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EXPLICIT SOLUTIONS OF A CLASS OF DIFFERENTIAL AND DIFFERENCE BOUNDARY VALUE PROBLEMS*

BY

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1. Introduction, motivation and preliminaries. This paper deals with boundary value problems of the type

$$\left\{\begin{array}{l} X^{(p)}(t) + A_{p-1}X^{(p-1)}(t), \cdots + A_0X(t) = 0, \\ X^{(i)}(0) = C_i, \quad 1 \le i \le p-1, \quad X(t) \to 0 \quad \text{as } t \to \infty \end{array}\right. \tag{1.1}$$

and

$$\left\{\begin{array}{l} X_{m+p} + A_{p-1}X_{m+p-1} \cdots + A_0X_m = 0, \quad m \ge 0, \\ X_i = C_i, \quad 1 \le i \le p-1, \quad X_m \to 0 \quad \text{as } m \to \infty, \end{array}\right. \tag{1.2}$$

where the unknown $X(t)$ for the differential problem, and X_m for the difference case, are n–dimensional compex vectors, elements of $\mathbf{C}^n, C_i \in \mathbf{C}^n$, for $1 \le i \le p-1$, and A_j for $0 \le j \le p-1$, are matrices in $\mathbf{C}^{n \times n}$. For the case $p = 2$, problem (1.1) appears in the solution of Clarke's equations related to heat diffusion random laminates problems [15]. Apart from case where by discretization of the type (1.1), discrete problems of the type (1.2) appear in the description of many economic, biological and sociological problems [1], [2].

Note that problems (1.1) and (1.2) are not typical in the sense that the initial value $X(0)$ for the continuous case and X_0 for the discrete problem, are free vectors. Thus, if we consider the classical approach, see [12], chap. 4, for instance, an easy answer of the problem is that if all the eigenvalues of the companion matrix

$$C = \begin{bmatrix} 0 & I & 0 & \cdots & 0 \\ 0 & 0 & I & \cdots & 0 \\ \vdots & & & I & \vdots \\ -A_0 & -A_1 & -A_2 & \cdots & -A_{p-1} \end{bmatrix} \tag{1.3}$$

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have a negative real part, then problem (1.1) has a solution for any initial vector $X(0)$, and if all the eingenvalues of C have module strictly smaller than one, then, for any initial vector X_0, problem (1.2) is solvable. These classical answers of the problem have some drawbacks. Firstly, the spectral condition required to the companion matrix is too restrictive. Secondly, we do not take advantage of the freedom coming from the lack of initial condition. Finaly, the classical approach increase the computational cost due to the consideration of an extended first order system.

The aim of this paper is to find more general existence conditions for solving problems (1.1) and (1.2), as well as to find explicit and computable solutions of them without increase of the data dimension. In this sense our approach follows the ideas developed in [6, 7, 8, 9].

We recall that for any matrix S in $\mathbf{C}^{m\times n}$, its Moore–Penrose generalized inverse, denoted by S^+, belongs to $\mathbf{C}^{n\times m}$ and can be efficiently computed by using MATLAB system [11]. An account of uses and properties of this generalized inverse may be found in [3], [13].

2. Existence conditions and explicit solutions. We begin this section with the problem of obtaining the general solution of the differential equation

$$X^{(p)}(t) + A_{p-1}X^{(p-1)}(t) + \cdots + A_0X(t) = 0 \tag{2.1}$$

and of the difference equation

$$X_{m+p} + A_{p-1}X_{m+p-1} + \cdots + A_0X_m = 0. \tag{2.2}$$

Theorem 1. *Let C be the companion matrix defined by (1.3), let M be an invertible matrix in $\mathbf{C}^{pn\times pn}$ and $J = \mathrm{diag}(J_1, J_2, \ldots, J_k)$ with $J_j \in \mathbf{C}^{m_j\times m_j}$, for $1 \le j \le k$, $m_1 + \cdots + m_k = np$, such that*

$$MJ = CM \tag{2.3}$$

Let us consider a block partition of M such that $M = (M_{ij})$ where $M_{ij} \in \mathbf{C}^{n\times m_j}$ for $1 \le i \le p$, $1 \le j \le k$. Then the following result hold:

(i) The general solution of system (2.1) is given by

$$X(t) = \sum_{j=1}^{k} M_{1j}\exp(tJ_j)d_j, \tag{2.4}$$

where d_j is an arbitrary vector in $\mathbf{C}^{m_j}$ for $1 \le j \le k$.

