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PARALLELIZABLE GENERALIZED HAMILTON SPACES

BY

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Introduction. Recently R. Miron [2, 4] proposed the notion of generalized Hamilton space which is the dual notion of generalized Lagrange space. And also he have the notion of N-almost Hermitian model of the generalized Hamilton space. On the other hand, the author [7] introduced the notion of parallelizable generalized Lagrange space and gave the d-connection with some properties.

The purpose of the present paper is to investigate the parallelizable generalized Hamilton space which is the dual of parallelizable generalized Lagrange space. In §1 we give the preliminaries about the generalized Hamilton space. In §2 we investigate the d-parallelizable spaces and give the d-connection without curvature. In §3 we define the parallelizable generalized Hamilton space and give the metrical d-connection without curvature. In §4 we give some notions which are used in §5. Last section is devoted to some geometrical meaning of parallelizable generalized Hamilton space.

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§1. **Generalized Hamilton spaces ([2,4]).** Let M be a real C^∞-manifold of dimension n and $\pi^*: T^*M \longrightarrow M$ its cotangent bundle. If (U, x^i) is a local chart on M, then $(\pi^{*-1}(U), x^i, p_i)$ is a local chart on T^*M. A coordinate transformation on cotangent bundle T^*M is given by

$$(1.1) \qquad \bar{x}^i = \bar{x}^i(x^1, \ldots, x^n), \quad \bar{p}_i = \frac{\partial x^j}{\partial \bar{x}^i} p_j, \quad \det\left\| \frac{\partial \bar{x}^i}{\partial x^j} \right\| \neq 0.$$

The indices $i, j, k, \ldots$ run over the set $(1, \ldots, n)$ and Einstein convention of summation is adopted.

Let V be the vertical distribution on T^*M. Thus $\{\dot{\partial}^1, \ldots, \dot{\partial}^n\}$, $\dot{\partial}^i = \partial/\partial p_i$, is a local basis adapted to V. Denote by N a C^∞-distribution on T^*M supplementary to V. Then we have

$$T_u(T^*M) = N_u \oplus V_u, \quad \text{for any } u \in T^*M.$$

N is caled a non-linear connection on T^*M. We can give a local basis $\{\delta_1, \ldots, \delta_n\}$ adapted to the distribution N in the form

$$(1.2) \qquad \delta_i = \partial_i + N_{ji}(x,p)\,\dot{\partial}^j, \quad \partial_i = \partial/\partial x^i \quad (i = 1, \ldots, n)$$

The system of functions $N_{ji}(x,p)$ given on each domain of chart $(\pi^{*-1}(U), x^i, p_i)$ represents "the coefficients" of the non-linear connection N. With respect to (1.1) the coefficients $N_{ji}(x,p)$ of the non-linear connection N have the following transformation law:

$$(1.3) \qquad \overline{N}_{ji}(\overline{x},\overline{p}) = \frac{\partial x^k}{\partial \overline{x}^j}\frac{\partial x^h}{\partial \overline{x}^i} N_{kh}(x,p) + p_k \frac{\partial^2 x^k}{\partial \overline{x}^j \partial \overline{x}^i}.$$

It is known that when the base manifold M is paracompact, there exist non-linear connections on T^*M.

With respect to (1.1) we have

$$(1.4) \qquad \overline{\delta}_i = \frac{\partial x^j}{\partial \overline{x}^i}\delta_j, \quad \dot{\overline{\partial}}^i = \frac{\partial \overline{x}^i}{\partial x^j}\dot{\partial}^j.$$

