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STABILITY AND CONDTINUOUS DEPENDENCE IN LINEAR VISOELASTICITY

BY

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To the memory of my beloved daughter IRINA (13.II.1972–26.XII.1987)

Introduction. This paper is concerned with some aspects regarding the behaviour results for generalized solutions of the linear dynamic theory of viscoelasticity under certain initial–boundary value problem. The non–homogeneous and anisotropic linear viscoelastic materials of non–convolutive relaxation type of constitutive equations (1.1) are considered. The initial–boundary problem of dynamic theory of viscoelasticity is defined in Sec.1, and in Sec.2 it is specified the suitable functional framework, the generalized solution of dinamic viscoelasticity is defined [6], and sufficient conditions for the existence of the generalized solution are specified [6], [7]. The basic inequality and fundamental hypotheses playing an important role in deriving the main resluts of paper are given in Sec.3. Sufficient Liapunov stability conditions of viscoelastic equilibrium [see.[11],§35] are obtained in Sec.4, the most extensive section of the paper. The results of this part of the paper are partially contained in author's paper [3] and [4].

Considering materials of relaxation [see constitutive equations (1.1)] and of creep type in exterior domain, and assuming the existence of classical solution, Guerriero [9] obtains an elegant generalization of the results in Sec.4 under somewhat larger hypotheses than ours.

Some interesting relations between the norms $||\cdot||_0$ and $||\cdot||_+$ of the generalized solution in the case of null initial and boundary conditions, but in the presence of the body forces, are rendered evident in Sec. 5. In Sec. 6 we find some results and improve the others results obtained in [5] on stability and behaviour of dynamic viscoelastic solution when the traction is prescribed on a proper part of the boundary and the displacement is null on the complementary part.

In the last section of the paper one find some results regarding the behaviour of dynamic viscoelastic solution in the case of mixed boundary conditions when the displacement is prescribed on a proper part of the boundary and the traction is null on the complementary part of the boundary.

Some of the obtained results of this paper are extending to the linear viscoelasticity the results of the reference work [11] of Knops and Wilkes on the theory of elastic stability.

1. Problem of dynamic linear viscoelasticity. Consider the non–homogeneous and anisotropic linear viscoelastic material of relaxation type having the constitutive equations.

$$\sigma_{ij}(x,t) = C_{ijkl}(\mathbf{x},t)u_{k,l}(\mathbf{x},t) + \int_0^\infty G_{ijkl}(\mathbf{x},t,\tau)u_{k,l}(\mathbf{x},\tau)\mathrm{d}\tau, \tag{1.1}$$

where $\sigma_{ij}(\mathbf{x},t)$ are the Cartesian components of the stress tensor, $\mathbf{u}(\mathbf{x},t) = (u_1(\mathbf{x},t), u_2(\mathbf{x},t), u)3(\mathbf{x},t))$ is the displacement vector at the point $\mathbf{x} = (x_1, x_2, x_3)$ and instant t, C_{ijkl} are the *elasticities*, and G_{ijkl} are the *relaxation moduli* of the material. All indices appearing in (1.1) range over value 1,2,3, and the usual convention of summation over repeated is adopted. A comma followed by the index i indicates differentiation with respect to x_i.

It is assumed that the following symmetry relations are satisfied.

$$G_{ijkl} = C_{jikl} = C_{klij}; \quad G_{ijlk} = G_{jikl} = G_{klij}. \tag{1.2}$$

The considered viscoelastic material occupies the regular domain [see p.13 of [8]] and compact $B \subset \mathbf{R}^3$ in the time interval $I = [0,T]$, $0 < T < \infty$. Let $\mathbf{n} = (n_1, n_2, n_3)$ be the unit outward normal on ∂B. The boundary ∂B is generally assumed smooth enough to allow application of the divergence theorem to domain B.

Let $\mathbf{F}(\mathbf{x},t) = (F_1(\mathbf{x},t), F_2(\mathbf{x},t), F_3(\mathbf{x},t))$, $(\mathbf{x},t) \in B \times I$, the body force and $\rho(\mathbf{x}), \mathbf{x} \in B$, the material density which is supposed to be independent of t.

Consider the initial and boundary value problem of the linear viscoelasticity: Find a solution of the system

$$\sigma_{ij,j}(\mathbf{x},t) + \rho(\mathbf{x})F_i(\mathbf{x},t) = \rho(\mathbf{x})\ddot{u}_i(\mathbf{x},t), \quad (\mathbf{x},t) \in B \times I, \tag{1.3}$$

satisfaing the initial conditions

$$\mathbf{u}(\mathbf{x},0) = \overset{\circ}{\mathbf{u}}(\mathbf{x}); \quad \dot{\mathbf{u}}(\mathbf{x},0) = \overset{\circ}{\mathbf{v}}(\mathbf{x}), \mathbf{x} \in B, \tag{1.4}$$

and the boundary conditions

$$\mathbf{u}(\mathbf{x},t) = \mathbf{h}(\mathbf{x},t), \quad (\mathbf{x},t) \in \partial B_1 \times I, \tag{1.5}$$

$$\sigma_{ij}(\mathbf{x},t)\eta_j = s_i(\mathbf{x},t), \quad (\mathbf{x},t) \in \partial B_2 \times I,$$

where $\partial B_1, \partial B_2 \subset \partial B$ are open subsets relative to ∂B satisfying the conditions $\partial B_1 \cap \partial B_2 = \emptyset$, $\partial B_1 \cup \partial B_2 = \partial B$. In the following considerations the situation $\partial B_2 = \emptyset$ is not excluded.

