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ON SUPER ORDER PRINCIPAL MODULES

BY

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1. Introduction. The sequential analogue of the notion of order principal module introduced in [4] will be called super order principal module. In the present paper we shall characterize super order principal modules and extend an abstract Egoroff type theorem [5, Theorem 74.3] to this more general framework.

Let M be a lattice ordered module over an Archimedean unital f–algebra A. Then M has also a natural Riesz space structure. We assume throughout this paper that M is Archimedean. M is said to be super order principal A–module, if for every $x \in M^+$ the Riesz subspace $Ax = \{ax : a \in A\}$ of M is super order dense in the band $\{x\}^{dd}$ generated by x.

Let L be an Archimedean Riesz space and $\mathrm{Orth}(L)$ the f–algebra of all orthomorphisms on L. Denote by $\mathcal{P}(L)$ the Boolean algebra of all band projections on L and by $\mathcal{M}(L)$ the Archimedean f–algebra generated by $\mathcal{P}(L)$. Evidently L is an Archimedean lattice ordered $\mathcal{M}(L)$– module. With this notation we can formulate the well known Freudenthal's spectral theorem [5, Theorem 40.3(unbounded case)] simple by saying that a Riesz space with the principal projection property is a super order principal $\mathcal{M}(L)$–module. Therefore, our characterization of super order principal modules describes (as a special case) all Riesz spaces in which the unbounded Freudenthal's spectral theorem holds.

For the terminology concerning Riesz spaces we refer the reader to [5], and for elementary f–algebra theory we refer to [1] and [7].

2. Preliminaries. Let L be an Archimedean Riesz space. If a sequence $(u_n) \subset L$ is increasing we shall write $u_n \uparrow$ while $u_n \uparrow u$ (with $u \in L$) means that also $u = \sup_n u_n$. The meanings of $u_n \downarrow$ and $u_n \downarrow u$ are analogous. Let D be a nonempty subset of L. Then D^1 denotes the set

$$D^1 = \{u \in L : \text{ there exists a sequence } (u_n) \subset D \text{ with } u_n \uparrow u\}.$$

Given the element $u \in L^+$, $\mathcal{B}_u$ will denote the Boolean algebra of all components of u, that is

$$\mathcal{B}_u = \{v \in L : \quad v \wedge (u - v) = 0\}.$$

The Riesz subspace of L generated by $\mathcal{B}_u$ is denoted by $\mathcal{M}_u$. An order bounded band preserving linear endomorphism of L is called an orthomorphism.

A Riesz algebra (lattice ordered algebra) A is said to be an f-algebra whenever

$$a, b, c \in A^+, a \wedge b = 0 \text{ implies } ac \wedge b = ca \wedge b = 0.$$

Any Archimedean f-algebra is commutative. The set $\mathrm{Orth}(L)$ of all orthomorphisms on L is (under pointwise defined operations) an Archimedean f-algebra with unit $I = \mathrm{id}_L$. The same holds for the order ideal $Z(L)$ generated by I in $\mathrm{Orth}(L)$ and called the center of L, and for its subalgebra $\mathcal{M}(L)$ as well.

Let A be an Archimedean f-algebra with unit e. Then A is said to be zero-dimensional if $\mathcal{M}_e$ is e-uniformly dense in the principal order ideal A_e generated by e. It is shown in [2] that A is zero-dimensional if and only if the Kakutani representation space of A_e is a zero-dimensional topological space.

Let M be an Archimedean lattice ordered module over an Archimedean f-algebra A with unit e. Then M inherits in an obvious way a Riesz space structure. It is shown in [4] that the mapping $H : A \longrightarrow \mathrm{Orth}(M)$ defined by

$$H(a)x = ax, \qquad a \in A, x \in M$$

is an algebraic Riesz homomorphism, hence the mapping $(a, x) \mapsto ax$ from $A \times M$ to M is a Riesz bimorphism. Order continuity of the multiplication by $a \in A^+$ in M is implied by the same property of the orthomorphism $H(a)$.

3. Super order principal modules. We shall start with a characterization of super order principal modules.

Theorem 3.1. *Let M be an Archimedean lattice ordered A-module. Then the following conditions are equivalent.*

(i) M is super order principal A-module.

(ii) M is super order principal A_e-module.

(iii) For any pair of disjoint elements $x, y \in M^+$ there exists an increasing sequence $(a_n) \subset [0, e]$ such that $a_n x \uparrow x$ and $a_n y = 0$ for all $n \in \mathbf{N}$.

(iv) $([0, e]x)^1 = [0, x]$ holds for every $x \in M^+$.

