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A MATHEMATICAL MODEL OF POPULATION DYNAMICS WITH DIFFUSION DEPENDING ON DENSITY AND RESOURCES

BY

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1. Introduction, Statement of the Problem. In a recent paper [4], we have considered a mathematical model of population dynamics, in which the population is structured on ages, and is supposed to move in R^2, the diffusion coefficient depending on the density of population. For such a model, we have proved existence theorems and, in a particular case, also an uniqueness one.

The equations of the model are analogous to those used in a more general setting in [1], [6], [3], [7].

It seems more natural to suppose that the diffusion coefficient, as well as the death coefficient, depend not only on the density, but also on a vector r of resources, vector which, in turn, depends on the point $x \in R^2$ on the time t, and, moreover, is a functional on the density.

The study of such a model will be the aim of this paper. Moreover, we shall consider the population moving in a bounded domain $\Omega \subset R^2$, with sufficiently smooth frontier, in the time interval $[0,T]$, between the ages 0 and B, and supposing that, on the frontier of Ω, the flow of population is proportional to the difference of the densities of the inner and outer populations; this hypothesis leads — like in a diffusion process — to the relation:

$$\frac{\partial u}{\partial n} = c(u-\theta) \text{ on } \partial\Omega \times [0,T] \times [0,B],$$

c — the constant proportionality factor and θ the outer density.

Concerning the resources, we shall suppose that their speed of variation is composed by a linear functional depending on the density of the population, and a second term, representing their autonomous variation. For the sake of simplicity of the calculations, without any lose of the theoretical problem, we will suppose that r is a scalar, which — taking into account, that a given material, can be in some quantity a resource, and in another one a polluant can be positive or negative.

For the theorems used in this paper, see [2], [5].

We denote by u — the density of the age a population in x, at the moment t,

b — the density of the offspring in x, at t,

u_0 — the initial distribution of the population of age a in x,
Λ — the death coefficient,
μ — the birth coefficient,
ν — a weight coefficient, describing the dependence of the resources on the population,
A — the diffusion coefficient,
B - the maximum age possible.

So our system will contain:

1. The diffusion equation:

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial a} - A^2(u,r)\Delta u + \Lambda(x,t,a,r)u = 0, \quad \text{on } \Omega \times [0,T] \times [0,B]; \tag{1.1}$$

2. The offspring equation:

$$b(x,t) = \int_0^B \mu(x,t,\beta)u(x,t,\beta)\, d\beta, \quad \text{on } \Omega \times [0,T]; \tag{1.2}$$

3. The initial distribution:

$$u(x,0,a) = u_0(x,a), \quad \text{on } \Omega \times [0,B]; \tag{1.3}$$

4. The boundary condition:

$$\frac{\partial u}{\partial n} + cu = h(x,t,a), \quad \text{on } \partial\Omega \times [0,T] \times [0,B]; \tag{1.4}$$

5. The equation of the resources:

$$\frac{dr}{dt} = \int_0^B \nu(x,t,\beta)u(x,t,\beta)db + g(x,t), \quad \text{on } \Omega \times [0,T]; \tag{1,5}$$

6. The initial situation of the resources:

$$r(x,0) = r_0(x); \tag{1.6}$$

7. A compatibility equation :

$$u_0(x,0) - \int_0^B \mu(x,0,\beta)u_0(x,\beta)d\beta. \tag{1.7}$$

Remark. It is easily seen, that, by the intermediate of the resources, the diffusion coefficient depends not only on the density of the age a population , but also on the density of the entire population.

2. **Hypotheses.** We suppose Ω a bounded domain in R^2, the frontier of which has continuous tangents, and denote:

$$Q = \Omega \times [0,T] \times [0,B], \qquad S = \partial\Omega \times [0,T] \times [0,B],$$
$$Q_T = \Omega \times [0,T], \qquad S_T = \partial\Omega \times [0,T],$$
$$Q_A = \Omega \times [0,B], \qquad S_A = \partial\Omega \times [0,B],$$

and by λ a real $0 < \lambda < \frac{1}{2}$.

With this notation and with those introduced in **1**, we suppose:

a) **About the diffusion coefficient** A:

a_1) $A: R_+ \times R \to R_+$ is bounded, continuous, positive in his domain of definition, and

$$A(0,r) = 0, \quad A(u,r) > 0 \quad \text{for } u > 0,$$
$$A(x,t) \geq p_1 > 0, \text{ for } x > p > 0 \quad A(x,v) \leq M_1;$$

a_2) A^2 is Lipschitz continuous, i.e.

$$|A_2(u_1,r_1) - A^2(u_2,r_2)| \leq L_1(|u_1 - u_2| + |r_1 - r_2|);$$

a_3) A^2 is increasing with respect to u, and decreasing with respect to r.

b) **About the coefficient** Λ:

b_1) $\Lambda: R^2 \times R_+^2 \times R \to R_+$ is bounded, continuous, positive in his domain of definition

$$0 \leq \Lambda \leq M_2;$$

b_2) Λ belongs to the Hölder class of functions $H^{(\lambda,\lambda/2,\lambda/2)}(Q)$ for each r, i.e., there exists:

$$\| \Lambda \|_Q^{(\lambda,\lambda/2,\lambda/2)} = \max_Q |\Lambda| + \sup_Q \frac{|\Lambda(x',t',a'r) - \Lambda(x,t,a,r)|}{(|x'-x|^\lambda + |t'-t|^{\lambda/2} + |a'-a|^{\lambda/2})};$$

$$(x',t',a'),(x,t,a) \in Q,$$

(For notation and properties of Hölder classes, see[2]).

b_3) Λ is Lipschitz continuous with respect to r: for each $(x,t,a) \in Q$:

$$|\Lambda(x,t,a,r_1) - \Lambda(x,t,a,r_2)| \le L_2|r_1 - r_2|.$$

c) **About the birth coefficient μ:**

c_1) $\mu: R^2 \times R_+^2 \to R_+$ is bounded, continuous, positive, $0 \le \mu \le M_3$;

c_2) $\mu \in C^2(\Omega)$, for each (t,a);

c_3) $\mu \in H^{(\lambda,\lambda/2)}(Q_T)$ for each $a \in [0,B]$, i.e., there exists

$$\| \mu \|_{Q_T}^{(\lambda,\lambda/2)} = \max \mu + \sup \frac{|\mu(x',t',a) - \mu(x,t,a)|}{|x'-x|^\lambda + |t'-t|^{\lambda/2}},$$

for each $a \in [0,B]$, and is integrable on $[0,B]$, for each $(x,t) \in Q_T$.

