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ON THE RATE OF CONVERGENCE IN THE GLIVENKO - CANTELLI THEOREM

BY

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1. Introduction. Let X be a random variable valued in a non empty measurable subset E of $\mathbb{R}^p$, $p \in \mathbb{N}^*$. We denote by F its distribution function and f its density function. Let $(X_n)_{\mathbb{N}}$ be a sequence of realizations of X.

We estimate F by the empirical distribution of the sample $X_1, X_2, \ldots, X_n$:

$$F_n(x) = n^{-1}\sum_{i=1}^{n} 1_{[X_i \le x]}, \quad \forall x \in E$$

where $\mathbb{R}^p$ is arranged in regular order by the relation:

$$(U_j)_{j=1,p} \le (V_j)_{j=1,p} \text{ iff for every } j = 1, \ldots, p, \quad U_j \le V_j.$$

Several authors have studied the estimator F_n under mixing assumptions but in the context of univariate variables (see e.g. Berkes and Philipp [2], Deo [8], Philipp [15]).

Most recently, Sarda and Vieu [20], using analytic techniques and applying Collomb's inequality [6], derived rate of convergence of F_n on a compact set of $\mathbb{R}^p$ when the sequence (X_n) satisfies the φ - mixing condition (see e.g. Billingsley [3]).

However, this mixing condition is rather restrictive notably in the nonparametric analysis of time series; for instance a stationary Gaussian process is φ - mixing if and only if it is m -dependent i.e.

$$\exists m \in \mathbb{N}, \quad \forall n > m, \quad \varphi_n = 0$$

(see Ibragimov and Linnik, theorem 17.3.2. [12]).

It is therefore reasonable to consider less restrictive mixing conditions; for a review of the several measures of dependence and the corresponding mixing conditions, we mention the articles of Takahata [21], Bradley [5] and the thesis of Doukhan [9] which give more complete discussions.

In what follows we focus our attention on the absolute regularity and the α - mixing condition which are less restrictive than the above condition.

These two dependence structures are more suitable in the analysis of time series and prediction problems. In connection with the present paper it seems useful to mention some available works. Using the α - mixing concept Robinson [17], Truong and Stome [22], [23], Roussas [19] and Rahmania [16] have shown some consistency results in nonparametric regression estimation. Berbee [1], Ibragimov and Solev [11], Mokkadem [13] and Pham [14] have obtained interesting properties of same absolutely regular processes.

The main aim of this paper is to establish rates of convergence of F_n on a compact set of $\mathbb{R}^p$ when the obsevations are α - mixing or absolutely regular.

2. Results. For $a \leq b$, let $\mathcal{M}_a^b$ denote the σ-algebra of events generated by $X_a, \ldots, X_b$.

The following set of assumption will be needed.

(H1) (a) $(x_n)_{\mathbb{N}}$ is absolutely regular (or weak Bernoulli), see Volkonskii and Rozanov [24], if :

$$\beta(n) = \sup_k E\{\sup_{A \in \mathcal{M}_{n+k}^{\infty}} |P(A/\mathcal{M}_{-\infty}^k) - P(A)|\} \downarrow 0 \text{ as } n \to \infty. \tag{2.1}$$

(b) $(x_n)_{\mathbb{N}}$ is α - mixing, see Rosenblatt [18], if:

$$\alpha(n) = \sup_k \sup_{\substack{A \in \mathcal{M}_1^k \\ B \in \mathcal{M}_{n+k}^{\infty}}} |P(A \cap B) - P(B)| \downarrow 0 \text{ as } n \to \infty \tag{2.2}$$

Since $\alpha(n) \leq \beta(n)$ the first concept is stronger.

(H2) In the absolute regularity case, we denote by $(m_n)_{\mathbb{N}}$ a sequence of positive integers such that $m_n \leq n$, $\lim_{n\to\infty} m_n = \infty$ and verifying with $(\beta(n))_{\mathbb{N}}$ the condition

$$\sum_{n \geq 1} n^{1+\lambda p} \beta(m_n) < \infty \tag{2.3}$$

for some $\lambda \geq \frac{1}{2}$.