2. Generalized solution of dynamic viscoelasticity. Let $\mathbf{C}_0^\infty(B)$ be the set of vector function $\mathbf{x} \to \mathbf{f}(\mathbf{x}) = (f_1(\mathbf{x}), f_2(\mathbf{x}), f_3(\mathbf{x})) \in \mathbf{R}^3, \mathbf{x} \in B$, with compact support in B and having the components f_i in $\mathbf{C}_0^\infty(B)$. Let us note $\mathbf{H}_0$ and $\mathbf{H}_+$ the Hilbert spaces obtained by the closure of the set $\mathbf{C}_0^\infty(B)$ in the norms $||\cdot||_0$ and $||\cdot||_+$, respectively, induced by the scalar products

$$\langle \mathbf{u}, \mathbf{v}\rangle_0 = \int_B u_i v_i \mathrm{d}v, \quad \langle \mathbf{u}, \mathbf{v}\rangle_+ = \int_B u_{i,j} v_{i,j} \mathrm{d}v, \tag{2.1}$$

and let $\mathbf{H}_-$ be the closure of the same set $\mathbf{C}_0^\infty(B)$ with respect to the norm

$$||\mathbf{u}||_= \sup_{\mathbf{v} \in \mathbf{H}_+} \frac{|\langle \mathbf{u}, \mathbf{v}\rangle_0|}{||\mathbf{v}||_+} \tag{2.2.}$$

Definition [6]. By a generalized solution of the problem (1.3)–(1.5) on the segment I we mean a function $\mathbf{u} \in \mathbf{L}^2(I; \mathbf{H}_+)$ with $\dot{\mathbf{u}} \in \mathbf{L}^2(I; \mathbf{H}_0), \rho\ddot{\mathbf{u}} \in \mathbf{L}^2(I, \mathbf{H}_-)$ which satisfices the system (1.3) and (1.5) almost everywhere and the conditions (1.4) in the sense

$$\mathbf{u}(\mathbf{x},0) = \overset{\circ}{\mathbf{u}}(\mathbf{x}) \text{ in } [\mathbf{H}_+, \mathbf{H}_0]_{\frac{1}{2}}; \quad \dot{\mathbf{u}}(\mathbf{x},0) = \overset{\circ}{\mathbf{v}}(\mathbf{x}) \text{ in } [\mathbf{H}_0, \mathbf{H}_-]_{\frac{1}{2}}. \tag{2.3}$$

Now we present, without proof, the following theorem giving sufficient conditions for the existence of generalized solution of the problem (1.3)–(1.5). It is easy to prove that the theorem is a consequence of the more general result obtained by Dafermos [6], [7].

Theorem. *If the following conditions are satisfied:*

1°. *The elasticities C_{ijkl} and $\dot{C}_{ijkl}$, their derivatives with respect to $t \in I$, are Lebesgue measurable essentially bounded functions on B for any fixed $t \in I$.*

2°. *The relaxation moduli C_{ijkl} and $\frac{\partial G_{ijkl}}{\partial \tau}$, their derivatives with respect to $\tau \in I$, are Lebesque measurable and essemtially bounded functions on B for any $(t,\tau) \in I \times I$.*

3°. *There exists the strictly positive constant $a > 0$ such that*

$$\int_B C_{ijkl}(\mathbf{x},t)u_{i,j}(\mathbf{x},t)u_{k,l}(\mathbf{x},t)\mathrm{d}v \geq a||\mathbf{u}(\cdot,t)||_+^2, \tag{2.4}$$

for any $\mathbf{u}(\cdot,t) \in \mathbf{H}_+$, uniformly with respect to $t \in I$.

4°. *The density ρ is an essentially bounded function on B and there exists $\rho_0 > 0$ such that*

$$\mathrm{essinf}_B \rho(\mathbf{x}) \geq \rho > 0. \tag{2.5}$$

5°. $\mathbf{F} = (F_1, F_2, F_3) \in \mathbf{L}^2(I, \mathbf{H}_-)$.

6°. $\overset{\circ}{\mathbf{u}} \in \mathbf{H}_+; \overset{\circ}{\mathbf{v}} \in \mathbf{H}_0$.

7°. $\mathbf{h} = (h_1, h_2, h_3), \dot{h} \in \mathbf{L}^2(I, \partial B_1); \mathbf{s} = (s_1, s_2, s_3), \dot{\mathbf{s}} \in \mathbf{L}^2(I, \partial B_2)$

then there exists a unique generalized solution of the problem (1.3)–(1.5).

3. Fundamental Inequalities and hypotheses. Multiplying (1.3) by $\dot{u}_i$, summing with respect to i from 1 to 3, integrating on B, applying the divergence theorem, and taking into account (1.5) we obtain

(3.1),

$$\frac{dE}{dt}(t) = \frac{1}{2}\int_B \dot{C}_{ijkl}\,(\mathbf{x},t)u_{i,j}(\mathbf{x},t)u_{k,l}(\mathbf{x},t)dv - \int_B \sigma_{ij}^{(v)}(\mathbf{x},t)\,\dot{u}_{i,j}\,(\mathbf{x},t)dv + \mathcal{P}(t)$$

where

$$E(t) = \frac{1}{2}\int_B \left[C_{ijkl}(\mathbf{x},t)u_{i,j}(\mathbf{x},t)u_{k,l}(\mathbf{x},t) + \rho(\mathbf{x})\,\dot{u}^2\,(\mathbf{x},t)\right] dv \tag{3.2}$$

is the *sum of the elastic energy of deformation and the kinetic energy,*

$$\sigma_{ij}^{(v)}(\mathbf{x},t) = \int_0^t G_{ijkl}(\mathbf{x},t,\tau)u_{k,l}(\mathbf{x},\tau)d\tau \tag{3.3}$$

is the *viscous* part of the stress tensor (1.1),

$$\mathcal{P}(t) = \int_B \rho(\mathbf{x})F_i(\mathbf{x},t)\,\dot{u}_i\,(\mathbf{x},t)dv + \tag{3.4}$$

$$+\int_{\partial B_1} [\sigma_{ij}(\mathbf{x},t)\eta_j]\,\dot{h}_i\,(\mathbf{x},t)d\sigma + \int_{\partial B_2} s_i(\mathbf{x},t)\,\dot{u}_i\,(\mathbf{x},t)d\sigma.$$

in the power of body forces and of the loading (1.5) applied on the boundary. Integrating (3.1) on $[0,t]$ we get

$$E(t) - E(0) = \frac{1}{2}\int_0^t \left[\dot{C}_{ijkl}\,(\mathbf{x},t)u_{i,j}(\mathbf{x},\eta)\dot{u}_{k,l}(\mathbf{x},\eta)dv\right] d\eta - \tag{3.5}$$

$$-\int_0^t \left[\int_B \sigma_{ij}^{(v)}(\mathbf{x},\eta)\,\dot{u}_{i,j}\,(\mathbf{x},\eta)dv\right] d\eta + \int_0^t \mathcal{P}(\eta)d\eta, \quad t \in I,$$

where

$$E(0) = \frac{1}{2}\int_B \left[C_{ijkl}(\mathbf{x},0)\,\overset{\circ}{u}_{i,j}\,(\mathbf{x})\,\overset{\circ}{u}_{k,l}\,(\mathbf{x}) + \rho(\mathbf{x})\overset{\circ}{\mathbf{v}}^2(\mathbf{x})\right] dv. \tag{3.6}$$

Next, besides the hypotheses 1 – 7 of Sec.2, we make and the following two important assumptions:

(I) There exists a strictly positive constant $b > 0$ such that for any $\mathbf{u} \in L(I; \mathbf{H}_+)$ it holds the inequality

$$-\int_B \dot{C}_{ijkl}\,(\mathbf{x},t)u_{i,j}(\mathbf{x},t)u_{k,l}(\mathbf{x},t)dv \geq b\|\mathbf{u}(\cdot,t)\|_+^2. \tag{3.7}$$

uniformly with respect to $t \in I$.

