Theorem D. *If conditions (5.5) and (5.6) are satisfied, the steady state solution of (4.1) is stable.*

REFERENCES

1. Iu. A. Dubinskij — *Slabaja Shodimosti v nelineinih ellipticeskih i paraboliceskih uravnenia*, Matematiceski Sbornik, T.67(109), (1965) p.609–642.
2. A. Friedman — *Partial Differential Equations of Parabolic Type*, Prentice Hall, 1964.
3. B.H. Gilding — *A non linear Parabolic Equation*, Ann. dela Sc. Normale Superiore di Pisa, Classe di Scientze, vol. IV serie IV, (1977) p. 393–432.
4. A. Haimovici — *A mathematical model of age structured population dynamics, with density dependent diffusion* (to appear).
5. O.A. Ladyshenskaja, V.A. Solonnikov, N.N. — *Ural'ceva, Lineinie i quasilineinie uravneniia, paraboliceskogo tipa*, Moskva, Nauka, 1967.
6. O.A. Oleainik, A.S. Kalasshniikov, Czhen You-Lin — *Zadaca Cauchy i kraevie zadaci dlea uravnenia type nestazionarnoi filtracii*, Izvestijy Akad. Nauk., Seria Mat. T. 22(1958), p.667–704.
7. P.A. Raviart — *Sur le résolution et l'approximation de certaines équations paraboliques nonlinéares dégénérées*, Arch.for Rat. Mechanics and Analysis, vol.25 (1967) p.64–80.

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Seminarul Matematic
"A. Myller"
6600 - Iaşi, România

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A NEHARI–TYPE RESULT FOR TIME–VARYING NODES

BY
A.HALANAY

Dedicated to Professor Viorel Barbu for his 50^{th} aniversary.

Introduction This paper contains essentially an extension to the case of time varying coefficients of some results of Glover [1], [2]. Given a time varying node $(A, B, C, D; X, U, Y)$ we consider, assuming an exponentially dichotomic evolution, input–output operators from $L^2(\mathbf{R}, U)$ to $L(\mathbf{R}, Y)$ and associate corresponding Hankel operators; it is proved that the norm of the Hankel operator equals the largest singular value, and then that this norm equals the distance form a node with exponentially stable evolution to the set of all nodes with antistable evolution. By duality a corresponing result holds for the distance from a node with antistable evolution to the set of all nodes with exponentially stable evolution. By using an idea of Jonckheere, Juang and Silverman [3], an explicit construction is given for a node with antistable evolution approximating a node with exponentially stable evolution.

2. Exponential dichotomy. Input–output operators. Hankel operators. Let $A : \mathbf{R} \to \mathcal{M}_{n\times n}(\mathbf{R})$, $B : \mathbf{R} \to \mathcal{M}_{n\times m}(\mathbf{R})$, $C : \mathbf{R} \to \mathcal{M}_{p\times n}(\mathbf{R})$, $D : \mathbf{R} \to \mathcal{M}_{p\times m}(\mathbf{R})$ be continous and bounded operators ; $\mathcal{M}_{k\times y}(\mathbf{R})$ denotes the set of matrices with k rows and j columns with real entries. With $X = \mathbf{R}^n$, $U = \mathbf{R}^m$, $Y = \mathbf{R}^p$, $(A, B, C, D; X, U, Y)$ will be called a time–varying node (see [3]). Associate the system

$$x' + A(t)x + B(t)u, \quad y = C(t)x + D(t)u$$

We shall denote by $X_A(\cdot,\cdot)$ the evolution operator associated to A defined by $X(s, s) = I$, $\frac{d}{dt}X_A(t, s) = A(t)X_A(t, s)$. The evolution defined by X_A is exponentially dichotomic if there exist a projection $\Pi^A_{t_0}$, $\beta \geq 1, \alpha > 0$, such that

$$|X_A(t, t_0)\Pi^A_{t_0}X_A(t_0, s)| \leq \beta e^{-\alpha(t-s)} \quad \text{for } t > s$$

$$|X_A(t, t_0)(I - \Pi^A_{t_0})X_A(t_0, s)| \leq \beta e^{-\alpha(s-t)} \quad \text{for } s > t.$$