Consequently $\{\delta_1, \ldots, \delta_n, \dot{\partial}^1, \ldots, \dot{\partial}^n\}$ is a local basis of the $\mathcal{F}(T^*M)$-module of vector fields $\mathcal{X}(T^*M)$, adapted to the distributions N and V. If we put

$$(1.5) \qquad \delta p_i = dp_i - N_{ij}(x,p)\,dx^j,$$

then $\{dx^1, \ldots, dx^n, \delta p_1, \ldots, \delta p_n\}$ gives the dual basis of the local basis $\{\delta_1, \ldots, \delta_n, \dot{\partial}^1, \ldots, \dot{\partial}^n\}$ and with respect to (1.1) we have

$$(1.6) \qquad d\overline{x}^i = \frac{\partial \overline{x}^i}{\partial x^j}dx^j, \qquad \delta\overline{p}_i = \frac{\partial x^j}{\partial \overline{x}^i}\delta p_j.$$

We call a d-connection on M a triple $H\Gamma(N) = (N_{ij}, H_j{}^i{}_k, C_j{}^{ik})$ formed by a non-linear connection $N_{ij}(x,p)$ and $H_j{}^i{}_k(x,p), C_j{}^{ik}(x,p)$ whose transformation laws with respect to (1.1) are given by

$$(1.7) \qquad \begin{aligned} \overline{H}_j{}^i{}_k(\overline{x},\overline{p}) &= \frac{\partial \overline{x}^i}{\partial x^r}\frac{\partial x^s}{\partial \overline{x}^j}\frac{\partial x^t}{\partial \overline{x}^k} H_s{}^r{}_t(x,p) + \frac{\partial \overline{x}^i}{\partial x^r}\frac{\partial^2 x^r}{\partial \overline{x}^j \partial \overline{x}^k}, \\ \overline{C}_j{}^{ik}(\overline{x},\overline{p}) &= \frac{\partial \overline{x}^i}{\partial x^r}\frac{\partial x^s}{\partial \overline{x}^j}\frac{\partial \overline{x}^k}{\partial x^t} C_s{}^{rt}(x,p). \end{aligned}$$

Then we can define the h- and v-covariant derivatives with respect to the d-connection $H\Gamma(N)$, e.g. for a d-tensor field $K_j^i(x,p)$ as follows:

$$(1.8) \quad K^i_{j|h} = \delta_h K^i_j + K^r_j H_r{}^i{}_h - K^i_r H_j{}^r{}_h, \qquad K^i_j|^h = \dot{\partial}^h K^i_j + K^r_j C_r{}^{ih} - K^i_r C_j{}^{rh}.$$

The Ricci identities contain the curvatures and torsions of the d-connection $H\Gamma(N)$:

$$(1.9) \qquad \begin{aligned} K^i_{j|k|h} - K^i_{j|h|k} &= K^r_j R_r{}^i{}_{kh} - K^i_r R_j{}^r{}_{kh} - K^i_{j|r} T_k{}^r{}_h - K^i_j|^r R_{rkh}, \\ K^i_{j|k}|^h - K^i_j|^h{}_{|k} &= K^r_j P_r{}^i{}_k{}^h - K^i_r P_j{}^r{}_k{}^h - K^i_{j|r} C_k{}^{rh} - K^i_j|^r P_{rk}{}^h, \\ K^i_j|^k|^h - K^i_j|^h|^k &= K^r_j S_r{}^{ikh} - K^i_r S_j{}^{rkh} - K^i_j|^r S_r{}^{kh}, \end{aligned}$$

where the torsion tensor fields are as follows:

$$(1.10) \qquad \begin{aligned} &T_j{}^i{}_k = H_j{}^i{}_k - H_k{}^i{}_j, \qquad S_j{}^{ik} = -(C_j{}^{ik} - C_j{}^{ki}), \\ &P_{jk}{}^i = -\left(\dot{\partial}^i N_{jk} - H_j{}^i{}_k\right), \qquad C_j{}^{ik}, \qquad R_{ijk} = \delta_j N_{ik} - \delta_k N_{ij}, \end{aligned}$$