(II) For any $\mathbf{u} \in \mathbf{L}^2(i; \mathbf{H}_+)$ with $\dot{\mathbf{u}} \in \mathbf{L}^2(I; \mathbf{H}_-)$ it holds the inequality

$$\int_0^t \left[\int_B \sigma_{ij}^{(v)}(\mathbf{x},\eta)\,\dot{u}_{i,j}\,(\mathbf{x},\eta)dv\right] d\eta \leq 0, \quad t \in I. \tag{3.8}$$

Remark 3.1. The inequality (3.7) is essential in the study of the elastic stability when elasticities are time dependent [see (36.3) of [11]]. Inequalities similar to (3.7) appears in Dafermos' papers [6] and [7] and they have great importance in the study of Liapunov or asymtotic stability of viscoelastic solutions.

Remark 3.2. The inequality (3.8) is expressing that the total power of the viscous part of the stress is non–negative on any interval $[0,t] \subset [0,T] = I$.

4. Stability of viscoelastic equilibrium [3], [4]. In this section we are placed in the case of null initial data, null boundary condition and in the absence of body force. In these situation the dynamic system [see p.128 of [11]] under consideration in the set of all vector functions $(\mathbf{x},t) \mapsto \mathbf{u}(\mathbf{u},t)$ satisfying the conditions $\mathbf{u}(\mathbf{x},t) \in \mathbf{L}^2(I; \mathbf{H}_+), \dot{\mathbf{u}}(\mathbf{x},t) \in \mathbf{L}^2(I; \mathbf{H}_0), \rho(\mathbf{x})\ddot{\mathbf{u}}(\mathbf{x},t) \in \mathbf{L}^2(I, \mathbf{H}_-)$ verifying almost everywhere system (1.3) and the boundary conditions (1.5).

In the above conditions we have $\mathcal{P}(t) = 0, \quad t \in I$, and from (3.7) (3.8), and (3.5) we have

$$E(t) \leq E(0), \quad t \in I. \tag{4.1}$$

Thus we have obtain the following rather crude

Theorem 4.1. The equilibrium viscoelastic solution is uniform Liapunov stable with respect to the measures

$$M(\mathbf{u}) = E(t) \quad \text{and} \quad M_0(\mathbf{u}) = E(0). \tag{4.2}$$

This means that it holds the implication

$$\forall \varepsilon > 0, \quad \exists \delta = \delta(\varepsilon)(= \varepsilon) \quad : \quad M_0(\mathbf{u}) < \delta \to M(\mathbf{u}) < \varepsilon. \tag{4.3}$$

Using the hypotheses 3° and 4° od Sec.2 we obtain

$$2E(t) \geq m\left[\|\dot{\mathbf{u}}(\cdot,t)\|_0^2 + \|\mathbf{u}(\cdot,t\|_+^2\right], \quad m = \min\{\rho_0, a\}, \tag{4.4}$$

and therefore we have

$$\|\dot{\mathbf{u}}(\cdot,t)\|_0^2 + \|\mathbf{u}(\cdot,t)\|_+^2 \leq \frac{2}{m}E(0). \tag{4.5}$$

From here we get

Theorem 4.2. The equilibrium viscoelastic solution is unifrom Liapunov stable with respect to the measures

$$\mu(\mathbf{u}) = ||\dot{\mathbf{u}}(\cdot,t)||_0^2 + ||\mathbf{u}(\cdot,t)||_+^2 \quad \text{and} \quad M_0(u) = E(0) \quad [see (4.3)] \tag{4.6.}$$

On the other hand the application of Schwarz inequality to the integral

$$\int_B C_{ijkl}(\mathbf{x},0)\ \dot{u}_{i,j}\ (\mathbf{x}\ \dot{u}_{k,l}\ (\mathbf{x})dv$$

leads to

$$2E(0) \leq K\left[||\ \dot{u}\ (\cdot,0)||_0^2 + ||\mathbf{u}(\cdot,0)||_+^2\right], \tag{4.7}$$

where $K = \max\left\{\operatorname*{ess\,sup}_{x\in B} \rho(x), \operatorname*{ess\,sup}_{x\in B} \sqrt{\operatorname{tr} C^2(x,0)}\right\}$, and tr $C^2(x,0)$ is the trace of the square of the symmetric matrix $C(x,0)$ of the type 6×6 of components $C_{ijkl}(\mathbf{x},0)$, the indices ij and kl taking the values $11;22;33;12;13;23$.

Therefore from (4.5) and (4.7) it follows

$$||\dot{\mathbf{u}}(\cdot,t)||_0^2 + ||\mathbf{u}(\cdot,t)||_+^2 \leq \frac{K}{m}\left[||\dot{\mathbf{u}}(\cdot,0)_0^2 + ||u(\cdot,0)||_+^2\right]. \tag{4.8}$$

In this way we obtain

Theorem 4.3. *The equilibrium viscoelastic solution is uniform Liapunov stable with respect to measures*

$$\mu(\mathbf{u}) = ||\dot{\mathbf{u}}(\cdot,t)||_0^2 + ||\mathbf{u}(\cdot,t)||_+^2 \quad \text{and} \quad \overline{\mu}_0(\mathbf{u}) = ||\dot{\mathbf{u}}(\cdot,0)||)_0^2 + ||\mathbf{u}(\cdot,0)||_+^2. \tag{4.9}$$

Remark 4.1. Theorems (4.2) and (4.3) hold both in the case when the elasticities are depending of time and in the case the elasticities does not depend of time. In the last case the first integral from the left hand side of the inequality (3.1) is zero.

Remark 4.2. The above results hold and in the case $\partial B_2 = \emptyset$.