If A defines an eponentially dischotomic evolution and $\Pi^A_{t_0} = I$ the evolution is exponentially stable, and if $\Pi^A_{t_0} = 0$, the evolution is called antistable. If A defines an exponentially dichotomic evolution we may associate an input–output operator from $L^2(\mathbf{R},U)$ to $L^2(\mathbf{R},Y)$

$$(G^{A,B,C,D})u(t) = D(t)u(t) + \int_{-\infty}^{t} C(t)X_A(t,t_0)\Pi^A_{t_0}X_A(t_0,s)B(s)u(s)ds -$$

$$-\int_t^{\infty} C(t)X_A(t,t_0)(I-\Pi^A_{t_0})X_A(t_0,s)B(s)u(s)ds.$$

If we write $L^2(\mathbf{R}) = L^2(-\infty,t_0)\oplus L^2(t_0,\infty)$, we associate the Hankel operator $\Gamma^{(A,B,C,D)}_{t_0} = P_2G^{(A,B,C,D)}P_1$ where P_1, P_2 are projections corresponding to the decomposition above. Remark that if the node has an antistable evolution, then $\Gamma^{(A,B,C,D)}_{t_0} = 0$. Given a node (A,B,C,D) with exponentially stable evolution and a node $(\widetilde{A},\widetilde{B},\widetilde{C},D)$ with antistable evolution, we shall have

$$\|G^{(A,B,C,D)} - G^{(\widetilde{A},\widetilde{B},\widetilde{C},D)}\| \geq \|G^{\left(\begin{pmatrix} A & 0 \\ 0 & \widetilde{A}\end{pmatrix},\begin{pmatrix} B \\ \widetilde{B}\end{pmatrix},(C-\widetilde{C}),0\right)}P_1\| \geq$$

$$\geq \Gamma_{t_0}^{\left(\begin{pmatrix} A & 0 \\ 0 & \widetilde{A}\end{pmatrix},\begin{pmatrix} B \\ \widetilde{B}\end{pmatrix},(C-\widetilde{C}),0\right)}\| = \|\Gamma^{(A,B,C,D)}_{t_0}\|$$

hence $\inf\{\|G^{(A,B,C,D)} - G^{(\widetilde{A},\widetilde{B},\widetilde{C},D)}\|, \widetilde{A}$ with antistable evolution $\} \geq \sup_{t_0}\|\Gamma^{(A,B,C,D)}_{t_0}\|$. The Nehari–type result asserts that we have in fact equality.

3. Hankel operators and singular values. Let $(A,B,C,D)_t$ be a node with exponentially stable evolution and define

$$P(t) = \int_{-\infty}^{t} X_A(t,s)B(s)B^*(s)X_A^*(t,s)ds$$

$$Q(t) = \int_t^{\infty} X_A^*(s,t)C^*(s)C(s)X_A(s,t)ds;$$

then $P(t)\geq 0$, $Q(t)\geq 0$, P,Q are bounded, and satisfy the Liapunov equations

$$P' = A(t)P + PA^*(t) + B(t)B^*(t)$$

$$Q' + QA(t) + A^*(t)Q + C^*(t)C(t) = 0.$$

It is proved in [5] that the nonzero eigenvalues of $[\Gamma^{(A,B,C,D)}_{t_0}]^*[\Gamma^{(A,B,C,D)}_{t_0}]$ are the nonzero eigenvalues of $P(t_0)Q(t_0)$, called the singular values (squared) of the node. From here we deduce the important for us fact that

$$\sup_{t_0}\|\Gamma^{(A,B,C,D)}_{t_0}\|^2 = \sup_{t_0}\gamma(P(t_0)Q(t_0))$$

where $r(M)$ means the spectral radius of M.