d) **About the coefficient ν of the resources:**

d_1)$\nu: R^2 \times R_+^2 \to R$ is bounded, continuous in Q: $|\nu| \le M_4$;

d_2) $\nu \in H^{(\lambda,\lambda/2)}(Q_T)$ for each $a \in [0,B]$, i.e. there exists

$$\| \nu \|_{Q_T}^{(\lambda,\lambda/2)} = \max_{QT} |\nu| + \sup_{QT} \frac{|\nu(x',t',a) - \nu(x,t,a)|}{|x-x'|^\lambda + |t'-t|^{\lambda/2}},$$

for each $a \in [0,B]$, and is integrable on $[0,B]$, for each$(x,t) \in Q_T$.

e) **About the initial conditions:**

e_1) $u_0: R^2 \times R_+ \to R_+$ is bounded, continuous, strictly positive

$$0 < cp < u_0 < M_5 \text{ on } \Omega \times [0,B];$$

e_2) $u_0 \in H^{(\lambda+2,\lambda/2+1)}(Q_A)$, i.e. there exists

$$\| u_0 \|_{Q_A}^{(\lambda+2,\lambda/2+1)} = \max_{Q_A} u_0 + \sum_i \max_{Q_A} \left|\frac{\partial u_0}{\partial x_i}\right| + \sup_i \max \left|\frac{\partial^2 u_0}{\partial x_i^2}\right| +$$

$$+\sup \frac{\left|\frac{\partial^2 u_0}{\partial x^2}(x',a') - \frac{\partial^2 u_0}{\partial x^2}(x,a)\right|}{|x'-x|^\lambda + |a'-a|^{\lambda/2}};$$

e_3) $r_0: R^2 \to R$ is bounded, continuous on Ω,

$$|r_0| \le M_6;$$

e_4) $g: R^2 \times R_+ \to R$ is bounded, integrable with respect to t for each x.

f) **About the boundary condition h:**

f_1) h : defined on S is bounded, continuous, strongly positive

$$0 < Cp < h \le M_7;$$

f_2) $h \in H^{(\lambda+2,\lambda/2+1/2,\lambda/2+1/2)}(S)$.

g) **The compatibility condition on the frontier, between the boundary and initial condition:**

$$\frac{\partial u_0}{\partial n} = h(x,0,a) \quad \text{on } \partial\Omega.$$

h) We denote moreover

$$C(\| u_0 \|^{(\lambda+2,\lambda/2+1)} + \| h \|^{(\lambda+1,\lambda/2+1/2)}) = P_1,$$

c — a given numerical constant, which will be precized in the following.

For the study of this system, it is convenient to split the domain $Q = \Omega \times [0,T] \times [0,B]$, in two parts

$$Q_1 = Q \cap \{a \ge t\} \quad \text{and} \quad Q_2 = Q \cap \{a < t\}.$$

In Q_1, i.e. for $a \ge t$, we substitute

$$a - t = \alpha, \qquad t = \tau$$

and denote

$$u(x,t,a) = u(x,\tau,\alpha+\tau) = v(x,\tau,\alpha) = v(x,t,a-t), \quad r(x,t) = \overline{r}(x,\tau).$$

In $Q_2)(i.e. fora < t)$, we substitute

$$t - a = \tau, \quad a = \alpha$$

and denote:

$$u(x,t,a) = u(x,\alpha+\tau,\alpha) = w(x,\tau,\alpha) = w(x,t-a,a),$$

$$r(x,t) = r(x,\alpha_\tau) = \overline{\overline{r}}(x,\tau,\alpha) = r(x,t-a,a).$$

The systems (1.1) — (1.7) becomes:

$$Lv = \frac{\partial v}{\partial \tau} - A^2(v,r)\Delta v + \overline{\Lambda} v = 0, \quad \text{on } Q_1, \tag{2.1}$$

$$v(x,0,\alpha) = u(x,0,\alpha) = u_0(x,\alpha), \quad \text{on } Q_A, \tag{2.2}$$

$$\frac{\partial \overline{r}}{\partial \tau}(x,\tau) = \int_0^\tau \nu(x,\tau,\beta) w(x,\tau-\beta,\beta) d\beta + \tag{2.3}$$

$$+\int_{\tau}^{B}\nu(x,\tau,\beta)v(x,\tau,\beta-\tau)d\beta+g(x,\tau),\quad \text{on } Q_T,$$

(2.4)
$$\frac{\partial v}{\partial n}+cv=h(x,\tau,\alpha+\tau),\quad \text{on } S$$

(2.5)
$$\bar{r}(x,0)=r_0(x),\quad \text{on } \Omega,$$

(2.6)
$$\frac{\partial w}{\partial \alpha}-A^2(w,\bar{\bar{r}})\Delta w+\bar{\bar{\Lambda}}w=0,\quad \text{on } Q_2,$$

(2.7)
$$b(x,\tau)=\int_0^{\tau}\mu(x,\tau,\beta)w(x,\tau-\beta,\beta)d\beta+\int_{\tau}^{B}\mu(x,\tau,\beta)v(x,\tau,\beta-\tau)d\beta,\quad \text{on } Q_T,$$

(2.8)
$$\frac{\partial \bar{r}}{\partial t}=\int_0^{\tau}\mu(x,\tau,\beta)w(x,\tau-\beta,\beta)d\beta+$$

(2.8)
$$+\int_{\tau}^{B}\nu(x,\tau,\beta)v(x,\tau,\beta-\tau)d\beta+g(x,\tau)\ \text{on } Q_T,$$

(2.9)
$$\frac{\partial w}{\partial n}+cw=h(x,\alpha+\tau,\alpha),\quad \text{on } S,$$

where

(2.10) $\bar{\Lambda}=\bar{\Lambda}(x,\tau,\alpha,\bar{\bar{r}})=\Lambda(x,\tau,\alpha+\tau,\bar{\bar{r}}),\quad \bar{\bar{\Lambda}}=\bar{\bar{\Lambda}}(x,\tau,\alpha,\bar{\bar{r}})=\Lambda(x,\alpha+\tau,\alpha,\bar{r}).$

Obviously, the coefficients $A(v,\bar{r}),\quad A(w,\bar{\bar{r}}),\bar{\Lambda},\bar{\bar{\Lambda}},\ldots$ satisfy the same conditions as those stated for $A(u,r),\Lambda,\ldots$.

Finally, the compatibility condition remains unchanged.

3. Existence of the solution. With the above notations, we can state:

Theorem. *If the conditions of the previous paragraph are satisfied, the system (1.1) — (1.7), or, equivalently (2.1) — (2.9) has a solution, which is unique.*

Proof. The proof will be performed in fife steps.

First Step. We consider the system of equations (2.1) — (2.5), in which we suppose that w is a known fixed function, belonging to $H^{(\lambda,\lambda/2,\lambda/2)}(Q)$ and moreover $\| w \|_Q^{(\lambda,\lambda/2,\lambda/2)}\le P$ (see hypothesis h). We shall prove that this system has a unique solution.