and the curvature tensor fields are defined by

$$(1.11) \qquad \begin{aligned} R_j{}^i{}_{kh} &= \delta_h H_j{}^i{}_k - \delta_k H_j{}^i{}_h + H_j{}^r{}_h H^i_{rk} + C_j{}^{ir} R_{rkh}, \\ P_j{}^i{}_k{}^h &= \dot{\partial}^h H_j{}^i{}_k - C^{ih}_{j|k} + C_j{}^{ir} P_{rk}{}^h, \\ S_j{}^{ikh} &= \dot{\partial}^h C_j{}^{ik} - \dot{\partial}^k C_j{}^{ih} + C_j{}^{rk} C_r{}^{ih} - C_j{}^{rh} C_r{}^{ik}. \end{aligned}$$

There exist eleven Bianchi identities for $H\Gamma(N)$.

A d-tensor field $g^{ij}(x,p)$ on T^*M of type (2,0) which is symmetric and non-degenerate is called a fundamental tensor (or metric structure) of the manifold M. The pair $M^{*n} = (M, g^{ij}(x,p))$ is called a generalized Hamilton space.

If there exists a C^∞-function $H: T^*M \longrightarrow R$ such that $g^{ij}(x,p) = (1/2)\,\partial^2 H/\partial p_i \partial p_j$ is non-degenerate, then $H^n = (M, H(x,p))$ is called a Hamilton space. In the case where H is positively homogeneous of degree 2 with respect to p_i, H^n is a Cartan space C^n. If the fundamental tensor field $g^{ij}(x,p)$ does not depend on p_i, C^n is a Riemannian space V^n. The inclusion relation among these spaces is as follows:

$$\{V^n\} \subset \{C^n\} \subset \{H^n\} \subset \{M^{*n}\}$$

Let $g_{ij}(x,p)$ be the d-tensor field given by $g^{ij}(x,p)\,g_{jh}(x,p) = \delta^i_k$. A d-connection $H\Gamma(N)$ is called metric with respect to the fundamental tensor $g^{ij}(x,p)$ if

$$(1.12) \qquad g^{ij}{}_{|k} = 0, \qquad g^{ij}|^k = 0$$

The following remarkable theorems were proved by R. Miron [2,4].

Theorem 1.1. *There exists a unique metric d-connection* $C\Gamma(N) = (\overset{c}{H}_j{}^i{}_k, \overset{c}{C}_j{}^{ik})$, *for a fixed non-linear connection* N, *having the property* $\overset{c}{T}_j{}^i{}_k = \overset{c}{S}_j{}^{ik} = 0$. *This d-connection is given by the "Christoffel symbols":*

$$(1.13) \qquad \begin{aligned} \overset{c}{H}_j{}^i{}_k &= \frac{1}{2} g^{ih}(\delta_k g_{hj} + \delta_j g_{hk} - \delta_h g_{jk}), \\ \overset{c}{C}_j{}^{ik} &= -\frac{1}{2} g_{jh}\left(\dot{\partial}^k g^{hi} + \dot{\partial}^i g^{hk} - \dot{\partial}^h g^{ik}\right). \end{aligned}$$

We call this d-connection $C\Gamma(N) = \left(\overset{c}{H}_j{}^i{}_k, \overset{c}{C}_j{}^{ik}\right)$ the N-canonical metric d-connection.

Theorem 1.2. *There exists a unique metric d-connection $H\Gamma(N) = (H_j{}^i{}_k, C_j^{ik})$, for a fixed non-linear connection N, whose torsion tensor fields $T_j{}^i{}_k$ and $S_j{}^{ik}$ are given a priori. This d-connection is given by*

$$\begin{aligned} H_j{}^i{}_k &= \overset{c}{H}_j{}^i{}_k + \frac{1}{2} g^{ir}\left(g_{rh}T_j{}^h{}_k - g_{jh}T_r{}^h{}_k + g_{kh}T_j{}^h{}_r\right), \\ C_j{}^{ik} &= \overset{c}{C}_j{}^{ik} - \frac{1}{2} g_{jr}\left(g^{rh}S_h{}^{ik} - g^{ih}S_h{}^{rk} + g^{kh}S_h{}^{ir}\right), \end{aligned} \tag{1.14}$$

where $C\Gamma(N) = \left(\overset{c}{H}_j{}^i{}_k, \overset{c}{C}_j{}^{ik}\right)$ is the N-canonical metric d-connection.