4. Antistable approximation of a stable node The following construction is inspired from [3].

Theorem. *Consider a node (A,B,C,D) with A defining an exponentially stable evolution and (A,B) uniformly controllable. Let P, Q be as above and assume that $\sup_{t_0} r(P(t_0)Q(t_0)) < \gamma^2$. Let $\widetilde{P}(t) = \frac{1}{\gamma^2}P(t)[I-\frac{1}{\gamma^2}Q(t)P(t)]^{-1}$; define $\widetilde{A} = -A^* + C^*C\widetilde{P}$, $\widetilde{B} = QB$, $\widetilde{C} = -C\widetilde{P}$. Then $\widetilde{A}$ defines an antistable evolution and $\|G^{(A,B,C,D)} - G^{(\widetilde{A},\widetilde{B},\widetilde{C},D)}\| \leq \gamma$.*

Proof. *a)* Because of the uniform controllablitity of (A,B), P is invertible with a bounded inverse and we may write $\widetilde{P} = (\gamma^2P^{-1}-Q)^{-1}$. To prove that $\widetilde{A}$ defines an antistable evolution we show that $-\widetilde{A}^* = A - \widetilde{P}C^*C$ defines an exponentially stable evolution. Denote $W = \widetilde{P}^{-1}$; we have $\nu I \leq W(t) \leq \mu I$, $\nu > 0$. Let x be a solution to $x' = (A-\widetilde{P}C^*C)x$; then a direct calculation gives

$$\frac{d}{dt}(x^\alpha Wx) = -x^\alpha C^\alpha Cx - \frac{1}{\gamma^2}x^\alpha(W+Q)BB^\alpha(W+Q)x$$

and from here it follows that $\int_\tau^\infty |C(s)x(s)|^2ds \leq \mu|x(\tau)|^2$. Since A defines an exponentially stable evolution and $-\widetilde{P}C^*Cx$ may be now viewed as an L^2–input, we deduce in a standard way that x is L^2 and $\int_\tau^\infty |x(s)|^2ds \leq \widetilde{\mu}|x(\tau)|^2$. Such property implies that $A - \widetilde{P}C^*C$ defines an exponentially stable evolution.

b) The node $(A,B,C,D) - (\widetilde{A},\widetilde{B},\widetilde{C},D)$ may be written as

$$(\mathcal{A},\mathcal{B},\mathcal{C},0),\qquad \mathcal{A} = \begin{pmatrix} A & 0 \\ 0 & -A^\alpha + C^\alpha C\widetilde{P}\end{pmatrix}$$

$$\mathcal{B} = \begin{pmatrix} B \\ BQ\end{pmatrix},\qquad \mathcal{C} = \begin{pmatrix} C & C\widetilde{P}\end{pmatrix}.$$

To this node we associate the Riccati equation

$$\mathcal{R}' + \mathcal{R}\mathcal{A} + \mathcal{A}^\alpha\mathcal{R} + \mathcal{C}^\alpha\mathcal{C} + \frac{1}{\gamma^2}\mathcal{R}\mathcal{B}\mathcal{B}^\alpha\mathcal{R} = 0$$

A direct calculation shows that this equation has the bounded on $\mathbf{R}$ solution $\mathcal{R} = \begin{pmatrix} Q & -I \\ -I & -\widetilde{P}\end{pmatrix}$.

A new calculation gives

$$\int_{-\infty}^{+\infty}\left|\left[G^{(A,B,C,D)}u - G^{(\widetilde{A},\widetilde{B},\widetilde{C},D)}u\right](t)\right|^2dt = \int_{-\infty}^{+\infty}\mathcal{X}^\alpha\mathcal{C}^\alpha\mathcal{C}\mathcal{X}dt =$$

$$= -\int_{-\infty}^{+\infty} \gamma^2 \left| \frac{1}{\gamma^2} B^\alpha \mathcal{R} \mathcal{X} - u \right|^2 dt + \gamma^2 \int_{-\infty}^{+\infty} u^\alpha u dt$$

hence $||G^{(A,B,C,D)} - G^{(\widetilde{A},\widetilde{B},\widetilde{C},D)}|| \leq \gamma^2$ and the theorm is proved.