To this end, we write first the auxiliary, in V and $\bar{R}$ system:

$$\mathcal{L}V=\frac{\partial V}{\partial \tau}-A^2(v,\bar{R})\Delta V+\bar{\Lambda}(x,\tau,\alpha,\bar{R})V=0,\quad \text{on } Q_1,$$

$$V(x,0,\alpha)=u_0(x,\alpha),\quad \text{on } Q_A,$$

(3.1)
$$\frac{\partial v}{\partial n}(x,\tau,\alpha)=h(x,\tau,\alpha),\quad \text{on S},$$

$$\bar{R}(x,\tau)=r_0(x)+\int_0^{\tau}\left\{\int_0^{\sigma}\nu(x,\sigma,\beta)w(x,\sigma-\beta,\beta)d\beta+\right.$$

$$\left.+\int_{\sigma}^{B}\nu(x,\sigma,\beta)v(x,\sigma,\beta-\sigma)d\beta+g(x,\sigma)\right\}d\sigma,$$

where we take for v a function of the Hölder class $H^{(\lambda,\lambda/2,\lambda/2)}(Q)$ with norm

(3.2)
$$\| v \|_Q\ (\lambda,\lambda/2,\lambda/2)\le P_1,$$

and which, satisfies the conditions:

$$v(x,0,\alpha)=u_0(x,\alpha),\quad \text{on } Q_A,$$

(3.2′)
$$\frac{\partial v}{\partial n}(x,\tau,\alpha)+cv(x,\tau,\alpha)=h(x,\tau,\alpha)\ \text{on } S,$$

(3.3)
$$C_1(\min u_0,\min h)\le v\le C_1(\max u_0,\max h)e^{(\min s)\tau},$$

c_1 being a numerical constant which will be determined in the following, and for $\bar{R}$ the second member of the fourth equation (3.1).

Denote by C the class of the above defined functions v. We shall prove that the system (3.1) has a solution — which is unique — belongs to the same class C, and that the function $\overline{R}$ belongs to $H^{(\lambda,\lambda/2)}(Q_T)$.

Indeed, from the fourth equation (3.1), taking into account the hypotheses on ν and r_0, and the fact that v and w belong to $H^{(\lambda,\lambda/2)}(Q_T)$ for each $\alpha \in [0,B]$, and are integrable for each $(x,t) \in Q_T$ we deduce

$$\overline{R} \in H^{(\lambda,\lambda/2)}(Q_T)$$

and moreover, that

$$\| R \|_Q^{(\lambda,\lambda/2)} \leq M_4 BTP_1.$$

On the other hand, since A^2 is Lipschitz continuous, and v and R belong to the mentioned Hölder classes, it follows that

$$A^2(v(x,t,\alpha);$$

$$\overline{R}(x,t)) \in H^{(\lambda,\lambda/2,\lambda/2)}(Q).$$

The same conclusion follows for $\overline{\Lambda}$.

We use now the theorem 5.3 of [5] pag. 364, adapted to our case, namely

Theorem A. *Let $0 < \lambda < 1$, $S \in H^{(\lambda+2)}$, the coefficients of the operator uniformly parabolic L belong to the class $H^{(\lambda,\lambda/2)}(Q_T)$ (for each $\alpha \in [0,B]$), the coefficient of the boundary condition (in our case a constant)belong to $H^{(\lambda,\lambda/2)}(S)$. Then for each $f \in H^{(\lambda,\lambda/2)}$ (in our case it is zero), for each $u_0 \in H^{(\lambda+2)}(\Omega)(\alpha \in [0,B])$ and each $h \in H^{(\lambda+1,(\lambda+1)/2,(\lambda+1)/2)}(S)$, satisfying the compatibility condition between the boundary and initial conditions, our problem has a unique solution V, which belongs to $H^{(\lambda+2,(\lambda+2)/2)}(Q_T)$ and so that*

$$\| V \|_{Q_T}^{(\lambda+2,\lambda/2+1)} \leq c_2\{\| f \|_{Q_T}^{(\lambda)} \| u_0 \|_\Omega^{(\lambda+2)} + \| h \|_{S_T}^{(\lambda+1)}\}, \tag{3.4}$$

where c_2 is a constant, independent on f, u_0, h, but dependent increasing on A^{-1}, Λ, T, mes Ω, and vanishing for $T = 0$.

Since $V \in H^{(\lambda+2,\lambda/2+1)}(Q_T)$, it belongs also to $H^{(\lambda,\lambda/2)}(Q_T)$; we choose further conveniently c_2 (eventualy, also the bounds of $\| u_0 \|_\Omega^{(\lambda)}$, and $\| h \|_S^{(\lambda+1)}$, and T), to obtain (see the hypothesis h)

$$\| V \|_{Q_1}^{(\lambda,\lambda/2)} \leq P_1.$$

To prove that $V \in C$, we must prove that $V \in H^{(\lambda,\lambda/2,\lambda/2)}(Q)$. To this end, we consider two values α_1 and α_2 of α and denote:

$$v^{(i)} = v(x,t,\alpha_i), \qquad \overline{\Lambda}^{(i)} = \overline{\Lambda}(x,\tau,\alpha_i,\overline{r}_i)\ldots \quad (i = 1,2)$$

by V^i, the corresponding solutions of (3.1), and finally:

$$Z = \frac{V(1) - V(2)}{|\alpha_1 - \alpha_2|^{\lambda/2}}, \qquad Y = \frac{\overline{R}^{(1)} - \overline{R}^{(2)}}{|\alpha_1 - \alpha_2|^{1/2}}.$$

These last functions satisfy the system:

$$\begin{aligned}
&\frac{\partial Z}{\partial \tau} - A^2(v^{(1)}, R^{(1)})\Delta Z + \overline{\Lambda}^i Z = F, \\
&Z|_{\tau=0} = \frac{u_0^{(1)} - u_0^{(2)}}{|\alpha_1 - \alpha_2|^{\lambda/2}}, \quad \text{on } \Omega, \\
&\frac{\partial Z}{\partial n} + cZ = \frac{h^{(1)} - h^{(2)}}{|\alpha_1 - \alpha_2|^{\lambda/2}}, \quad \text{on } S_T, \\
&Y(x,\tau) = \int_0^\tau \{\int_\tau^B \nu(x,\sigma,\beta) Z(x,\sigma,\beta) d\beta,
\end{aligned} \tag{3.5}$$

where:

$$|\alpha_1 - \alpha_2|^{\lambda/2} F = [A^2(v^{(1)}, \overline{R}^{(1)}) - A^2(v^{(2)}, \overline{R}^{(2)})]\Delta V^{(2)} - (\overline{\Lambda}^{(1)} - \overline{\Lambda}^{(2)})V^{(2)}.$$