The relations between the two d-connection $H\Gamma(N) = (N_{ij}, H_j{}^i{}_k, C_j{}^{ik})$ and $H\overset{\circ}{\Gamma}\left(\overset{\circ}{N}\right) = \left(\overset{\circ}{N}_{ij}, \overset{\circ}{H}_j{}^i{}_k, \overset{\circ}{C}_j{}^{ik}\right)$ are given by the following equations:

$$N_{ij} = \overset{\circ}{N}_{ij} + A_{ij}, \quad H_j{}^i{}_k = \overset{\circ}{H}_j{}^i{}_k + \overset{\circ}{C}_j{}^{im}A_{mk} + B_j{}^i{}_k, \quad C_j{}^{ik} = \overset{\circ}{C}_j{}^{ik} + D_j{}^{ik}, \tag{1.15}$$

where $A_{ij}(x,p), B_j{}^i{}_k(x,p)$ and $D_j{}^{ik}(x,p)$ are d-tensor fields.

§2. d-Parallelizable spaces.

Let M be an n-dimensional differentiable manifold and T^*M its cotangent bundle.

Definition 2.1. *M is called a d-parallelizable space if there exist n globaly linearly independent d-covector fields on M.*

In the d-parallelizable space we have n globally linearly independent d-covector fields $Z_i^1(x,p), \ldots, Z_i^n(x,p)$, thus $\det\|Z_i^\alpha\| \neq 0$. Hence we have the inverse matrix of $\|Z_i^\alpha\|$, which is denoted by $\|Z_\alpha^i\|$ that is,

$$Z_i^\alpha Z_\beta^i = \delta_\beta^\alpha, \qquad Z_j^\alpha Z_\alpha^i = \delta_j^i. \tag{2.1}$$

Remark 2.1. Because p_i is a non-zero d-covector field on M, we always take $Z_i^1 = p_i$.

Theorem 2.1. *In a d-parallelizable space $\{M, Z_i^\alpha\}$, we fix the d-connection $H\overset{\circ}{\Gamma}\left(\overset{\circ}{N}\right) = \left(\overset{\circ}{N}_{ij}, \overset{\circ}{H}_j{}^i{}_k, \overset{\circ}{C}_j{}^{ik}\right)$. Then any d-connection $H\Gamma(N) = (N_{ij}, H_j{}^i{}_k, C_j{}^{ik})$ with the property:*

(A) *Z_i^α $(\alpha = 1, \ldots, n)$ are parallel with respect to $H\Gamma(N)$, that is, $Z^\alpha_{i|k} = 0$ and $Z_i^\alpha|^k = 0$, is given by*

$$\begin{aligned} N_{ij} &= \overset{\circ}{N}_{ij} + X_{ij}, \quad H_j{}^i{}_k = \overset{\circ}{H}_j{}^i{}_k + Z_\alpha^i Z^\alpha_{\overset{\circ}{j|k}} + X_{mk}\left(\overset{\circ}{C}_j{}^{im} + Z_\alpha^i Z_j^\alpha \overset{\circ}{|}{}^m\right), \\ C_j{}^{ik} &= \overset{\circ}{C}_j{}^{ik} + Z_\alpha^i Z_j^\alpha \overset{\circ}{|}{}^k, \end{aligned} \tag{2.2}$$

where $\overset{\circ}{|}$ and $\overset{\circ}{|}$ denote the h- and v-covariant derivatives with respect to $H\overset{\circ}{\Gamma}\left(\overset{\circ}{N}\right)$ and X_{ij} is an arbitrary d-tensor field.