Proof. $(i) \Rightarrow (ii)$. Let $0 \leq y \in \{x\}^{dd}$, $x \in M^+$, and choose a double sequence $a_{n,k}$ in A^+ such that

$$a_{n,k} \uparrow_n, \quad a_{n,k} \uparrow_k, \quad a_{n,k} \uparrow_k (y \wedge nx) \text{ for all } n \in \mathbf{N}.$$

Set $b_{n,k} = a_{n,k} \wedge (ne) \in A_e$, observe that $b_{n,k}x \uparrow_k (y \wedge nx)$ and put $c_n = b_{n,n}$. Then we have $c_n \in A_e^+$ and $c_n x \uparrow y$.

$(ii) \Rightarrow (iii)$. Let $x \wedge y = 0$ and take an increasing sequence $(b_n) \subset A^+$ such that $b_n(x + y) \uparrow x$. Then the sequence $a_n = b_n \wedge e$ satisfies the required conditions.

$(iii) \Rightarrow (iv)$. Assume that $y \in [0, x]$, $n \in N$, $k \in \{1, 2, \ldots, n\}$, and set

$$y_{n,k} = y \wedge (k/n)x - y \wedge ((k - 1/n)x)$$

$$z_{n,k} = (y - (k/n)x)^- - (y - ((k - 1)/n)x)^-.$$

Observe that

$$0 \leq y_{n,k} \downarrow_k, \qquad 0 \leq z_{n,k} \uparrow_k,$$

$$y_{n,k} + z_{n,k} = (1/n)x, \quad y_{n,k+1} \wedge z_{n,k} = 0 \quad (k < n).$$

Fix $n \in \mathbf{N}$ and for each $k < n$ choose a sequence $(a_{k+1,i} : i \in \mathbf{N})$ such that

$$0 \leq a_{k+1,i} \uparrow \leq e, \qquad a_{k+1,i}y_{n,k+1} \uparrow_i y_{n,k+1}, \quad a_{k+1,i}z_{n,k} = 0$$

for all $i \in \mathbf{N}$. Put

$$b_{n,i} = (1/n) \sum_{k=1}^{n-1} a_{k+1,i} \in [0, e],$$

note that

$$(1/n)a_{k+1,i}x = a_{k+1,i}(y_{n,k} + z_{n,k}) = a_{k+1,i}y_{n,k} \leq y_{n,k}$$

and therefore

$$(1) \qquad 0 \leq b_{n,i}x \leq \sum_{k=1}^{n-1} y_{n,k} \leq y.$$

On the other hand we have

$$(2) \qquad b_{n,i}x = \sum_{k=1}^{n-1} a_{k+1,i}y_{n,k} \geq \sum_{k=1}^{n-1} a_{k+1,i}y_{n,k+1} \uparrow_i \sum_{k=1}^{n-1} y_{n,k+1} = $$
$$= y - y_{n,1} \geq y - (1/n)x.$$

The sequence $a_n = \sup\{b_{m,n} : m \leq n\}$ satisfies $0 \leq a_n \uparrow \leq e$ and by (1) $a_n x \leq y$ holds for all n. We claim that $a_n x \uparrow y$. Indeed, if $a_n x \leq z$ for all n, then $b_{n,i}x \leq z$ for all n, i. It follows from (2) that $y - (1/n)x \leq z$ for all n, thus $y \leq z$ and consequently $a_n x \uparrow y$.

$(iv) \Rightarrow (i)$. Obvious.

If A is zero-dimensional, the following result holds.

Theorem 3.2. *Let M be an Archimedean lattice ordered module over a zero-dimensional unital f-algebra A. Then the following statements are equivalent.*

(i) M is super order principal A-module.

(ii) M is super order principal $\mathcal{M}_e$-module.

(iii) For any pair of disjoint elements $x, y \in M^+$ there exists an increasing sequence $(p_n) \subset \mathcal{B}_e$ such that $p_n x \uparrow x$ and $p_n y = 0$ for all $n \in \mathbf{N}$.

Proof. $(i) \Rightarrow (iii)$. By Theorem 3.1 we can choose an increasing sequence $(a_n) \subset A \cap [0, e]$ such that $a_n x \uparrow x$, $a_n y = 0$ for all n. Since A is zero-dimensional, there is a sequence $(b_n) \subset \mathcal{M}_e$ such that $0 \leq b_n \uparrow \leq e$ and $0 \leq a_n - b_n \leq (1/n)e$. Clearly $b_n x \uparrow x$ and $b_n y = 0$ for all n. The sequence of elements $(p_n) \subset \mathcal{B}_e$ satisfying $\{p_n\}^d = \{b_n\}^d$ requires the conditions in (iii).