Since $V^{(2)} \in H^{(\lambda,\lambda/2)}(Q_T)$ for each α and is a solution of (3.1), it results that it exists the constant P_1, so that

$$|V^{(2)}|, \quad |\Delta V^{(2)}| \leq P_1.$$

From the fourth equation (3.5), we get:

$$\| Y \| \leq M_4 BT \max|Z|.$$

It follows:

$$|F| \leq \left\{[L_1 + (L_1 + L_2)M_4 BT] \max|Z| + \| \overline{\Lambda}^{(2)} \|_{Q_T}^{(\lambda,\lambda/2)}\right\} P_1$$

and since V belongs to $H^{(\lambda,\lambda/2)}(Q_T)$, we deduce

$$|F| \leq K, \quad K = \left\{(L_1 + (L_1 + L_2)M_4 BT) \| V \|_{Q_T}^{(\lambda,\lambda/2)} + \| \overline{\Lambda} \|_{Q_T}^{(\lambda,\lambda/2)}\right\} P_1.$$

We refer now to the maximum principle, applied to a problem of type (3.5) (see [5] theorem 2.3), and deduce that:

$$\max|Z| = \max \frac{|V(x,\tau,\alpha_1) - V(x,\tau,\alpha_2)|}{|\alpha_1-\alpha_2|^{\lambda/2}} \le$$

$$\le C_1\left\{\max|u_0^{(1)}-u_0^{(2)}|, \max|h^{(1)}-h^{(2)}|, \max|F|\right\}, \quad C_1 = \text{ const.}$$

Taking into account the hypotheses on u_0 and h,

$$\frac{|u_0^{(1)}-u_0^{(2)}|}{|\alpha_1-\alpha_2|^{\alpha/2}} \le P_2, \qquad \frac{|h^{(1)}-h^{(2)}|}{|\alpha_1-\alpha_2|^{\lambda/2}} \le P_2$$

and the result on F, we obtain

$$\sup \frac{|V(x,\tau,\alpha_2) - V(x,\tau,\alpha_1)|}{|\alpha_1-\alpha_2|^{\lambda/2}} \le P_3$$

which proves the statement.

It remains to prove the V has the same bounds that those of v in (3.3). Or, this is a direct consequence of the same above maximum principe, namely:

$$c_1(\min u_0, \min h) \le V \le c_1(\max u_0, \max h)e^{\overline{\Lambda}\tau},$$

and by a suitable choise of c_1, the result follows.

We conceive now (3.5) as a mapping $\mathcal{T}$ of the class $\mathcal{C}$ of functions into itself, since to each $v \in \mathcal{C}$, it corresponds $V \in \mathcal{C}$. We use Banach's fixed point theorem, in the space $H^{(\lambda,\lambda/2,\lambda/2)}(Q)$ with the metric induced by this norm, — this is a complete metric space — (see [2]).

As $\mathcal{T}$ transforms $\mathcal{C}$ into $\mathcal{C}$,it remains to prove that $\mathcal{T}$ is a contraction. In order to prove this assertion, we consider two functions $v^{(1)}$ and $v^{(2)}$ of $\mathcal{C}$, and denote by $V^{(i)}(i=1,2)$ the corresponding solutions of (3.5). Denote again:

$$Z = V^{(1)} - V^{(2)}, \quad Y = \overline{R}^{(1)} - \overline{R}^{(2)}.$$

The system which determines these functions is :

$$\frac{\partial Z}{\partial \tau} - A^2(v^{(1)}, \overline{R}^{(1)})\Delta Z + \overline{\Lambda}(x,\tau,\alpha,\overline{R}^{(1)})Z = G,$$

$$\text{(3.7)} \qquad Z|_{\tau=0} = 0, \quad \frac{\partial Z}{\partial n} + cZ|_S = 0,$$

$$Y = \int_0^\tau \left\{ \int_\sigma^B \nu(x,\sigma,\beta)(v^{(1)} - v^{(2)})d\beta \right\} d\sigma,$$

with

$$G = (A^2(u^{(1)}, \overline{R}^{(1)}) - A^2(v^{(2)}, \overline{R}^{(2)}))\Delta V^{(2)} - (\overline{\Lambda}^{(1)} - \overline{\Lambda}^{(2)})V^{(2)},$$

with analogous notations as in the previous paragraph; and using the same remarks on $V^{(2)}$, A^2, $\Delta V^{(2)}$, we get

$$|G| \le (L_1 + (L_1+L_2)M_2BT)P_1 \max|v^{(1)} - v^{(2)}|.$$

Obviously (3.7) satisfies the conditions of theorem A, from which it follows that it is has a unique solution satisfying

$$||Z||_{Q_T}^{(\lambda+2,\lambda/2+1)} \le k_1||G||_{Q_T}^{(\lambda,\lambda/2)}, \quad k_1 = a \text{ constant },$$

from which it results also:

$$||Z||_{Q_T}^{(\lambda,\lambda/2)} = ||V_1 - V_2||_{Q_T}^{(\lambda,\lambda/2)} \le K_1||v^1 - v^2||_{Q_T}^{(\lambda,\lambda/2)},$$

$$K_1 = k_1\left[(L_1 + (L_1+L_2)M_4BT)P_1\right]^\lambda.$$

Supposing

$$k_1[(L_1 + (L_1+L_2)m_4BT)P_1]^\lambda < 1,$$

it results that $\mathcal{T}$ is a contraction, and, as a consequence of Banach's fixed point theorem, it has a fixed point which is the unique solution of the system (2.1)-(2.5). Obviously this solution belongs to the class $\mathcal{C}$, i.e. it satisfies the system.