(ii) The general solution of the difference system (2.2) is given by

$$X_m = \sum_{j=1}^{k} M_{1j}J_j^m d_j, \quad m \ge 0, \tag{2.5}$$

where d_j is an arbitrary vector in $\mathbf{C}^{m_j}$ for $1 \le j \le k$.

Proof. From (2.3) it follows that

$$M_{pj}J_j = -\sum_{h=1}^{p} A_{h-1}M_{hj} \quad \text{and} \quad M_{ij}J_j = M_{i+1,j}, \quad 1 \le i \le p-1 \tag{2.6}$$

From (2.6) one gets

$$M_{ij} = M_{1j}J_j^{i-1}, \quad 1 \le i \le p \tag{2.7}$$

and from (2.3), (2.4) we have

$$M_{1j}J_j^p + \sum_{i=1}^{p} A_{i-1}M_{1j}J_j^{i-1} = 0. \tag{2.8}$$

Note that from (2.8), for each vector $d_j \in \mathbf{C}^{m_j}$, $X_j(t) = M_{1j}\exp(tJ_j)d_j$, is a solution of (2.1), for $1 \le j \le k$.

Let $Y(t)$ be any solution of (2.1) and let $Y^{(i)}(0) = D_i$, $0 \le i \le p-1$. Taking derivatives in the expression of $X(t)$ given by (2.4) and taking account (2.7) it follows that

$$X^{(i)}(t) = \sum_{j=1}^{k} M_{1j}J_j^i\exp(tJ_j)d_j = \sum_{j=1}^{k} M_{i+1\,j}\exp(tJ_j)d_j.$$

By imposing that $X^{(i)}(0) = D_i$, for $0 \le i \le p-1$, it follows that vectors $d_j \in \mathbf{C}^{m_j}$ must verify

$$M\begin{bmatrix} d_1 \\ \vdots \\ d_k \end{bmatrix} = \begin{bmatrix} D_0 \\ \vdots \\ D_{p-1} \end{bmatrix}. \tag{2.9}$$

Since M is invertible, taking the vectors d_j, $1 \le j \le k$, satisfying (2.9), one gets a solution $X(t)$ defined by (2.4) that for these values of d_j, satisfies the same initial conditions as $Y(t)$. From the uniqueness, $X(t)$ and $Y(t)$ coincide. This proves part (i).

(ii) Let Y_m be a solution of (2.2) and let $D_i = Y_i$ for $0 \le i \le p-1$. Note that from (2.8), for any vector $d_j \in \mathbf{C}^{m_j}$, the sequence $X_{m,j} = M_{1j}{J_j}^m d_j$is a solution of (2.2), for $1 \le j \le k$. Taking vectors d_j, for $1 \le j \le k$, defined by (2.9), the corresponding vector sequence $\{X_m\}$ defined by (2.5) satisfies $X_i = Y_i = D_i$, for $i = 0, 1, \ldots, p-1$. From the uniqueness, one concludes that $\{X_m\}$ and $\{Y_m\}$ coincide. Thus the result is establised.

Remark 1. It is important to note that any block diagonalization of the companion matrix C is appropriate to obtain the general solution of equations (2.1) and (2.2) by means of theorem 1. In this sense, when one uses as J the Jordan canonical form, one has the advantage that the expressions of $\exp(tJ_j)$as well as of the powers J_j^n are well known and they can be computed in an easy way, see [12], pag. 15 and 66. Although an interesting algorithm for computing the Jordan canonical form of a matrix may be found in [10], from a computational point of view it is more interesting to use the procedure given in [4, chap. 7], based on the Schur form of a matrix because such approach is more stable numerically than the Jordan form. The procedure given in [4] gives us the matrix M satisfyng (2.3).