The condition (2.3) is fulfilled, for example in the following cases:

i) The process $(X_n)_{\mathbb{N}}$ is m-dependent i.e. $\exists m \in \mathbb{N}$, $\beta(n) = 0$ for $n > m$.

ii) The process $(X_n)_{\mathbb{N}}$ is geometrically β - mixing (i.e. $\exists a, \rho$, $0 < a < \infty$, $0 < \rho < 1$, $\beta(n) \leq a\,\rho^n$, $\forall n \in \mathbb{N}$) and $m_n = c \log n$, $c > -\dfrac{2+\lambda p}{\log \rho}$.

iii) $m_n = n^a$, $0 < a < 1$, and $\beta(n) \leq v\,n^{-\delta}$, $0 < v < \infty$, $0 < \delta < \infty$ with $a\,\delta > 2 + \lambda\,p$.

(H3) In the α - mixing case we require that there exists an increasing sequence of positive integers $(k_n)_{\mathbb{N}}$ satisfying with $\alpha(n)_{\mathbb{N}}$ the condition:

$$\exists A < \infty, \quad \forall n \in \mathbb{N}, \quad n \cdot [\alpha(k_n)]^{2k_n/3n} \cdot k_n^{-1} \leq A \text{ and } 1 \leq k_n \leq n. \tag{2.4}$$

Rahmania [16] gives some examples of sequences $(k_n)_{\mathbb{N}}$ which satisfies (2.4).

(H4) We assume that the density function f is such that:

$$\exists \gamma < \infty, \quad \forall x \in E, \quad f(x) \leq \gamma \tag{2.5}$$

and let C be a compact subset of $\mathbb{R}^p$.

Theorem 1. *Under the condition (H1b), (H3) and (H4) and if the sequence* $\theta_n = (n^{-1} k_n \log n)^{1/2}$ *satisfies* $\lim_{n\to\infty} \theta_n = 0$, *then we have:*

$$\theta_n^{-1} \sup_{x \in C} |F_n(x) - F(x)| = 0(1), \quad \textit{with probability one} \tag{2.6}$$

Theorem 2. *Under the conditions (H1a), (H2) and (H4) we have for* $0 < a < \frac{1}{2}$

$$n^{a-1/2} \sup_{x \in C} |F_n(x) - F(x)| = 0(1), \quad \textit{with probability one} \tag{2.7}$$

3. Proofs. The proofs involve mainly an extension of the Bernstein inequality to α - mixing sequences (see e.g. Bosq and Lecoutre [4]) and to absolute regular sequences (see Yoschira [25]), and analytic techniques used by Sarda and Vieu [20] and Collomb and Härdle [7]; we need the following lemmas.

Lemma 1. *Let* $(\xi_i)_{i \in \mathbb{N}}$ *be a sequence of real random variables satisfying* $(\forall i \in \mathbb{N})$ $E(\xi_i) = 0$; $|\xi_i| \leq d$; $E|\xi_i|^2 < D$, *where* d *and* D *are constants. We assume* $(\xi_i)_{i \in \mathbb{N}}$ α *- mixing; then:*

$$(\forall \varepsilon > 0), \ (\forall n \geq 4), \ (\forall k \in \{1, \ldots, \left[\tfrac{n}{2} - 1\right]\}), \ (\forall \eta \in]0, (4kde)^{-1}]),$$

$$P[|\sum_{i=1}^{n} \xi_i| > \varepsilon] \leq 2 \exp\left[-\eta\varepsilon + 6\,\eta^2\, e(D + 8\,d^2 \sum_{\ell=1}^{k} \alpha_\ell)n + 2\sqrt{e}(\alpha_k)^{2k/3n}\frac{n}{k}\right] \tag{3.1}$$

($[\frac{n}{2} - 1]$ denotes the greatest integer less than $\frac{n}{2} - 1$). The complete proof is given by Bosq and Lecoutre [4].