Second Step. The dependence of the solution of (3.1), on w. Consider two functions w, namely $w^{(1)}$, $w^{(2)}$, belonging to the Hölder class $H^{(\lambda,\lambda/2,\lambda/2)}(Q)$, denote again by $v^{(i)}(i = 1.2)$ the corresponding solutions of the system (2.1)−(2.5), then:

$$v^{(1)} - v^{(2)} = z, \qquad \overline{r}^{(1)} - \overline{r}^{(2)} = y,$$

and by the index i the coefficients of the system i. The above functions will satisfy:

$$\frac{\partial Z}{\partial \tau} - \Lambda^2(v^{(1)}, \overline{r}^{(1)})\,\Delta Z + \overline{\Lambda}^{(1)}Z = H,$$

$$Z|_{\tau=0} = 0,$$

$$\frac{\partial Z}{\partial n} + Z|_S = 0,$$

$$y(x,\tau) = \int_0^\tau \left\{ \int_0^\sigma \nu(x,\sigma,\beta)(w^{(1)} - w^{(2)})d\beta + \int_\sigma^B \nu(x,\sigma,\beta)zd\beta \right\} d\sigma,$$

with

$$H = (A^2(v^{(1)}, \overline{r}^{(1)} - A^2(v^{(2)}, \overline{r}^{(2)})\Delta v^2 - (\overline{\Lambda}^{(1)} - \overline{\Lambda}^{(2)}v^2.$$

After calculations analogous with those of the first step we get

$$|H| \le \left\{(L_1 + (L_1 + L_2)M_4BT)|Z| + (L_1 + L_2)M_4BT|w^{(1)} - w^{(2)}|\right\} P_1$$

and using theorem A, we obtain:

$$||v_1 - v_2||^{(\lambda,\lambda/2)}_{(Q_T)} = ||Z||^{(\lambda,\lambda/2)}_{(Q_T)} \le \left[(1-k_1)^{-1}(L_1 + L_2)M_4BTP_1\right]^{\lambda} ||w_1 - w_2||^{(\lambda,\lambda/2)}.$$

Again by a method used in the previous step, one can prove that

$$||v_1 - v_2||^{(\lambda,\lambda/2,\lambda/2)}_{Q_T} \le K_3||w_1 - w_2||^{(\lambda,\lambda/2,\lambda/2)}_{Q},$$

where the expresionn of K_3 can be easily deduced from the previous constants.

Third Step. We come now to the system (2.6)—(2.9)

$$\frac{\partial w}{\partial \alpha} - A^2(w, \bar{\bar{r}})\Delta w + \bar{\bar{\Lambda}}(x,\tau,\alpha,\bar{\bar{r}})w = 0, \quad \text{on } Q,$$

(3.9)
$$w(x,\tau,0) = \Psi(x,\tau), \quad \text{on } Q_T,$$

$$\frac{\partial w}{\partial n} + cw = h(x,\tau,\alpha), \text{ on } \partial\Omega \times [0,B] \text{ for each } t \in [0,T],$$

$$\bar{\bar{r}}(x,\tau) = \int_0^{\tau}\left\{\int_0^{\sigma} \nu(x,\tau,\beta)w(x,\sigma-\beta,\beta)d\beta + \right.$$

$$\left. + \int_{\sigma}^{B} \nu(x,\sigma,\beta)v(x,\sigma,\beta-\sigma)d\beta + g(x,\sigma)\right\}d\sigma + r_0(x),$$

where, for the moment, we suppose Ψ a given function satisfying on Q_T analogous conditions as u_0 on Q_A, i.e.

(3.10)
$$0 \le \Psi(x,\tau) < M_5,$$

$$\Psi \in H^{(\lambda+2,\lambda/2+1)}(Q_T),$$

and v is the solution of (2.1) – (2.5) corresponding to w.

We remark now that Ψ is the initial repartition of offspring, so that Ψ and b have the same meaning.

The system is analogous to (2.1) – (2.5), the only difference results from the fourth equation, from we deduces, with a notation which can be understood by itself

$$||\bar{\bar{r}}||^{(\lambda/2)}_{Q_T} \le M_4BT(||v||^{(\lambda,\lambda/2)}_{Q_T} + ||w||^{(\lambda,\lambda/2)}_{Q_T})$$

and, using (3.8):

$$||\bar{\bar{r}}^{(1)} - \bar{\bar{r}}^{(2)}||_{Q_T} \le M_4BT(1 + K_3)||w^{(1)} - w^{(2)}||_{Q_T}.$$

So, we can deduce that (3.9) admits a unique solution satisfying

(3.11)
$$||w||^{(\lambda,\lambda/2,\lambda/2)}_{Q} \le k_4(||\Psi||^{(\lambda+2,1+\lambda/2)}_{Q_T} + ||h||^{(\lambda,\lambda/2,\lambda/2)}_{S})$$

(like V in (3.4)).

(3.4′)
$$||v||^{(\lambda,\lambda/2,\lambda/2)}_{Q} \le c_2(||u_0||^{(\lambda+2,\lambda/2+1)}_{Q_A} + ||h||^{(\lambda+1,\lambda/2+1,\lambda/2+1)}_{S}).$$

Fourth Step. For that which will follow, we need the estimation of the difference of two solutions $w^{(1)}$, $w^{(2)}$ of (3.9), corresponding, respectively to functions $\Psi^{(1)}$, $\Psi^{(2)}$, which satisfy (3.10).

Denoting

$$w^{(1)} - w^{(2)} = W, \qquad \bar{\bar{r}}^1 - \bar{\bar{r}}^2 = \bar{\bar{R}},$$

we easily see that W and $\bar{\bar{R}}$ satisfy the system

$$\frac{\partial W}{\partial \alpha} - A^2(w^{(1)}, \quad \bar{\bar{r}}^{(1)})\Delta W + \bar{\bar{\Lambda}}(x,\tau,\alpha,\bar{\bar{r}}^{(1)})W = \Phi,$$

$$W(x,\tau,0) = \Psi^{(1)} - \Psi^{(2)},$$

$$\frac{\partial W}{\partial n} + CW = 0,$$

$$\bar{\bar{R}}(x,\tau) = \int_0^{\tau}\left\{\int_0^{\nu}(x,\sigma,\beta)W(x,\sigma-\beta,\beta)d\beta\right\}d\sigma,$$

where

$$\Phi = (A^2(w^{(1)},\bar{\bar{r}}^{(1)}) - A^2(w^{(2)},\bar{\bar{r}}^{(2)})\Delta w^{(2)} - (\bar{\bar{\Lambda}}(x,\tau,\alpha,\bar{\bar{r}}^{(1)}) - \bar{\bar{\Lambda}}(x,\tau,\alpha,\bar{\bar{r}}^{(2)})w_2,$$

and by a reasoning analogous with that of the previous steps,already some times used, we obtain

(3.12)
$$||w^{(1)} - w^{(2)}||^{(\lambda,\lambda/2,\lambda/2)} \le K_5(||\Psi^{(1)} - \Psi^{(2)}||)^{(\lambda,\lambda/2)},$$

K_5 being a constant, analogous with that from (3.8).