Remark 4.3. The previous results hold if the condition $\mathbf{F}(\mathbf{x},t) = 0, (\mathbf{x},t) \in BxI$ is replaced by the weaker condition

$$\int_B \rho(\mathbf{x})F_i(\mathbf{x},t)\ \dot{u}_i\ (\mathbf{x},t)dv = 0, \quad t\in I.$$

Taking into account the Poincare's inequality [see (35.13) of [11]]

$$||\mathbf{u}(\cdot,t)||_0 \leq c||\mathbf{u}(\cdot,t)||_+, \quad c>0, \quad t\in I, \tag{4.11}$$

from (4.8) we obtain

$$||\dot{\mathbf{u}}(\cdot,t)||_0^2 + ||\mathbf{u}(\cdot,t)||_0^2 \leq \frac{K}{mm_1}\left[||\dot{\mathbf{u}}(\cdot,0)||_0^2 + ||\mathbf{u}(\cdot,0)_+^2\right], \quad t\in I, \tag{4.12}$$

where $m_1 = \min\{1,c^2\}$.

Therefore we have

Theorem 4.4. *The equilibrium viscoelastic solution is uniform Liapunov stable with respect to measures*

$$\nu(\mathbf{u}) = ||\dot{\mathbf{u}}(\cdot,t)||_0^2 + ||\mathbf{u}(\cdot,t)||_0^2 \quad \text{and} \quad M_0(\mathbf{u}) = E(0), \tag{4.13}$$

or with respect to measures

$$\nu(\mathbf{u}) = \dot{\mathbf{u}}(\cdot,t)||_0^2 + ||\mathbf{u}(\cdot,t)||_0^2 \quad \text{and} \quad \overline{\mu}(\mathbf{u}) = ||\dot{\mathbf{u}}(\cdot,0)||_0^2 + ||\mathbf{u}(\cdot,0)||_+^2. \tag{4.14}$$

If the elasticities C_{ijkl} are time dependent, from (3.5) and (3.7), we get the inequality

$$m\left[||\dot{\mathbf{u}}(\cdot,t)||_+ + a||\mathbf{u}(\cdot,t)||_+^2\right] + \frac{b}{2}\int_0^t ||\mathbf{u}(\cdot,\tau)||_+^2 d\tau \leq 2E(0), \quad t\in I. \tag{4.15}$$

From here we have

$$ma||\mathbf{u}(\cdot,t)||_+^2 + \frac{b}{2}\int_0^t ||\mathbf{u}(\cdot,\tau)||_+^2 d\tau \leq 2E(0), \quad t\in I.$$

which leads to

$$\int_0^t ||\mathbf{u}(\cdot,t)||_+^2 d\tau \leq \frac{4}{b}\left(1 - e^{-\frac{b}{2ma}t}\right) E(0) \leq \frac{4}{b}E(0), \quad t\in I. \tag{4.16}$$

This inequality together with (4.11) gives

$$\int_0^t ||\mathbf{u}(\cdot,\tau)||_0^2 d\tau \leq \frac{4c^2}{b}\left(1 - e^{-\frac{b}{2ma}t}\right) E(0) \leq \frac{4c^2}{b}E(0), \quad t\in I. \tag{4.17}$$

It follows

Theorem 4.5. *If the elasticities C_{ijkl} are time dependent then the equilibrium viscoelastic solution is uniform Liapunov stable with respect to the measures:*

$$\varphi(\mathbf{u}) = \int_0^t ||\mathbf{u}(\cdot,\tau)||_+^2 d\tau \quad \text{and} \quad M_0(\mathbf{u}) = E(0) \tag{4.18}$$

or with respect to the measures

$$\psi(\mathbf{u}) = \int_0^t ||\mathbf{u}(\cdot,\tau)||_0^2 d\tau \quad \text{and} \quad M_0(\mathbf{u}) = E(0). \tag{4.19}$$

Remark 4.4 If we make $t \to T$ in (4.16) and (4.17) we obtain the following a priori estimations of the equilibrium solution of liniar viscoelasticity in the absence of body force

$$\int_0^T \|\mathbf{u}(\cdot,\tau)\|_+^2 d\tau \le \frac{4}{b}E(0), \tag{4.20}$$

$$\int_0^T \|\mathbf{u}(\cdot,\tau)\|_0^2 d\tau \le \frac{4c^2}{b}E(0). \tag{4.21}$$

In the same conditions if $I = [0,\infty)$ from (4.16) and (4.17) we have

$$\int_0^\infty \|\mathbf{u}(\cdot,\tau)\|_+^2 d\tau \le \frac{4}{b}E(0), \tag{4.22}$$

$$\int_0^\infty \|\mathbf{u}(\cdot,\tau)\|_0^2 d\tau \le \frac{4c^2}{b}E(0). \tag{4.23}$$

Remark 4.5 Guerriero [9], using the results of [4], obtaines in 1987 an interesting extension regarding the stability of equilibrium solution for the exterior of a domain for the case of classical solution. In some less restrictive hypotheses than ours he show that the equilibrium viscoelastic solution is uniform Liapunov stable with respect to the measure.

(4.24)

$$N(\mathbf{u}) = \int_B \left[C_{ijkl}(\mathbf{x},t)u_{i,j}(\mathbf{x},t)u_{k,l}(x,t) + \rho(\mathbf{x})\dot{\mathbf{u}}^2(x,t)\right] dv \text{ and } N_0(\mathbf{u}) = N(\overset{0}{u}\overset{0}{}),$$

as well as in the norms

$$\varphi(\mathbf{u}) = \int_0^t \|\mathbf{u}(\cdot,\tau)\|_+^2 d\tau \quad \text{and} \quad N_0(\mathbf{u}) = N(\overset{\circ}{\mathbf{u}}) \quad (\overset{\circ}{\mathbf{u}} = \mathbf{u}(\cdot,0)). \tag{4.25}$$

Guerriero also obtains sufficient stability conditions for the equilibrium classical solution in the norm of uniform convergence.