In the following we denote by $\sigma(C)$ the set of all eigenvalues of the matrix C. We recall that if B is a matrix whose eigenvalues have a negative real part, then B is similar to its Jordan canonical form J_B, that is, there exist an invertible matrix S such that $B = S^{-1}J_BS$ and thus $\exp(tB) = S^{-1}\exp(tJ_B)S$. Now, from [12, p.66], it follows that $\exp(tB) \to 0$ as $t \to \infty$.

On the other hand, if B is a matrix whose spectral radius $\rho(B)$ satisfied $\rho(B) < 1$, then from theorem 1.3.9 of [12,p.25], it follows that $B^n \to 0$ as $n \to \infty$.

Theorem 2. *Let us suppose that the set $\sigma(C)$ of the eigenvalues of the companion matrix C, may be decomposed in two parts*

$$\sigma(c) = \sigma_1 \cup \sigma_2; \quad \mathrm{Re}(z) < 0 \text{ for } z \in \sigma_1 \text{ and } \sigma_1 \text{ is nonempty.} \tag{2.10}$$

Let M be an invertible matrix $\mathbf{C}^{np\times np}$ satisfying (2.3) where $J = \mathrm{diag}(J_1, \dots, J_k)$ with $J_j \in \mathbf{C}^{m_j\times m_j}, m_1 + m_2 + \dots + m_k = np$ and let $J_k, J_{k-1}, \dots, J_s$ the diagonal blocks whose eigenvalues are contained in σ_1, and let $m = m_s + m_{s+1} + \dots + m_k \geq 1$. Let us consider the partition of the matrix $M = (M_{ij})$ with $M_{ij} \in \mathbf{C}^{n\times m_j}$ and let M^{-1} be partitioned in the form

$$M^{-1} = V = \begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix}; \quad \begin{matrix} V_{11} \in \mathbf{C}^{(np-m)\times n}, & V_{12} \in \mathbf{C}^{(np-m)\times(n-1)p} \\ V_{21} \in \mathbf{C}^{m\times n}, & V_{22} \in \mathbf{C}^{m\times(n-1)p}. \end{matrix} \tag{2.11}$$

Then problem (1.1) is solvable if

$$V_{11}V_{11}^{+}V_{12}\begin{bmatrix} D_1 \\ \vdots \\ D_{p-1} \end{bmatrix} = V_{12}\begin{bmatrix} D_1 \\ \vdots \\ D_{p-1} \end{bmatrix}. \tag{2.12}$$

Under the condition (2.12), a set of solutions of problem (1.1) is given by

$$X(t) = \sum_{j=s}^{k} M_{1j}\exp(tJ_j)d_j, \tag{2.13}$$

where $d_s, d_{s+1}, \dots, d_k$, are given by

$$\begin{bmatrix} d_s \\ \vdots \\ d_k \end{bmatrix} = V_{22}\begin{bmatrix} D_1 \\ \vdots \\ D_{p-1} \end{bmatrix} - V_{21}\left(V_{11}^{+}\begin{bmatrix} D_1 \\ \vdots \\ D_{p-1} \end{bmatrix} + (I - V_{11}^{+}V_{11})Q\right) \tag{2.14}$$

and Q is an arbitrary vector in $\mathbf{C}^{(np-m)}$.

Proof. From theorem 1, given initial conditions $X^{(i)}(0) = D_i$, where D_i for $1 \leq i \leq p-1$ are data problem and D_0 is a free vector, the unique solution of equation (2.1) satisfyng these initial conditions is given by (2.4) where vectors $d_1, \dots, d_k$ are given by

$$\begin{bmatrix} d_1 \\ \vdots \\ d_s \\ \vdots \\ d_k \end{bmatrix} = M^{-1}\begin{bmatrix} D_0 \\ \vdots \\ D_{p-1} \end{bmatrix} = \begin{bmatrix} V_{11}D_0 + V_{12}\begin{bmatrix} D_1 \\ \vdots \\ D_{p-1} \end{bmatrix} \\ V_{21}D_0 + V_{22}\begin{bmatrix} D_1 \\ \vdots \\ D_{p-1} \end{bmatrix} \end{bmatrix}. \tag{2.15}$$