Fifth Step. We came finally to the equation

$$b(x,\tau)=\int_0^{\tau}\mu(x,\tau,\beta)w(x,\tau-\beta,\beta)d\beta+\int_{\tau}^{B}\mu(x,\tau,\beta)v(x,\tau,\beta-\tau)d\beta, \tag{3.13}$$

in which ν is the solution of the system (2.1)—(2.5) corresponding to w, and w is the solution of (3.9) corresponding to Ψ, and we must prove that it exists a function Ψ such that $\Psi=b$. To this end, we will use again Banach's fixed point theorem. Remark first that

$$\|b(x,t\|_{Q_T}^{(\lambda,\lambda/2)}\le M_3T\left(\|w\|_Q^{(\lambda,\lambda/2,\lambda/2)}+\|v\|_Q^{(\lambda,\lambda/2,\lambda/2)}\right), \tag{3.14}$$

and taking into account the estimates (3.11) and (3.4) for w and v, we get

$$\|b(x,t)\|_{Q_T}^{(\lambda,\lambda/2)}\le M_3T\left(K_4\|\Psi\|_{Q_T}^{(\lambda+1,(\lambda+1)/2)}+\|h\|_S^{(\lambda+1,(\lambda+1)/2,(\lambda+1)/2)}\right)+$$

$$+c_2\left(\|u_0\|_{Q_T}^{(\lambda+2,(\lambda+2)/2)}+\|h\|_S^{(\lambda+1,(\lambda+1)/2,(\lambda+1)/2)}\right),$$

and by imposing to u_0, h, T suitable, but obvious conditions, we deduce

$$\|b(x,t)\|_{Q_T}^{(\lambda,\lambda/2)}\le k'\|\Psi\|_{Q_T}^{(\lambda,\lambda/2)},$$

so that the operator defined by (3.13) maps the class of functions Ψ into itself.

To prove that this mapping is a contraction, we remark first, by an obvious notation

$$b_1-b_2=\int_0^{\tau}\mu(x,\tau,\beta)\left[w^{(1)}(x,\tau-\beta,\beta)-w^{(2)}(x,\tau-\beta,\beta)\right]d\beta+$$

$$+\int_{\tau}^{B}\mu(x,\tau,\beta)\left[v^{(1)}(x,\tau,\beta-\tau)-v^{(2)}(x,\tau,\beta-\tau)\right]d\beta.$$

Then using (3.8) and (3.12), we have

$$\|b_1-b_2\|_{Q_T}^{(\lambda,\lambda/2)}\le M_3T(1+K_3)K_5\|\Psi^{(1)}-\Psi^2\|_{Q_T}^{(\lambda,\lambda/2)},$$

and by a suitable choise of the constants (especially T) we deduce that the above operator is a contraction and this ends the proof of our theorem.

4. A Steady State Solution. We will suppose now, that Λ, μ, ν, g, h do not depend on t, and look for a solution of the system (1.1)—(1.7) independent on t. Moreover, we will suppose Λ dependent only on the age a, and A^2 of class C^1, so our system will be

$$\frac{\partial u}{\partial a}-A^2(u,r)\frac{\partial^2 u}{\partial x^2}+\Lambda(a)u=0,\qquad u=u(x,a), \tag{4.1$_1$}$$

$$b(x)=u(x,0)=\int_0^B\mu(x,a)u(x,a)da, \tag{4.1$_2$}$$

$$\frac{\partial u}{\partial n}+cu=h(x,a), \tag{4.1$_3$}$$

$$r=r_0(x), \tag{4.1$_4$}$$

$$0=\int_0^B\nu(x,a)u(x,a)da+g(x). \tag{4.1$_5$}$$

We will be concerned in this § with the existence of a solution of the above system, the stability of which will be investigated in the following one.

To this end, we consider first the system (4.1$_1$) — (4.1$_4$) in which we perform the substitution

$$u=v\exp\left(-\int_0^a\Lambda(\alpha)d\alpha\right).$$

We obtain for v the system :

$$\frac{\partial v}{\partial a}-A^2\left(vexp\left(-\int_0^a(\Lambda(\alpha)d\alpha\right),r\right)\Delta v=0,$$

$$v(x,0)=\varphi_1(x), \tag{4.2}$$

$$\frac{\partial v}{\partial n} + cv = h(x,a)\exp\left(-\int_0^a \Lambda(\alpha)d\alpha\right),$$

$$r = r_0(x),$$

(4.3) $$b(x) = \int_0^B \mu(x,\alpha)\exp\left(-\int_0^\alpha \Lambda(\beta)d\beta\right)v(x,\alpha)d\alpha,$$

$$b(x) = \varphi_1(x),$$

(4.4) $$\int_0^B \nu(x,\alpha)v(x,\alpha)\exp\left(-\int_0^\alpha \Lambda(\beta)d\beta\right)d\alpha + g(x) = 0,$$

in which $\varphi_1(x)$ is an auxiliary, for the moment arbitrary function, which satisfies the conditions

(4.5) $$m_1 \leq \varphi_1(x) \leq m_2, \qquad m_i = \text{ const .},$$

$$\varphi_1(x) \in H^{(\lambda+2)}(\Omega),$$

and r_0 is another function

(4.5′) $$r_0(x) \in H^{(\lambda+2)}(\Omega).$$

For the system (4.2) we consider the mixed problem: replacing it, by the set of systems:

$$\frac{\partial v^{(k)}}{\partial a} - A^2\left(v^{(k-1)}\exp\left(\int_0^a \Lambda(\alpha)d\alpha\right), \gamma_0\right)\Delta v^{(k)} = 0,$$

(4.6) $$v^{(k)}(x,0) = \varphi_1(x),$$

$$\frac{\partial v^{(k)}}{\partial n} + cv^{(k)} = h(x,a)\exp\left(-\int_0^a \Lambda(\alpha)d\alpha\right), \quad k = 1,2,\ldots$$

where $v^{(0)}$ is a function belonging to the class K defined by:

$$v^{(0)}(x,0) = \varphi_1(x),$$

$$\frac{\partial v^{(0)}}{\partial n} + cv^{(0)} = h(x,a)\exp\left(-\int_0^a \Lambda(\alpha)d\alpha\right),$$

$$v^{(0)} \in H^{(\lambda+2)}(\Omega),$$

$$||v^0||_{Q_A}^{(\lambda+2,\lambda/2+1)} \leq c_1||\varphi_1||_\Omega^{(\lambda+2)} + ||h||_{S_A}^{(\lambda+2,\lambda/2+1)};$$

c_1 is a constant which will be determined in the following.

The system (4.6) with $k = 1$, has a unique solution $v^{(1)}$ (see p.364), which is positive, and belongs to the same class K, with c_1 depending only on the coefficients of the equation, and not on φ_1 and h, the same c_1 is taken in (4.7). The positiveness of $v^{(1)}$ results from the fact that, as a consequence of the maximum principle, the values of $v^{(1)}$ lie in the interval

$$\left\{\min\left[\varphi_1(x), h(x,a)\exp\left(\int_0^a \Lambda(\alpha)d\alpha\right)\right], \max\left[\varphi_1(x), h(x,a)\exp(\Lambda(\alpha)d\alpha)\right]\right\}.$$

Iterating this method, we obtain the sequence $v^{(1)}, v^{(2)}, \ldots$ of positive functions, belonging to K.

