Of course, $\mathrm{lin}(C)$ is the largest linear subspace included in C. And, $\mathrm{pt}(C)$ is a cone without origin; that is, (1.1) takes place — with $\mathrm{pt}(C)$ in place of C — but without its last part.

For a nonempty subset Y of E, denote by $\max(Y,C)$ the (eventual empty) subset of all $z \in Y$ with the maximal $(\bmod C)$ property

$$w \in Y, z \leq w(\bmod C) \Longrightarrow z \leq w(\bmod \mathrm{lin}(C)),$$

where, by $\leq (\bmod C)$ we understand the quasi–ordering over E induced by C, in the usual way

$$x \leq y(\bmod C) \quad \text{if and only if } y-x \in C.$$

We shall be interested in the sequel to determine (structural) conditions upon C so that the following property — referred to as C is a *comp–max cone* — be fulfilled

(1.2) $\max(H,C)$ is nonempty, for each (nonempty) compact part H of E.

(Here, as usually, the term "compact" is taken as in Kelley [9,ch.5,§1]: that is, each net in H admits an accumulation point in H). To give a practical motivation of this, we note (cf.Penot [12]) that an element of $\max(H,C)$ may be deemed as a *Pareto efficient point* of a certain multi–criterion optimization problem. But, our interest has also a theoretical motivation; this emerges from the fact that (1.2) may be interpreted as a (linear) topological version of the Zorn Maximality Principle. In this perspective, as a basic answer to the problem we dealt with, we must consider the one obtained in 1954 by Ward [18].