Note that if we choose D_0 so that

$$V_{11}D_0 + V_{12}\begin{bmatrix} D_1 \\ \vdots \\ D_{p-1} \end{bmatrix} = 0 \tag{2.16}$$

then the vector coefficients $d_1, d_2, \dots, d_{s-1}$ corresponding to the exponentials $\exp(tJ_j)$ with $1 \leq j \leq s-1$, that is, those block diagonal whose eingenvalues are not contained in the left half plane $\mathrm{Re}(z) < 0$ became zero. Thus if we take D_0 satisfying (2.16), then from (2.4) it follows that $X(t)$ given by (2.13) with $d_s, d_{s+1}, \dots, d_k$ defined by

$$\begin{bmatrix} d_s \\ \vdots \\ d_k \end{bmatrix} = V_{21}D_0 + V_{22}\begin{bmatrix} D_1 \\ \vdots \\ D_{p-1} \end{bmatrix} \tag{2.17}$$

is a solution of (2.1) such that $X(t) \to 0$ as $t \to \infty$.

Now, from theorem 2.3.2 of [13], p.24, equation (2.16) is solvable for D_0, if and only the condition (2.12) is satisfied. Furthermore, under the condition (2.12), the general solution for D_0 of equation (2.16), is given by

$$D_0 = -V_{11}^{+}V_{12}\begin{bmatrix} D_1 \\ \vdots \\ D_{p-1} \end{bmatrix} + (I - V_{11}^{+}V_{11})Q, \quad Q \in \mathbf{C}^{(np-m)}. \tag{2.18}$$

Thus taking any vector D_0 in $\mathbf{C}^n$ defined by (2.18) and placing the vectors $d_s, d_{s+1}, \dots, d_k$ in (2.13), one gets a solution of problem (1.1).

The proof of the following result is an easy consequence of the proof of theorem 2.

Theorem 3. *Let us suppose that the set $\sigma(C)$ of the eigenvalues of the companion matrix C, may be decompased in two parts*

$$\sigma(C) = \sigma_3 \cup \sigma_4; \quad |z| < 1 \text{ for } z \in \sigma_3 \text{ and } \sigma_3 \text{ is nonempty.} \tag{2.19}$$

Let $J = \mathrm{diag}(J_1, \ldots, J_k)$ with $J_j \in \mathbb{C}^{m_j \times m_j}$, $m_1 + \cdots + m_k = np$ and let M be an invertible matrix in $\mathbb{C}^{np \times np}$, with $M = (M_{ij})$, $M_{ij} \in \mathbb{C}^{n \times m_j}$, $1 \le i \le p$, $1 \le j \le k$, satisfyng (2.3). Let us suppose that $J_s, J_{s+1}, \ldots, J_k$, are the blocks corresponding to the eingenvalues of C contained in the set σ_3, and let us consider the partition of M^{-1} defined by (2.11). Then problem (1.2) is solvable if the condition (2.12) is satisfied and under the condition (2.12), a set of solutions of problem(1.2), is given by

$$X_m = \sum_{j=s}^{k} M_{1j} J_j^m d_j, \qquad m \ge 0, \tag{2.20}$$

where $d_s, d_{s+1}, \ldots, d_k$, are defined by (2.18).

Remark 1. The results included in this paper are related to those in [6,7,8,9] in the sense that our algebraic approach is based on algebraic properties of the companion matrix. However in the above papers we studied different boundary value problems and under different hypotheses. In fact, papers [6,7,8] are concerned with initial value problems or two–point boundary value differential problems. Papers [6] and [7] assume the existence of solutions for the associated algebraic matrix equation

$$Z^p + A_{p-1} Z^{p-1} + \cdots + A_0 = 0 \tag{2.21}$$

The existence of solutions of this equation is related to the companion matrix C given by (1.3), but equation (2.21) may be unsolvable, see [6]. In the present paper we do not assume the existence of solutions of equation (2.21).

The paper [7] does not suppose the existence of solutions of equation (2.21) but assumes that the companion matrix is similar to a block–diagonal matrix with all the block diagonals of size exactly equal to the dimension of the data, that is, of size $n \times n$. In the present paper we do not impose any restriction to the companion matrix C. The same restriction as the one used in [7] is assumed in [8] and in both papers the boundary value conditions are very different to the ones considered here. In particular, the conditions $X(t) \to 0$ as $m \to \infty$, of (1.2) are not considered in the papers [6,7,8,9].

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