For what follows, we need an estimate of the difference between two elements $v^{(k)}$ and $v^{(k-1)}$ of the above sequence. It is easily seen that this difference satisfies the system:

$$\frac{\partial(v^{(k)} - v^{(k-1)})}{\partial a} - A^2\left(v^{(k-1)}\exp\left(-\Lambda(\alpha)d\alpha\right), r_0\right)\Delta\left(v^{(k)} - v^{(k-1)}\right) =$$

$$= \left(\frac{\partial A^2}{\partial v}\right)^*\left(v^{(k-1)} - v^{(k-2)}\right)\Delta_v^{(k-1)},$$

$$v^{(k)}(x,0) - v^{(k-1)}(x,0) = 0,$$

$$\frac{\partial\left(v^{(k)}(x,0) - v^{(k-1)}(x,0)\right)}{\partial n} + c(v^{(k)}(v^{(k)}(x,0) - v^{(k-1)}(x,0)) = 0.$$

From the same theorem, already quoted, we have

(4.7)

$$||v^{(k)} - v^{(k-1)}||^{(\lambda+2,\lambda/2+1)} \leq c\left[\left(\frac{\partial A^2}{\partial v}\right)^*\Delta v^{(k-1)}\right]\cdot||v^{(k-1)} - v^{(k-2)}||^{(\lambda+2,\lambda/2+1)}.$$

As a consequence of the fact that $v^{(k-1)} \in K$, it results that $\Delta v^{(k-1)}$ is bounded, and that the bound depends only on the coefficients A^2 and Λ. Supposing satisfied the supplementary condition

$$c\left|\left(\frac{\partial A^2}{\partial u}\right)\right|\Delta v < 1,$$

it results that the sequence $\{v^{(k)}\}$ converges uniformly. The limit v is obviously the solution of (4.2).

Further we need an estimate of the difference between two solutions v' and v" of (4.2), corresponding to initial functions φ'_1, $\varphi"_1 \in K$. Denoting

$$U = (v' - v'')e^{\Gamma a}, \qquad \Gamma - \text{ constant },$$

this function satisfies:

$$\frac{\partial U}{\partial a} - A^2(v', r_0)\Delta U + \left(\Gamma - \left(\frac{\partial A^2}{\partial u}\right)^* \Delta v''\right) U = 0,$$

(4.9) $$U(x,0) = \varphi'_1 - \varphi''_1,$$

$$\frac{\partial U}{\partial n} + cU = 0.$$

If we choose $\Gamma > \left(\frac{\partial A^2}{\partial v}\right) \Delta v''$, from the maximum principle it follows

$$|U| = |v' - v''|e^{\Gamma_a} \le \max|\varphi'_1 - \varphi''_1|,$$

and finally

$$|v' - v''| \le \max|\varphi'_1, \varphi''_1|.$$

We come now to the system (4.3), where we conceive v as an operator on the space Φ of functions φ_1 defined by (4.5). Using Banach's fixed point theorem, and the above estimation of $v' - v$", we deduce that the transformation leading from φ_1 to b, transforms the space Φ into a set of functions b, so that:

$$M_1\overline{\mu}_1 \le M_1 \int_0^B \mu(x,a)da \le \int_0^B \mu(x,a)v(x,a)da \le M_2 \int_0^B \mu(x,a)da \le M_2\overline{\mu}_2,$$

where M_1 and M_2 are the lower ans upper bounds of v and $\overline{\mu}_1, \overline{\mu}_2$, are the lower and upper bounds of

$$\int_0^B \mu(x,a)da.$$

Now supposing that $\overline{\mu}_i$ satisfy:

(4.11) $$M_i\overline{\mu} = m_i,$$

it results that Φ is transformed into itself.

On the other hand, we have

$$|b' - b''| \le \int_0^B \mu(x,\alpha)|v' - v"|d\alpha \le \overline{\mu}_2 \max|\varphi'_1 - \varphi"_1|e^{\gamma}B.$$

We must also suppose a second restriction of μ, namely

(4.12) $$\overline{\mu}_2 e^{\Gamma B} < 1,$$

in order that the transformation (4.3) be a contraction.

It follows so that the system (4.2), (4.3) — and also $(4.1_1)-(4.1_4)$ has solution.

The uniqueness of this solutions results from the maximum principle, applied to a system analogous to (4.9) for the difference of two solutions, with the same initial and boundary values.

We can also state:

Theorem. *If $A^2 \in C^1$, and conditions (4.8) and (4.12) are satisfied, with $\Gamma > \left(\frac{\partial A^2}{\partial v}\right)^* \Delta v$, then system $(4.1_1) - (4.1_4)$ has a unique solution.*

The relation (4.1_5), must be considered as a compatibility condition implying the solution of the system $(4.1_1) - (4.1_4)$. Obviously, if this one is not satisfied then the system $(4.1_1) - (4.1_5)$ has no solution.

5. The stability of the steady state solution. We will suppose in this §, that the system $(4.1_1) - (4.1_5)$ has a unique solution $u \in H^{(\lambda+2,\lambda/2+1)}(Q_A)$, and consider a system

$$\frac{\partial U}{\partial t} + \frac{\partial U}{\partial a} - A^2(U,R)\Delta U + \Lambda(a)U = 0, \qquad U = U(x,t,a),$$

$$U(x,0,a) = u(x,a) + u_1(x,a),$$

$$\beta(x,t) = \int_0^B \mu(x,a)U(x,t,a)da,$$

$$\frac{\partial U}{\partial n} + cU = h(x,a),$$

(5.1) $$R(x,0) = r_0(x) + r_1(x),$$

$$\frac{dR}{dt} = \int_0^B \nu(x,a)U(x,t,a)da + g(x),$$

with the same coefficients as (4.1) — independent on t - and will prove, that — under certain conditions — the steady state solution is stable, i.e. given $\varepsilon > 0$, one can find δ so that, if,

(5.2) $$|u_1|, |r_1| < \delta,$$

then

(5.3) $$|U - u|, |R - r_1| < \varepsilon.$$

To prove this result, we put:

$$U(x,t,a) = u(x,a) + U_1(x,t,a),$$
$$R(x,t) = r_0(x) + R_1(x,t).$$

Tacking into account the fact that u and r_0 are the solutions of (4.1), it follows:

$$\frac{\partial U_1}{\partial t} + \frac{\partial u_1}{\partial_a} - A^2(u+U_1, r_0+R_1)\Delta U_1 + \Lambda(a)U_1 = \left[\left(\frac{\partial A^2}{\partial u}\right)^* U_1 + \left(\frac{\partial A^2}{\partial R}\right)^* R_1\right]\Delta u,$$