Remark 4.6 By considering linear viscoelastic materials of convolutive type of constitutive equations

$$\sigma_{ij}(\mathbf{x},t) = G_{ijkl}(\mathbf{x},0)u_{k,l}(\mathbf{x},t) + \int_0^t G_{ijkl}(\mathbf{x},t-\tau)u_{k,l}(\mathbf{x},\tau)d\tau \tag{4.26}$$

Beevers [2] obtained a result regarding the Hölder stability [see DEF.8.2. of [11]] of the equilibrium viscoelastic solution with respect to the measures

$$\psi(\mathbf{u}) = \int_B \rho(\mathbf{x})\mathbf{u}^2(x,t)dv \quad \text{and} \quad \psi_0(\mathbf{u}) = \psi(\overset{\circ}{\mathbf{u}}) + H(\overset{\circ}{\mathbf{u}}), \tag{4.27}$$

where

$$H(\overset{\circ}{\mathbf{u}}) = \frac{1}{2}\int_B \left[G_{ijkl}(\mathbf{x},0)u_{i,j}(\mathbf{x},0)u_{k,l}(\mathbf{x},0) + \rho(\mathbf{x})\dot{\mathbf{u}}^2(\mathbf{x},0)\right] dv. \tag{4.28}$$

Here is the Beevers' result: If G_{ijkl} satisfy the simmetrry relations (1.2) and if the tensor G_{ijkl} is strict positive, then if $H(\overset{\circ}{\mathbf{u}})$ is strictly positive and sufficiently small there exists $\eta \in (0,1)$ such that

$$\psi(\mathbf{u}) \le A\left[\psi_0(\mathbf{u})\right]^\eta, \quad t \in [0,t], \tag{4.29}$$

where $A = A(T)$ is a strict positive constant.

5. Initial and boundary null conditions in presence of body forces [3]. In tis case $E(0) = 0$ and from (3.5), (3.4), (3.6), and (3.8) we obtain

$$E(t) \le \int_0^t \left[\int_B \rho(\mathbf{x})F_i(\mathbf{x},\tau)\,\dot{u}_i\,(\mathbf{x},\tau)dv\right] d\tau, \quad t \in I. \tag{5.1}$$

Let us notice that the estimates (73.6) and (73.10) in [11] hold and in the case of dynamic linear viscoelasticity solution.

If we apply the Schwarz inequality to the integral in (5.1), assume that there exists $L > 0$ such that $\|\rho\mathbf{F}(\cdot,t)\|_{L^2(B)} \le L$, uniformly with respect to $t \in I$, and take into account (4.4), we find

$$\|\,\dot{u}\,(\cdot,t)\|_0^2 + \|\mathbf{u}(\cdot,t)\|_+^2 \le \frac{L}{m}\int_0^t \|\dot{\mathbf{u}}(\cdot,\tau)\|_0 d\tau, \quad t \in I, \tag{5.2}$$

which entails the evident inequality

$$\|\dot{\mathbf{u}}(\cdot,t)\|_0^2 \le \frac{L}{m}\int_0^t \|\dot{\mathbf{u}}(\cdot,\tau)\|_0 d\tau, \quad t \in I. \tag{5.3}$$

This leads to [see p.27 of [1]]

$$\|\dot{\mathbf{u}}(\cdot,t)\|_0 \le \frac{L}{m}t, \quad t \in I, \tag{5.4}$$

whence it results a bound for the norm of velocity of viscoelastic solution on every compact $I = [0,T]$

$$\|\dot{\mathbf{u}}(\cdot,t)\|_0 \le \frac{L}{m}T. \tag{5.5}$$

6. The case of prescribed tensions the boundary [5]. In this section we suppose the displacement is null on $\partial B_1 \ne \phi$ and the body force satisfy the condition (4.10), which evidently includes the case of null body force.

In these conditions, from (3.4), (3.5), and the hypothesis (3.8) we get

$$(6.1)\qquad E(t)-\frac{1}{2}\int_0^t\left[\int_B \dot{C}_{ijkl}\,(\mathbf{x},t)u_{i,j}(\mathbf{x},\tau)u_{k,l}(\mathbf{x},\tau)\mathrm{d}v\right]\mathrm{d}\tau\leq E(0)+$$

$$+\int_0^t\left[\int_{\partial B_2} s_i(\mathbf{x},\tau)\;\dot{u}_i\;(\mathbf{x},\tau)\mathrm{d}\sigma\right]\mathrm{d}\tau.$$

Supposing the conditions providing that the order of integration may be reversed and applying the Schwarz inequality to the last integral on the right hand side of (6.1) then yields

$$(6.2)\qquad \int_0^t\left[\int_{\partial B_2} s_i(\mathbf{x},\tau)\;\dot{u}_i\;(\mathbf{x},\tau)\mathrm{d}\sigma\right]\mathrm{d}\tau=\int_{\partial B_2} s_i(\mathbf{x},t)\;\dot{u}_i\;(x,t)\mathrm{d}\sigma-$$

$$-\int_0^t\left[\int_{\partial B_2}\dot{s}_i\;(\mathbf{x},\tau)u_i(\mathbf{x},\tau)\mathrm{d}\sigma\right]\mathrm{d}\tau\leq\left[\int_{\partial B_2}\mathbf{u}^2(\mathbf{x},t)\mathrm{d}\sigma\right]^{\frac{1}{2}}\left[\int_{\partial B_2}\mathbf{s}^2(\mathbf{x},t)\mathrm{d}\sigma\right]^{\frac{1}{2}}+$$

$$+\int_0^t\left[\int_{\partial B_2}\dot{s}^2\;(\mathbf{x},\tau)\mathrm{d}\sigma\right]^{\frac{1}{2}}\left[\int_{\partial B_2}\mathbf{u}^2(\mathbf{x},\tau)\mathrm{d}\sigma\right]^{\frac{1}{2}}\mathrm{d}\tau.$$

If we use (3.2), the Stekloff inequality to the right hand side of (2.4) [see (70.9) and (70.10) of [11]], and suppose that $E(t)$ is bounded on $I=[0,T]$, that is,

$$(6.3)\qquad \sup{}_{[0,T]}E(t)=\widetilde{E}(T)<\infty$$

we obtain

(6.4)

$$c\alpha\left[\int_{\partial B_2}\mathbf{u}^2(\mathbf{x},t)\mathrm{d}\sigma\right]^2\leq\int_B C_{ijkl}(\mathbf{x},t)u_{i,j}(\mathbf{x},t)u_{k,l}(\mathbf{x},t)\mathrm{d}v\leq 2E(t)\leq 2\widetilde{E}(T),$$

where $\alpha>0$ is the constant entering into Stekloff inequality.