Lemma 2. *Let* $(\xi_i)_{\mathbb{N}}$ *be a strictly stationary absolutely regular sequence of real random variables such that* $|\xi_i| \leq M_0$ *and* $E\,\xi_i = 0$; *then for* n *sufficiently large, the following inequalities hold:*

$$P(n^{-1/2}|\sum_{i=1}^{n} \xi_i| \geq \varepsilon) \leq 2m \exp\{-\frac{(m^{-1/2}\varepsilon)}{2\sigma_0^2}(1 - \frac{M_0 \epsilon}{\sigma_0^2 (mn)^{1/2}})\} + 4n\,\beta(m), \tag{3.2}$$

for $0 < \varepsilon < (\sigma_0^2/M_0) n^{1/2}$

(3.3)
$$P(n^{-1/2}|\sum_{i=1}^{n}\xi_i| \geq \varepsilon) \leq 2m\exp\{-\frac{\varepsilon n^{1/2}}{2M_0 m^{1/2}}\} + 4n\beta(m), \quad \text{for } \epsilon \geq (\sigma_0^2/M_0)n^{1/2}$$

where $\sigma_0^2 = \mathrm{Var}\,\xi_1 > 0$ *and* m *denotes any positive integer such that* $m \leq n$.

For the proof, see Yoshihara [25].

Proof of theorem 1. Firstly, the proof consists in establishing, for any sequence $(\varepsilon_n)_{\boldsymbol{N}}$ such that: $0 \leq \varepsilon_n \leq \beta < \infty$ and under the assumption $(H1b)$ the following inequality

(3.4)
$$\sup_{x\in E} P[|F_n(x) - F(x)| > \varepsilon_n] \leq c_1 \exp(-c_2 n\varepsilon_n^2 k_n^{-1}), \quad \forall n \geq n_0,$$

where c_1, c_2 are constants.

Define $\Delta_i = n^{-1}(1_{|X_i \leq x|} - F(x))$ and note that:

$F_n(x) - F(x) = \sum_{i=1}^{n} \Delta_i$.

For any $i \in \boldsymbol{N}$, we have:

$E\,\Delta_i = 0$; $|\Delta_i| \leq 2n^{-1}$; $E\Delta_i^2 = n^{-2}\cdot F(x)(1 - F(x)) \leq n^{-2}$.

Then, applying Lemma 1 with the choice: $\eta = n\epsilon_n(8keB)^{-1}$, we get:

(3.5)
$$P[|F_n(x) - F(x)| > \varepsilon_n] \leq c_k \exp\{-nk^{-1}t(\varepsilon_n, k)\}$$

where: $c_k = 2\exp(2\sqrt{e}\alpha(k)^{2k/3n}nk^{-1})$ and

$t(\varepsilon_n, k) = \varepsilon_n^2(8eB)^{-1}[1 - 3(4B)^{-1}(k^{-1} + 32k^{-1}\widetilde{\alpha}_k)]$,

where $\widetilde{\alpha}(k) = \sum_{\ell=1}^{k}\alpha_\ell$. There are two possibilities for the sequence $(k_n)_{\boldsymbol{N}}$.

First case: "$k_n \underset{n\to\infty}{\longrightarrow} \infty$"; we get by (2.1).

(3.6)
$$\exists n_0 \in \boldsymbol{N}, \quad \forall n \geq n_0, \quad k_n^{-1} \leq B/3 \text{ and } k_n^{-1}\widetilde{\alpha}_{k_n} \leq B/96$$

So that:

(3.7)
$$\exists n_0 \in \boldsymbol{N}, \quad \forall n \geq n_0, \quad t(\varepsilon_n, k_n) \geq (16eB)^{-1}\varepsilon_n^2$$

Applying (3.1) to $k = k_n$ and using (2.4), (3.6) and (3.7) we obtain the result of theorem (1) with

$C_1 = 2\exp 2\sqrt{e}A$

and

$C_2 = (16eB)^{-1}$

Second case: "$\exists n_1 \in \boldsymbol{N}, \quad \exists k_0 \in \boldsymbol{N}, \quad \forall n \geq n_1, \quad k_n = k_0$". (3.2) and (3.3) remain valid changing k_n in k_1, where k_1 is an integer independent of n and greater than k_0. Applying (3.1) to $k = k_1$, we obtain the inequality with:

$$c_1 = 2\exp 2\sqrt{e}A \text{ and } c_2 = (16eB^{-1})k_0k_1^{-1}.$$

Now, the compactness of C entails that we can cover it, for any $\ell > 0$, by a finit number d_ℓ of balls of radius $\ell/2$ (ℓ and d may depend on n),

(3.8)
$$C \subset \bigcup_{r=1}^{d_\ell} Br$$

where $Br = \{(y_j)_{j=1,\ldots,p} \in \boldsymbol{R}^p, \max_{j=1,\ldots,p}|x_r^j - y_r^j| \leq \ell/2\}$ $x_r = (x_r^j)_{j=1,p}$ being some point of C.