(5.4) $$U_1(x,0,a) = u_1(x,a),$$

$$\beta - b = \beta_1, \quad (x,t) = \int_0^B \mu(x,a)U_1(x,t,a)da,$$

$$\frac{\partial U_1}{\partial n} + cU_1 = 0,$$

$$\frac{dR_1}{dt} = \int_0^B \nu(x,a)U_1(x,t,a)da,$$

$$R_1(x,0) = r_1(x).$$

This system is analogous with that studied in §3. For $a \geq t$, by the intermediate of the substitution

$$a - t = \alpha, \quad t = \tau,$$

and using the already cited theorem 5.3, pag.364, [5], we deduce (with the notations of [2])

$$||U_1||^{(\lambda+2)} \leq \gamma||R||^{(\lambda)} + \beta||u_1||^{(\lambda+2)}, \qquad \gamma = c\Delta u\left(\frac{\partial A_1^2}{\partial R}\right)^*, \quad \beta = c_1$$

$$||R_1||^{(\lambda)} \leq m||U_1||^{(\lambda)} + ||r_1||^{(\lambda)}, \qquad m = \overline{\nu}BT, \quad \overline{\nu} = \max \nu,$$

$$\leq m||U_1||^{(\lambda+2)} + ||r_1||^{(\lambda+2)},$$

from which, supposing

(5.5) $$\gamma m < 1,$$

we obtain

$$||U_1||^{(\lambda+2)} \leq (1-\gamma m)^{-1}\left(\gamma||r_1||^{(\lambda+2)} + \beta||u_1||^{(\lambda+2)}\right),$$

$$||R_1||^{(\lambda)} \leq (1-\gamma m)^{-1}\left(||r_1||^{(\lambda+2)} + \beta m||u_1||^{(\lambda+2)}\right);$$

for $a < t$, using the substitution

$$t - a = \tau, \quad a = \alpha,$$

we obtain:

$$||U_1||^{(\lambda+2)} \leq ||R_1||^{(\lambda)} + \delta||\Psi_1||^{(\lambda+2)}, \quad \delta = c_2, \quad \Psi_1 = U_1(x,\tau,0)$$
$$||R_1||^{(\lambda)} \leq m||U_1||^{(\lambda+2)} + ||r_1||^{(\lambda+2)},$$
$$||\Psi_1||^{(\lambda+2)} \leq \overline{\mu}\left(\gamma||R_1||^{(\lambda)} + \delta||\psi_1||^{(\lambda+2)}\right) + \overline{\mu}\left(\gamma||R_1||^{(\lambda)} + \beta||u||^{(\lambda+2)}\right),$$

$$\overline{\mu} = \max\int_0^t \mu(x,a)da,$$

from which, supposing in addition

(5.6) $$m\gamma + \delta\overline{\mu} < 1,$$

we get

$$||\Psi||^{(\lambda+2)} \leq \overline{\mu}(1 - \delta\overline{\mu} - \gamma m)^{-1}\left(2\gamma||r_1||^{(\lambda+2)} + \beta||u_1||^{(\lambda+2)}\right),$$

$$||U_1||^{(\lambda+2)} \leq (1-\gamma m)^{-1}(1 - \delta\overline{\mu}\gamma - m\gamma)^{-1}$$
$$\left[\gamma(1 + \delta\overline{\mu} - \gamma m||r_1||^{(\lambda+2)} + \beta\delta\mu||u_1||^{(\lambda+2)}\right.$$

$$||R_1||^{(\lambda)} \leq (1-\gamma m)^{-1}(1 - \gamma m - \delta\overline{\mu})^{-1}$$
$$\left[(1 - \delta\overline{\mu} - \gamma m - 2m\gamma - \delta\overline{\mu})||r_1||^{(\lambda+2)} + m\beta\delta\mu||u_1||^{(\lambda+2)}\right]$$

from which the stated stability results immediately. So we can state.

Theorem D. *If conditions (5.5) and (5.6) are satisfied, the steady state solution of (4.1) is stable.*

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A NEHARI–TYPE RESULT FOR TIME–VARYING NODES

BY

A.HALANAY

Dedicated to Professor Viorel Barbu for his 50^{th} aniversary.

Introduction This paper contains essentially an extension to the case of time varying coefficients of some results of Glover [1], [2]. Given a time varying node $(A, B, C, D; X, U, Y)$ we consider, assuming an exponentially dichotomic evolution, input–output operators from $L^2(\mathbf{R}, U)$ to $L(\mathbf{R}, Y)$ and associate corresponding Hankel operators; it is proved that the norm of the Hankel operator equals the largest singular value, and then that this norm equals the distance form a node with exponentially stable evolution to the set of all nodes with antistable evolution. By duality a corresponing result holds for the distance from a node with antistable evolution to the set of all nodes with exponentially stable evolution. By using an idea of Jonckheere, Juang and Silverman [3], an explicit construction is given for a node with antistable evolution approximating a node with exponentially stable evolution.

2. Exponential dichotomy. Input–output operators. Hankel operators. Let $A : \mathbf{R} \to \mathcal{M}_{n\times n}(\mathbf{R})$, $B : \mathbf{R} \to \mathcal{M}_{n\times m}(\mathbf{R})$, $C : \mathbf{R} \to \mathcal{M}_{p\times n}(\mathbf{R})$, $D : \mathbf{R} \to \mathcal{M}_{p\times m}(\mathbf{R})$ be continous and bounded operators ; $\mathcal{M}_{k\times y}(\mathbf{R})$ denotes the set of matrices with k rows and j columns with real entries. With $X = \mathbf{R}^n$, $U = \mathbf{R}^m$, $Y = \mathbf{R}^p$, $(A, B, C, D; X, U, Y)$ will be called a time–varying node (see [3]). Associate the system

$$x' + A(t)x + B(t)u, \quad y = C(t)x + D(t)u$$

We shall denote by $X_A(\cdot,\cdot)$ the evolution operator associated to A defined by $X(s,s) = I$, $\frac{d}{dt}X_A(t,s) = A(t)X_A(t,s)$. The evolution defined by X_A is exponentially dichotomic if there exist a projection $\Pi^A_{t_0}$, $\beta \geq 1, \alpha > 0$, such that

$$|X_A(t,t_0)\Pi^A_{t_0}X_A(t_0,s)| \leq \beta e^{-\alpha(t-s)} \quad \text{for } t > s$$

$$|X_A(t,t_0)(I - \Pi^A_{t_0})X_A(t_0,s)| \leq \beta e^{-\alpha(s-t)} \quad \text{for } s > t.$$

If A defines an eponentially dischotomic evolution and $\Pi^A_{t_0} = I$ the evolution is exponentially stable, and if $\Pi^A_{t_0} = 0$, the evolution is called antistable. If A defines