Thus from (6.1), (6.2), and (6.4) we find

$$(6.5)\qquad E(t)-\frac{1}{2}\int_0^t\left[\int_B \dot{C}_{ijkl}\,(\mathbf{x},\tau)u_{i,j}(\mathbf{x},\tau)u_{k.l}(\mathbf{x},\tau)\mathrm{d}c\right]\mathrm{d}\tau\leq$$

$$\leq E(0)+\left[\frac{2\widetilde{E}(T)}{c\alpha}\right]^{\frac{1}{2}}\left\{\left[\int_{\partial B_2}\mathbf{s}^2(\mathbf{x},t)\mathrm{d}\sigma+\int_0^t\dot{s}^2\;(\mathbf{x},\tau)\mathrm{d}\sigma\right]^{\frac{1}{2}}\mathrm{d}\tau\right\}.$$

From the evident inequality

$$\frac{d}{\mathrm{d}t}\left[\int_{\partial B_2}\mathbf{s}^2(\mathbf{x},t)\mathrm{d}\sigma\right]=2\int_{\partial B_2}s_i(x,t)\;\dot{s}_i\;(\mathbf{x},t)\mathrm{d}\sigma\leq$$

$$\leq 2\left[\int_{\partial B_2}\mathbf{s}^2(\mathbf{x},t)\mathrm{d}\sigma\right]^{\frac{1}{2}}\left[\int_{\partial B_2}\dot{\mathbf{s}}^2\;(\mathbf{x},t)\mathrm{d}\sigma\right]^{\frac{1}{2}}$$

it follows

$$(6.6)\qquad \left[\int_{\partial B_2}\mathbf{s}^2(\mathbf{x},t)\mathrm{d}\sigma\right]^{\frac{1}{2}}\leq\int_0^t\left[\int_{\partial B_2}\dot{\mathbf{s}}^2(\mathbf{x},\tau)\mathrm{d}\sigma\right]^{\frac{1}{2}}\mathrm{d}\tau.$$

Now, (6.5) and (6.6) entail

$$E(t)-\frac{1}{2}\int_0^t\left[\int_B \dot{C}\;ijkl(\mathbf{x},\tau)u_{i,j}(\mathbf{x},\tau)u_{k,l}(\mathbf{x},\tau)\mathrm{d}v\right]\mathrm{d}\tau\leq$$

$$(6.7)\qquad \leq E(0)+\left[\frac{8\widetilde{E}(T)}{c\alpha}\right]^{\frac{1}{2}}\left[\int_0^t\left[\int_{\partial B_2}\dot{s}^2\;(\mathbf{x},\tau)\mathrm{d}\sigma\right]\mathrm{d}\sigma\right]^{\frac{1}{2}}.$$

Hence, by using the inequalities (2.4), (2.5), (3.7), and (4.11) it follows

$$(6.8)\qquad k\left\{\|\mathbf{u}(\cdot,t)\|_0^2+\|\;\dot{u}\;(\cdot,t)\|_0^2+\int_0^t\|\mathbf{u}(\cdot,\tau\|_0^2\mathrm{d}\tau\right\}\leq$$

$$\leq E(0)+\left[\frac{8\widetilde{E}(T)}{c\alpha}\right]^{\frac{1}{2}}\int_0^t\left[\int_{\partial B_2}\dot{s}^2(\mathbf{x},\tau)\mathrm{d}\right]^{\frac{1}{2}}\mathrm{d}\tau,$$

where

$$k=\min\left\{\frac{a}{2c^2},\frac{\rho_0}{2},\frac{b}{2c^2}\right\}\quad\text{and}\quad\dot{s}=(\dot{s}_1,\dot{s}_2,\dot{s}_3).$$

An application of Schwarz inequality to the last integral of (6.8) yields

$$\int_0^t\left[\int_{\partial B_2}\dot{\mathbf{s}}^2(\mathbf{x},\tau)\mathrm{d}\sigma\right]^{\frac{1}{2}}\mathrm{d}\tau\leq\sqrt{t}\left\{\int_0^t\left[\int_{\partial B_2}\dot{s}^2\;(\mathbf{x},\tau)\mathrm{d}\sigma\right]\mathrm{d}\tau\right\}^{\frac{1}{2}}.$$

From here and (6.8) it follows

$$(6.9)\qquad k\left\{\|\mathbf{u}(\cdot,t)\|_0^2+\|\dot{\mathbf{u}}(x,t)\|_0^2+\int_0^t\|\mathbf{u}(\cdot,\tau)\|_0^2\mathrm{d}\tau\right\}\leq$$

$$\leq E(0)+\left[\frac{8\widetilde{E}(T)}{c\alpha}\right]^{\frac{1}{2}}\sqrt{t}\left\{\int_0^t\left[\int_{\partial B_2}\dot{\mathbf{s}}^2(\mathbf{x},\tau)\mathrm{d}\sigma\right]\mathrm{d}\tau\right\}^{\frac{1}{2}}.$$

Thus we obtain

Theorem 6.1. *If the functions*

$$U(t)=\int_0^t\left[\int_{\partial B_2}\dot{\mathbf{s}}^2(\mathbf{x},\tau)\mathrm{d}\sigma\right]^{\frac{1}{2}}\mathrm{d}\tau \quad or \quad V(t)=\sqrt{t}\left\{\int_0^t\left[\int_{\partial B_2}\dot{\mathbf{s}}^2(\mathbf{x},\tau)\mathrm{d}\sigma\right]\mathrm{d}\tau\right\}^{\frac{1}{2}}$$

are sufficiently small on I [it is sufficient that the functions (6.10) do not exceed $E(0)$ on I] then the solution of the problem (1.3)-(1.5), with $\mathbf{h}(\mathbf{x},t)=0,(\mathbf{x},t)\in \partial B_1 x]$ and the body forces are satisfying (4.10), is uniform Liapunov stable with respect the measures

$$\chi(u)=\|\mathbf{u}(\cdot,t)\|_0^2+\|\dot{\mathbf{u}}(\cdot,t)\|_0^2+\int_0^t\|\mathbf{u}(\cdot,\tau)\|_0^2\mathrm{d}\tau \quad and \quad M_0(u)=E(0).(6.11)$$

Remark 6.1. If

$$\left[\frac{8\widetilde{E}(T)}{c\alpha}\right]^{\frac{1}{2}}\left\{\int_0^t\left[\int_{\partial B_2}\dot{\mathbf{s}}^2(\mathbf{x},\tau)\mathrm{d}\sigma\right]\mathrm{d}\tau\right\}^{\frac{1}{2}}\leq E(0), \quad t\in I, leqno(6.12)$$

then from (6.8) we have

$$\chi(\mathbf{u})\leq\frac{2}{k}E(0), \tag{6.13}$$

whence one obtain

$$\int_0^t\|\mathbf{u}(\cdot,\tau)\|_0^2\mathrm{d}\tau\leq\frac{2}{k}\left(1-\mathrm{e}^{-t}\right)E(0)\leq\frac{2}{k}E(0). \tag{6.14}$$