The Nehari-type result follows directly from this theorem.

5. Stable approximation of an antistable node. For a given node (A,B,C,D) with A defining an exponentially dichotomic evolution we associate the adjoint node $(-A^*, -C^*, B^*, D^*)$ with the property that $G^{(-A^\alpha, -C^\alpha, B^\alpha, D^\alpha)} = [G^(A,B,C,D)]^*$. Let now (A,B,C,D) be a node with A defining an antistable evolution; associate the adjoint node and apply to it the construction in the Theorem. The corresponding Liapunov equations are $\widehat{P}' = -A^*\widehat{P} - \widehat{P}A + C^*C$, $\widehat{Q}' - Q\widehat{A}^* - A\widehat{Q} + BB^* = 0$, and a similar condition on the spectral radius corresponds to the fact that the norm of the Hankel operator defined as $\mathcal{H}_{t_0}^{(A,B,C)} = P_1 G^{(A,B,C,0)} P_2$ is less than γ. We take now, assumming that (C,A) is uniformly observable, $\widetilde{Q}(t) = \frac{1}{\gamma^2}\widehat{P}(t)[I - \frac{1}{\gamma^2}\widehat{Q}(t)\widehat{P}(t)]^{-1}$, $\widehat{A} = A + BB^*\widetilde{Q}, \widetilde{C} = -B^*\widetilde{Q}$; $\widehat{A}$ has an antistable evolution, $||[G^{(A,B,C,D)}]^* - G^{(\widehat{A},\widehat{B},\widehat{C},D^\alpha)}|| \leq \gamma$ and we deduce that $||G^{(A,B,C,D)} - G^{-\widetilde{A}^\alpha, -\widetilde{C}^\alpha, \widetilde{B}^\alpha, D)}|| \leq \gamma$.

The stable approximation of the antistable node (A,B,C,D) is $(-(A+BB^*\widetilde{Q})^*, \widetilde{Q}B, -C\widetilde{Q}, D)$.

6. Double coprime factorization, a family of stabilizing compensators and robust stabilization. Let $(A,B,C,0)$ be a node with (A,B) uniformly controllable and (C,A) uniformly observable, let F, H be bounded and such that $A - BF$, $A - HC$ define exponentially stable evolutions. Define the nodes $M = (A - BF, B, -F, I)$, $\widetilde{M} = (A - HC, H, -C, I)$, $N = (A - FB, B, C, 0)$, $\widetilde{N} = (A - HC, B, C, 0)$, $\widetilde{U} = (A - HC, H, F, 0)$, $U = (A - BF, H, F, 0)$, $\widetilde{V} = (A - HC, B, F, I)$, $V = (A - BF, H, C, I)$. We have the double coprime factorization

$$G^{(A,B,C,D)} = (N + DM)M^{-1} = \widetilde{M}^{-1}(\widetilde{N} + \widetilde{M}D),\ \widetilde{V}M + U\widetilde{N} = I,\ \widetilde{N}U + \widetilde{M}V = I.$$