Denote, for any $r = 1, \ldots, d_\ell$

$$a_r = (a_r^j)_{j=1,p} \quad \text{with } a_r^j = x_r^j - \ell/2$$

and

$$b_r = (b_r^j)_{j=1,p} \quad \text{with } b_r^j = x_r^j + \ell/2$$

it is clear that

$$a_r \in B_r, \quad b_r \in B_r \qquad \text{and } \forall x \in B_r, \quad a_r \leq x \leq b_r.$$

This entails for $r = 1, \ldots, d_\ell$:

$$F(a_r) \leq \sup_{x\in B_r} F(x) \leq F(b_r)$$

and

(3.9)
$$F_n(a_r) \leq \sup_{x\in B_r} F_n(x) \leq F_n(b_r)$$

All the norms on $\boldsymbol{R}^p$ being equivalent, we obtain by the assumption (H4):

$$|F(u) - F(v)| \leq G\max_{j=1,p}|u^j - v^j|, \text{ for any } u \text{ and } v \text{ in } \boldsymbol{R}^p$$

and for some finite constant G.

Applying the results (3.8) and (3.9), we get:

(3.10)
$$\sup_{x\in C}|F_n(x) - F(x)| \leq \max_{r=1,\ldots,d_\ell}\max_{t_r=a_r,b_r}|F_n(t_r) - F(t_r)| + G\ell$$

Let ε_0 be a fixed positive number and choose

(3.11)
$$\ell = \gamma\theta_n \text{ where } \gamma < \varepsilon_0/2G$$

The inequality (3.10) gives:

$$\theta_n^{-1}\sup_{x\in C}|F_n(x) - F(x)| \leq \theta_n^{-1}\max_{r=1,\ldots,d_\ell}\max_{t_r=a_r,b_r}|F_n(t_r) - F(t_r)| + \varepsilon_0/2$$

To get the result (2.6), it suffices to note that

$$P[\theta_n^{-1} \max_{r=1,\dots,d_\ell} \max_{t_r=a_r,b_r} |F_n(t_r) - F(t_r)] > \varepsilon_0/2| \leq$$
$$\leq \sum_{r=1}^{d_\ell} \sum_{t_r=a_r,b_r} P[\theta_n^{-1}|F_n(t_r) - F(t_r)| > \varepsilon_0/2]$$

using (3.4) with: $\varepsilon_n = \theta_n \varepsilon_0$, we have

$$\sum_{r=1}^{d_\ell} \sum_{t_r=a_r,b_r} P[\theta_n^{-1}|F_n(t_r) - F(t_r)| > \varepsilon_0/2] \leq 2c_1 d_\ell n^{-c_2\varepsilon_0^2/4}.$$

By (3.11) d_ℓ may be such that: $d_\ell = h\theta_n^{-p}$ for some $0 < h < \infty$. To obtain the result of theorem (1), it suffices now to choose ε_0 such that: $c_2\varepsilon_0^2/4 > 1 + p/2$.

We remark at once that if we choose:

i) $k_n = O(\log n)$, then $\theta_n = O\left(\frac{\log n}{\sqrt{n}}\right)$

ii) $k_n = 1 + k$, $k \in N$ (i.e. $(X_i)_{i\in N}$ is k-dependent: $\alpha_j = 0 \quad \forall j \geq k+1$) then $\theta_n = O\left(\sqrt{(\log n)/n}\right)$

iii) $k_n = O(n^b)$, $0 < b < 1$, then $\theta_n = O\left[\left(\frac{\log n}{n^{1-b}}\right)\right]^{1/2}$

Proof of theorem 2. Let us show that under the condition (H1a) and for any $\varepsilon_0 \geq 1$, we have for n sufficiently large:

$$\sup_{x\in C} P(n^{a-1/2}|F_n(x) - F(x)| \geq \varepsilon_0) \leq 2m\exp[-\frac{\varepsilon_0}{n} n^{1-a}] + 4n\,\beta(m) \tag{3.12}$$

where m denotes any positive integer satisfying $m \leq n$ and $a \in]\frac{1}{2}, 1[$.