If the solution of the problem exists on $I=[0,\infty)$ we obtain the following bound of the solution

$$\int_0^\infty\|\mathbf{u}(\cdot,\tau)\|_0^2\mathrm{d}\tau\leq\frac{2}{k}E(0). \tag{6.15}$$

Remark 6.2. All the results of this section hold only in condition $\partial B_1\neq\phi$. If $\partial B_1=\phi$, i.e. $\partial B_2=\partial B$, certain supplimentary restrictions must be imposed on the solution in order to exclude any instabilities that may arrise from possible rigid motions of the body.

Remark 6.3. In the absence of the initial condition (1.4) we have $E(0)=0$. In this case we obtain a result expressing the *continuous dependence of the solution on the traction* $\mathbf{s}(\mathbf{x},t)$ **prescribed on** $\partial B_2\subset\partial B_2\neq\partial B$.

Indeed, if u_1 and u_2 are solutions of the problem (1.3)-(1.5) with $\mathbf{h}(\mathbf{x},t)=0$ on $\partial B_1\times I$, $\overset{\circ}{u}=\overset{\circ}{v}=0$, with the body forces satisfying (4.10), and the tractions $\mathbf{s}_1(\mathbf{x},t),\mathbf{s}_2(\mathbf{x},t)$ on $\partial B_2 x I$, respectively, then $\mathbf{w}=\mathbf{u}_1-\mathbf{u}_2$ is the solution of the problem (1.3)-(1.5) with $\mathbf{h}=0$, $\overset{\circ}{u}=\overset{\circ}{v}=0,\mathbf{F}=0$, and $\mathbf{s}=\mathbf{s}_1-\mathbf{s}_2$ the traction on $\partial B_2\neq\partial B$.

$$\chi(\mathbf{w})\leq\frac{1}{k}\left[\frac{8\widetilde{E}_{\mathbf{w}}(T)}{c\alpha}\right]^{\frac{1}{2}}\int_0^t\left[\int_{\partial B_2}\dot{\mathbf{s}}^{2}(\mathbf{x},\tau)\mathrm{d}\sigma\right]^{\frac{1}{2}}\mathrm{d}\tau, \quad t\in I, \tag{6.16}$$

$$\chi(\mathbf{w})\leq\frac{1}{k}\left[\frac{8\widetilde{E}_{\mathbf{w}}(T)}{c\alpha}\right]^{\frac{1}{2}}\sqrt{t}\left\{\int_0^t\left[\int_{\partial B_2}\dot{\mathbf{s}}^2(\mathbf{x},\tau)\mathrm{d}\sigma\right]\mathrm{d}\tau\right\}^{\frac{1}{2}}, \quad \tau\in I, \tag{6.17}$$

expressing the continous dependence of the solution on the treaction $\mathbf{s}(\mathbf{x},t)$ prescribed on $\partial B_2\subset\partial B$, $\partial B_2\neq\partial B$.

7. The case of prescribed displacement on the boundary. Here we suppose that the traction is null on ∂B_2 and the body forces satisfy (4.10). In these conditions and making use of the hypothesis (3.8) we get

$$E(t)-\frac{1}{2}\int_0^t\left[\int_B\dot{C}_{ijkl}(\mathbf{x},\tau)u_{i,j}(\mathbf{x},\tau)u_{k,l}(\mathbf{x},\tau)\mathrm{d}v\right]\mathrm{d}\tau\leq \tag{7.1}$$

$$\leq E(0)+\int_0^t\left[\int_{\partial B_1}(\sigma_{ij}(\mathbf{x},\tau)\eta_j)\,\dot{h}_i\,(\mathbf{x},\tau)\mathrm{d}\sigma\right]\mathrm{d}\tau$$

Supposing the conditions providing that the order of integration may be reversed and applying the Schwarz inequality to the integral

$$\int_{\partial B_1}(\sigma_{ij}(x,\tau)\eta_j)\,\dot{h}_i\,(x,\tau)\mathrm{d}\sigma$$

we obtain

(7.2)

$$\int_{\partial B_1}(\sigma_{ij}(\mathbf{x},\tau)\eta_j)\,\dot{h}_i\,(\mathbf{x},\tau)\mathrm{d}\sigma\leq\left[\int_{\partial B_1}(\sigma_{ij}(\mathbf{x},\tau)\eta_j)^2\mathrm{d}\sigma\right]^{\frac{1}{2}}\left[\int_{\partial B_1}\dot{\mathbf{h}}^2(\mathbf{x},\tau)\mathrm{d}\sigma\right]^{\frac{1}{2}}$$

We assume, what has mechanical justification, that the traction is bounded on ∂B_1. This means that there exists a strict positive constant $Q>0$ such that

$$\left[\int_{\partial B_1}(\sigma_{ij}(\mathbf{x},\tau)\eta_j)^2\mathrm{d}\sigma\right]^{\frac{1}{2}}\leq Q, \quad \tau\in I \tag{7.3}$$

If $\overline{m} = \min\{m, b\}$, where $m = \min\{\rho_0, a\}$ [see (4.4)], from (7.1) and (7.2) it follows

$$\|\mathbf{u}(\cdot,t)\|_+^2 + \|\dot{\mathbf{u}}(\cdot,t)\|_0^2 + \int_0^t \|\mathbf{u}(\cdot,\tau)\|_+^2 d\sigma \leq \tag{7.4}$$

$$\leq \frac{2}{\overline{m}}\left\{E(0) + Q\int_0^t \left[\int_{\partial B_1} \dot{\mathbf{h}}^2(\mathbf{x},\tau)d\sigma\right]^{\frac{1}{2}} d\tau\right\}.$$

Applying the Schwatz inequality to the integral in the right hand side of (7.4) we have

$$\int_0^t \left[\int_{\partial B_1} \dot{\mathbf{h}}^2(\mathbf{x},\tau)d\sigma\right]^{\frac{1}{2}} d\tau \leq \sqrt{t}\left\{\int_0^t \left[\int_{\partial B_1} \dot{\mathbf{h}}^2(\mathbf{x},\tau)d\sigma\right]^{\frac{1}{2}} d\tau\right\}, \quad t \in I. \tag{7.5}$$