$(iii) \Rightarrow (ii)$. This implication follows from Theorem 3.1.

$(ii) \Rightarrow (i)$. Clear.

Let L be an Archimedean Riesz space. If $u \in L^+$, denote by $\mathcal{N}_u$ the Riesz subspace of L generated by the Boolean algebra $\mathcal{P}_u = \{Pu : P \in \mathcal{P}(L)\}$. The Freudenthal's spectral theorem (unbounded case) states that if L has the principal projection property then the subspaces $\mathcal{N}_u$ is super order dense in $\{u\}^{dd}$ for any $u \in L^+$. Observing that L is a lattice ordered module over the zero-dimensional f-algebra $\mathcal{M}(L)$ we recognize that Theorem 3.2 characterizes Riesz space in which a kind of Freudenthal's spectral theorem holds. More precisely, L is a super order principal $\mathcal{M}(L)$-module if and only if it possesses the following projection property

(σP) *For every pair of disjoint nonzero elements u, $v \in L^+$ there exists an increasing sequence of band projections P_n such that $P_n u \uparrow u$ and $P_n v = 0$ for all $n \in \mathbf{N}$.*

Since every super SMP-space possesses (σP) [3, Lemma 1.1, 1.2], an Archimedean Riesz space L satisfying the property (σP) will be called a weak super SMP-space. In [6] we can find an Archimedean Riesz space L without sufficiently many projections and with the property that any two disjoint elements of L belong to disjoint projection bands. Hence, L is a weak super SMP-space which fails to be a super SMP-space.

On the other hand, we shall see that if a weak super SMP-space has a weak order unit, then it is a super SMP-space.

Let us call an Archimedean f-algebra A with unit locally algebraically super rich for an Archimedean lattice ordered A-module M, if for each $x \in M^+$ there exists a sequence $(a_n) \subset A^+$ such that $a_n x \uparrow x$ and $a_n M \subset \{x\}^{dd}$ for all $n \in \mathbf{N}$.

Using Theorem 3.1 (iii) and [4, Lemma 3.3] we can establish the following result.

Proposition 3.3. *Let M be an Archimedean lattice ordered A-module. Consider the following two assertions.*
(i) A is locally algebraically super rich for M.
(ii) M is super order principal A-module.
Then (i) implies (ii). If M has a weak order unit, then (i) and (ii) are equivalent.

4. Egoroff type theorem. Recall that a Riesz space L is said to have the Egoroff property if for every element $f \in L^+$ and any double sequence $(u_{n,k} : n, k \in \mathbf{N})$ in L such that $0 \le u_{n,k} \uparrow_k f$ there exists a sequence (v_m) satisfying $0 \le v_m \uparrow f$ and $v_m \ll u_{n,k}$ for all m (i.e., for every m and n there exists $k = k(m,n) \in \mathbf{N}$ such that $v_m \le u_{n,k}$).

We shall extend now an abstract Egoroff type theorem on super order principal A-modules.

Theorem 4.1. *Let M be a super order principal A-module with the Egoroff property, $x \in M^+$ and $y_k \downarrow 0$ in M. Then there exists an increasing sequence $(a_m) \subset [0,e] \subset A$ such that $a_m x \uparrow x$ and $a_m y_k \downarrow_k 0$ x-uniformly for every $m \in \mathbf{N}$. If A is zero-dimensional, $(a_m : m \in \mathbf{N})$ can be taken from $\mathcal{B}_e$.*

Proof. Put $x_{n,k} = (y_k - (1/n)x)^-$, $n, k \in \mathbf{N}$, note that $0 \le x_{n,k} \uparrow_k (1/n)x$, and choose by Theorem 3.1 the increasing sequences $(a_{n,k,i} : i \in \mathbf{N})$, $n, k \in \mathbf{N}$, in $[0,e]$ such that

$$a_{n,k,i} x_{n,k} \uparrow_i x_{n,k},$$
$$a_{n,k,i}(y_k - (1/n)x)^+ = 0 \text{ for all } i.$$

Since $y_k \downarrow 0$ we may assume that $a_{n,k,i} \uparrow_k$ for fixed n, i.

Set $a_{n,k} = a_{n,k,n+k}$. We claim that $a_{n,k}x \uparrow_k x$.

Evidently $a_{n,k}$ is increasing in k. It follows from $x - a_{n,k}x \ge w \ge 0$ for all k, that

$$x - n a_{n,k,i} x_{n,k} \ge x - a_{n,k,i}x \ge x - a_{n,k+i}x \ge w$$

holds for all $i, k \in \mathbf{N}$. Thus

$$(1/n)x - x_{n,k} = (1/n)x - \sup_i a_{n,k,i} x_{n,k} \ge (1/n)w$$

for all k, and consequently $w = 0$ (since $x_{n,k} \uparrow_k (1/n)x$), so the claim follows.