The node $G_c = (\widetilde{V} + L\widetilde{N})^{-1}(L\widetilde{N} - \widetilde{U}) = (ML - U)(V + NL)^{-1}$ defines a family of stabilizing compensators for $G^{(A,B,C,0)}$, parametrized by the node L with exponentially stable evolution. Assume that A defines an antistable evolution and that stabilization was performed with $F = B^*\widehat{Q}^{-1}$ and $H = \widehat{P}^{-1}C^*$; then $M^* = M^{-1}$ and $\widetilde{M}^* = \widetilde{M}^{-1}$. We deduce that

$$||G_c(I - GG_c)^{-1}||^2 = ||M(L - M^{-1}U)\widetilde{M}||^2 =$$

$$= \sup \sigma[\widetilde{M}^{-1}(L - M^{-1}U)^*(L - M^{-1}U)\widetilde{M}] =$$

$$= \sup \sigma[(L - M^{-1}U)^*(L - M^{-1}U)] = ||L - M^{-1}U||^2.$$

A calculation shows that the node $M^{-1}U = M^*U$ is equivalent to $(-(A - BF)^*, (\widehat{P}\widehat{Q})^{-1}C^*, B^*, 0)$, which is with antistable evolution, while L must have an exponentially stable evolution. A new calculation shows that M^*U is equivalent to $(A, \widehat{P}^{-1}C^*, B^*\widehat{Q}^{-1}, 0)$. Adding a stable node does not change the Hankel operator and

$$\max_{t_0} ||\mathcal{H}_{t_0}^{L-M^*U}||^2 = 1/\min_{t_0} \min \sigma(\widehat{P}(t_0)\widehat{Q}(t_0));$$

if $\gamma^2 > 1/\min_{t_0} \min \sigma(\widehat{P}(t_0)\widehat{Q}(t_0))$ we may construct the stable approximation L such that $||L - M^*U|| \leq \gamma$. A calculation leads to $L = (-(A + HC(\gamma^2\widehat{Q}\widehat{P} - I)^{-1})^*, (\gamma^2\widehat{P}\widehat{Q} - I)^{-1}C^*, -B^*(\widehat{P}\widehat{Q}^{-1})$ and the corresponding compensator is defined by

$$\widetilde{z}' = (A - HC - \widehat{P}^{-1}(\gamma^2\widehat{P}\widehat{Q} - I)^{-1}C^*C)\widetilde{z} + \gamma^2\widehat{Q}(\gamma^2\widehat{P}\widehat{Q} - I)^{-1}C^*u$$

$$z' = (A - HC - BF)z + BB^*\widehat{Q}^{-1}\widetilde{z}, \quad y = u + B^*\widehat{Q}^{-1}z.$$

For a node G and a stabilizing compensator G_c stabilization is related to the properties of the nodes $(I - GG_c)^{-1}$ and $(I - G_cG)^{-1}$. If we are looking for perturbations ΔG such that $I - (G + \Delta G)G_c$ is still invertible, we deduce that if $||\Delta G|| < 1/||G_c(I - GG_c)^{-1}||$ the condition is satisfied. We may thus consider $1/||G_c(I - GG_c)^{-1}||$ as an upper bound of admissible error for G associated to G_c. A robust compensator will be one for which $||G_c(I - GG_c)^{-1}||$ is as small as possible. We have constructed a stabilizing compensator for which $||G_c(I - GG_c)^{-1}||$ is "almost minimal".

REFERENCES

1. Glover, K. — *All optimal Hankel-norm approximations of linear multivariable systems and their L^∞-error bounds.* International Journal of Control, 39 (1984); 1115–1193.
2. Glover, K. — *Robust stabilization of linear multivariable systems: Relations to approximation.* International Journal of Control , 43 (1986), 741–766.
3. Jonckheeare, E.A., Juang, J.C., Silverman, L.M. — *Spectral Theory of the Linear-Quadratic and H^∞- Problems,* Linear Algebra and its Applications, 122/123/124 (1989), 273–300.
4. Bart, H., Gohberg, I., Kashoek, M.A. — *Minimal factorization of matrix and operator funktions,* Birkhüser, Basel, Boston-Stuttgart, 1979.
5. Halanay, A. — *Advances is Control Theory and Riccati equations* (to appear in Rendiconti Seminario Matematico, Torino).

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Faculty of Mathematics
University of Bucharest
România