Thus we obtain

Theorem 7.1. *If the function*

$$R(t) = \int_0^t \left[\int_{\partial B_1} \dot{\mathbf{h}}^2(\mathbf{x},\tau)d\sigma\right]^{\frac{1}{2}} d\tau \quad \text{or} \quad S(t) = \sqrt{t}\left\{\int_0^t \left[\int_{\partial B_1} \dot{\mathbf{h}}^2(\mathbf{x},\tau)d\sigma\right] d\tau\right\}^{\frac{1}{2}} \tag{7.6}$$

are sufficiently small on I [it is sufficient that the functions (7.6) do not exceed $E(0)$ on I], then solution of the problem (1.3)-(1.5), with $\mathbf{s}(\mathbf{x},t) = 0$, $(\mathbf{x},t) \in \partial B_2 \times I$ and the body forces satisfy (4.10), is uniformly Liapunov stable with respect to measures (6.11).

Remark 7.1. If

$$\int_0^t \left[\int_{\partial B_1} \dot{\mathbf{h}}^2(\mathbf{x},\tau)d\sigma\right]^{\frac{1}{2}} d\tau \leq E(0), \quad t \in I, \tag{7.7}$$

then from (7.4) it follows

$$\chi(\mathbf{u}) \leq \frac{3}{\overline{m}}E(0), \quad t \in I, \tag{7.8}$$

whence we obtain

$$\int_0^t \|\mathbf{u}(\cdot,\tau)\|_0^2 d\tau \leq \frac{3}{\overline{m}}\left(1 - e^{-t}\right) E(0) \leq \frac{3}{\overline{m}}E(0), \quad t \in I. \tag{7.9}$$

If the solution of the problem exists on $[0,\infty)$ it follows the next estimate

$$\int_0^\infty \|\mathbf{u}(\cdot,\tau)\|_0^2 d\tau \leq \frac{3}{\overline{m}}E(0). \tag{7.10}$$

Remark 7.2. All result of the this section hold if $\partial B_1 = \partial B$ i.e. $\partial B_2 = \phi$.

Remark 7.3. In the case of null initial condition we have $E(0) = 0$, and one obtain a result expressing the continuous *dependence of the solution on the displacement prescribed* on $\partial B_1 \subseteq \partial B$.

Indeed, if $\mathbf{u}_1$ and $\mathbf{u}_2$ are solution of the problem (1.3)-(1.5), with $\mathbf{s}(\mathbf{x},t) = 0$, $(\mathbf{x},t) \in \partial B_2 \times I$, $\overset{\circ}{u} = \overset{\circ}{v} = 0$, the body forces satisfying (4.10), and the displacements $\mathbf{h}_1(\mathbf{x},t)$ and $\mathbf{h}_2(\mathbf{x},t)$, $(\mathbf{x},t) \in \overline{\partial B_1} \times I$, respectively then $\mathbf{w} = \mathbf{u}_1 - \mathbf{u}_2$ is the solution of the problem (1.3)-(1.5) with $\tilde{\mathbf{h}} = \mathbf{h}_1 - \mathbf{h}_2$, $\mathbf{s} = 0, \mathbf{F} = 0$ (or satisfying (4.10)), and $\overset{\circ}{u} = \overset{\circ}{v} = 0$.

Therefore, from (7.4) and (7.5) we obtain

$$\chi(w) \leq \frac{2Q}{\overline{m}}\int_0^t \left[\int_{\partial B_1} \dot{\tilde{\mathbf{h}}}^2(\mathbf{x},\tau)d\sigma\right]^{\frac{1}{2}} d\tau, \quad t \in I, \tag{7.11}$$

$$\chi(w) \leq \frac{2Q}{\overline{m}}\sqrt{t}\left\{\int_0^t \left[\int_{\partial B_1} \dot{\tilde{\mathbf{h}}}^2(\mathbf{x},\tau)d\sigma\right] d\tau\right\}^{\frac{1}{2}}, \quad t \in I. \tag{7.12}$$

This inequalities express the continuuuous dependence of the solution on the displacement $\mathbf{h}(\mathbf{x},t)$ prescribed an $\partial B_1 \subset \partial B$.

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EXPLICIT SOLUTIONS OF A CLASS OF DIFFERENTIAL AND DIFFERENCE BOUNDARY VALUE PROBLEMS*

BY

LUCAS JÓDAR

1. Introduction, motivation and preliminaries. This paper deals with boundary value problems of the type

$$\begin{cases} X^{(p)}(t) + A_{p-1}X^{(p-1)}(t), \cdots + A_0X(t) = 0, \\ X^{(i)}(0) = C_i, \quad 1 \leq i \leq p-1, \quad X(t) \to 0 \quad \text{as } t \to \infty \end{cases} \tag{1.1}$$

and

$$\begin{cases} X_{m+p} + A_{p-1}X_{m+p-1} \cdots + A_0X_m = 0, \quad m \geq 0, \\ X_i = C_i, \quad 1 \leq i \leq p-1, \quad X_m \to 0 \quad \text{as } m \to \infty, \end{cases} \tag{1.2}$$

where the unknown $X(t)$ for the differential problem, and X_m for the difference case, are n–dimensional compex vectors, elements of $\mathbf{C}^n$, $C_i \in \mathbf{C}^n$, for $1 \leq i \leq p-1$, and A_j for $0 \leq j \leq p-1$, are matrices in $\mathbf{C}^{n\times n}$. For the case $p = 2$, problem (1.1) appears in the solution of Clarke's equations related to heat diffusion random laminates problems [15]. Apart from case where by discretization of the type (1.1), discrete problems of the type (1.2) appear in the description of many economic, biological and sociological problems [1], [2].

Note that problems (1.1) and (1.2) are not typical in the sense that the initial value $X(0)$ for the continuous case and X_0 for the discrete problem, are free vectors. Thus, if we consider the classical approach, see [12], chap. 4, for instance, an easy answer of the problem is that if all the eigenvalues of the companion matrix

$$C = \begin{bmatrix} 0 & I & 0 & \cdots & 0 \\ 0 & 0 & I & \cdots & 0 \\ \vdots & & & I & \vdots \\ -A_0 & -A_1 & -A_2 & \cdots & -A_{p-1} \end{bmatrix} \tag{1.3}$$

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