Let (ε_n) be a sequence such that:

$$\varepsilon_n = \varepsilon_0/n^a, \text{ with } \varepsilon_0 \geq 1$$

with this choice, we have $\varepsilon_n \geq \left(\frac{\sigma_0^2}{M_0}\right) \cdot n^{1/2}$ where

$$\sigma_0^2 = E\Delta_i^2 = n^{-2}F(x)(1 - F(x))$$

and

$$M_0 = 2n^{-1} \geq |\Delta_i|$$

Using Lemma 2, we get (3.12).

As pointed out in the proof of theorem 1, we have:

(3.13)
$$n^{a-1/2} \sup_{x\in C} |F_n(x) - F(x)| \leq n^{a-1/2} \max_{r=1,\dots,d_\ell} \max_{t_r=a_r,b_r} |F_n(t_r) - F(t_r)| + n^{a-1/2}G\cdot\ell$$

Let us choose

$$\ell = \gamma n^{1/2-a}, \text{ with } \gamma < \varepsilon_0/G \tag{3.14}$$

This gives using (3.12) and (3.13):

$$n^{a-1/2} \sup_{x\in C} |F_n(x) - F(x)| \leq n^{a-1/2} \max_{r=1,\dots,d_\ell} \max_{t_r=a_r,b_r} |F_n(t_r) - F(t_r)| + \varepsilon_0 \tag{3.15}$$

Applying (3.12) and (3.15), we get:

$$P(n^{a-1/2} \sup_{x\in C} |F_n(x) - F(x)| \geq \varepsilon_0) \leq \sum_{r=1}^{d_\ell} \sum_{t_r=a_r,b_r} P(n^{a-1/2}|F_n(t_r) - F(t_r)| > \varepsilon_0)$$
$$\leq 4d_\ell m \cdot \exp[\frac{-\varepsilon_0}{4} \cdot n^{1-a}] + 8nd_\ell\beta(m)$$

It suffices now to remark that by (3.14) d_ℓ is such that: $d_\ell = \rho n^{(a-1/2)p}$, for some $0 < \rho < \infty$.

The assumptions (2.3) leads to the result (2.7).

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NOTE SUR DES POSSIBILITES D'APPROXIMATION DE L'ENTROPIE D'UNE DISTRIBUTION CONTINUE

PAR

ION PURCARU

Soit X une variable aléatoire continue à la distribution probalistique définie par $f(x) \geq 0, \quad x \in D \subseteq R, \quad \int_D f(x)dx = 1$. Supposons que l'entropie Shannon de X ou de $f(x)$ définie par

$$H(X) = -\int_D f(x) \ln f(x) dx = H(f) \tag{1}$$

est connue (voir [1] , [2] , [4]). On sait que, généralment, même pour les distributions probabilistiques classiques, on ne peut pas donner des expressions bien déterminées de l'entropie (1) et par conséquent, d'un cas à l'autre, il faut l'approximer. Ça c'est le but de cette note. On supposera connues les fonctions gamma et beta d'Euler (voir [3] et [5]) utilisées parfois dans ce qui suit.

Proposition 1. *Si*

$$f(x) = \frac{na^m}{2\Gamma(m)} |x|^{mn-1} e^{-a|x|^n}, x \in R, \tag{2}$$

où $a > 0$, $m > 0$, $n > 0$ et $\Gamma(m)$ est la fonction gamma d'Euler, alors

$$H(X) = \frac{m(1-mn)}{n} \sum_{k=1}^{\infty} \frac{1}{k(k+m)} + \frac{(1+m\gamma)(mn-1)}{mn} + m + \ln \frac{2}{n} \Gamma(m) a^{\frac{-1}{n}} \tag{3}$$

où γ est la constante d'Euler ($\gamma \approx 0,5772$).

Preuve. Si on introduit (2) en (1), alors par calcul direct on trouve sans difficultés

$$H(X) = \ln \frac{2\Gamma(n)}{na^m} - \frac{n(mn-1)a^m}{\Gamma(m)} \int_0^{\infty} x^{mn-1} e^{-ax^n} \ln x dx + m. \tag{4}$$