The Egoroff property of M implies the existence of a sequence $(z_m) \subset M^+$ such that $z_m \uparrow x$ and $z_m \ll a_{n,k}x$, i.e., for any pair $m, n \in \mathbf{N}$ there exists $k(m,n)$ such that $z_m \le a_{n,k(m,n)}x$. We may assume that $k(m,n)$ increases in m.

As easily seen that $z_m \wedge (y_{k(m,n)} - (1/n)x)^+ = 0$ (Indeed, $a_{n,k}(y_k - (1/n)x)^+ = 0$ implies $a_{n,k}x \perp (y_k - (1/n)x)^+$), and by Theorem 3.1 we can find increasing sequences $(a_m^{(i)} \in [0,e] : i \in \mathbf{N})$, $m \in \mathbf{N}$, such that

$$(1) \qquad \begin{aligned} &a_m^{(i)} z_m \uparrow_i z_m \text{ for all } m, \\ &a_m^{(i)}(y_{k(m,n)} - (1/n)x)^+ = 0 \text{ for all } i, m. \end{aligned}$$

Put

$$a_m = \sup\{a_j^{(m)} : 1 \le j \le m\},$$

and note that a_m increases. We shall show that $a_m x \uparrow x$ and $a_m y_{k(m,n)} \le (1/n)x$, so the first part of the proof will be complete.

Since $0 \le w \le x - a_m x \downarrow$ implies

$$w \le x - a_m^{(i)}x \le x - a_m^{(i)} z_m \text{ for all } m, i,$$

we have $w \le x - z_m$ for every $m \in \mathbf{N}$, and $a_m x \uparrow x$ follows easily. Next, the condition $k(m,n) \uparrow_m$ together with (1) implies $a_m(y_{k(m,n)} - (1/n)x)^+ = 0$, thus $a_m y_{k(m,n)} \le (1/n)x$ as desired.

In the same way we can prove the remaining part of the theorem.

The following special case of Theorem 4.1 generalizes an abstract Egoroff type theorem [5, Theorem 74.3] and [3, Theorem 2.3].

Corollary 4.2. *Let L be a weak super SMP-space with the Egoroff property, $f \in L^+$, and $g_k \downarrow 0$ in L. Then there exists an increasing sequence P_m of band projections on L such that $P_m f \uparrow f$ and $P_m g_k \downarrow_k 0$ f-uniformly for every $m \in \mathbf{N}$.*

The space c of all real convergent sequences provides an example of a Riesz space without the principal projection property and satisfying the requirements of Corollary 4.2.

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ON THE THERMOELASTICITY OF MATERIALS WITH VOIDS

BY

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1. Introduction. Recently [1], Batra has proposed a principle of virtual work for the nonlinear theory of thermoelasticity. An account of the research on formulations of principle of virtual work for continuum mechanics can be found in [2-4].

The purpose of the present paper is to extend the result established by Batra in [1] to the case of nonlinear theory of thermoelastic materials with voids. In this theory, the bulk density is written as the product of two fields, the matrix material density field and the volume fraction field (cf. [5.6]). This representation introduces an additional degree of kinematic freedom. The intended applications of the theory of elastic materials with voids are to geological materials like rock and soils and to manufactured porous materials (ceramics and pressed powders). The nonlinear theory of thermoelastic materials with voids has been studied in various paper (see [5], [7-9]).

In section 2 we give a resume of the basic equations of the nonlinear theory of thermoelastic materials with voids and some preparatory results. The next section is devoted to a formulation of the principle of virtual work. From it are derived the equations of motion, the balance of equilibrated force, the constitutive equations for the temperature and the boundary conditions for stress and equilibrated stress.

2. Basic equations. Preliminaries. We consider a body $\mathcal{B}$ which occupies a closed, bounded, regular region of Euclidean three-dimensional space in a fixed reference configuration B_R. The motion of $\mathcal{B}$ is given by a smooth, differentiable map $\mathcal{X}$ in $B_R \times (0, t_0)$,

$$\mathbf{x} = \mathcal{X}(\mathbf{X}, t). \tag{2.1}$$

We denote by B_t the current configuration of the body $\mathcal{B}$.

We suppose that the map $\mathcal{X}$ is invertible with a smooth, differentiable inverse on $B_R \times (0, t_0)$. Let $\mathbf{F} = \nabla_X \mathcal{X}$ be the deformation gradient. Then

$$\det\mathbf{F} > 0. \tag{2.2}$$

The principle of balance of mass can be expressed in the form