Proof. From (1.15) we have

$$Z^\alpha_{j|k} = Z^\alpha_{\overset{\circ}{j|k}} + A_{mk}Z_j^\alpha \overset{\circ}{|}{}^m - B_j{}^m{}_k Z_m^\alpha = 0, \qquad Z_j^\alpha|^k = Z_j^\alpha \overset{\circ}{|}{}^k - D_j{}^{mk}Z_m^\alpha = 0.$$

Hence, contracting the above two equations with Z_α^i and using (2.1) we get

$$B_j{}^i{}_k = Z_\alpha^i Z^\alpha_{\overset{\circ}{j|k}} + A_{mk}Z_\alpha^i Z_j^\alpha \overset{\circ}{|}{}^m, \qquad D_j{}^{ik} = Z_\alpha^i Z_j^\alpha \overset{\circ}{|}{}^k.$$

Substituting these equations into (1.15) with $A_{ij} = X_{ij}$ we obtain (2.2).

If we fix the non-linear connection N we have

Theorem 2.2. *In a d-parallelizable space $\{M, Z_i^\alpha\}$ we fix the d-connection $H\overset{\circ}{\Gamma}(N) = \left(\overset{\circ}{H}_j{}^i{}_k, \overset{\circ}{C}_j{}^{ik}\right)$. Then the d-connection $H\Gamma(N) = (H_j{}^i{}_k, C_j{}^{ik})$ with the property (A) is given by*

$$H_j{}^i{}_k = \overset{\circ}{H}_j{}^i{}_k + Z_\alpha^i Z^\alpha_{\overset{\circ}{j|k}}, \qquad C_j{}^{ik} = \overset{\circ}{C}_j{}^{ik} + Z_\alpha^i Z_j^\alpha \overset{\circ}{|}{}^k. \tag{2.3}$$

Theorem 2.3. *All the d-connection $H\Gamma(N)$ [(2.2), (2.3)] have the properties:*
1°. $R_j{}^i{}_{kh} = 0$, $P_j{}^i{}_k{}^h = 0$, $S_j^{ikh} = 0$,
2°. $D_{ij} = 0$, $\delta_j^i - C_j{}^{hi}p_h = 0$, where $D_{ij} = N_{ij} - p_m H_i{}^m{}_j$ is the deflection tensor field of $H\Gamma(N)$.

Proof. From the Ricci identities of $H\Gamma(N)$ written for Z_i^α, we can easily obtain by $Z^\alpha_{i|k} = 0$ that $R_j{}^i{}_{kh} = P_j{}^i{}_k{}^h = S_j^{ikh} = 0$. The second property 2° is obvious from $Z_i^1 = p_i$ and $p_{i|j} = D_{ij} = 0$, $p_i|^k = \delta_i^k - C_i^{hk}p_h = 0$.

We can find the torsion tensor fields and the Bianchi identities of these d-connections [(2.2, (2.3)].

§3. Parallelizable generalized Hamilton spaces.

Let $M^{*n} = (M, g^{ij}(x,p))$ be a generalized Hamilton space of dimension n.

Definition 3.1. *A generalized Hamilton space $M^{*n} = (M, g^{ij})$ is called a parallelizable generalized Hamilton space if M is d-parallelizable.*

Remark 3.1. From Definition 3.1, Theorems 2.1 – 2.3 are also valid in the parallelizable generalized Hamilton space.

Proposition 3.1. *If there exist $n-1$ globally linearly independent d–covector fields in the generalized Hamilton space M^{*n}, then M^{*n} is parallelizable.*

Proof. The so–called ε–tensor is defined by $\varepsilon_{ij\dots k} = (\sqrt{g})^{-1}\delta^{12\dots n}_{ij\dots k}$, where $\delta^{12\dots n}_{ij\dots k}$ is a generalized Kronecker delta and $g = \det\|g^{ij}\|$. When g is negative, $-g$ may be used, instead of g.

By assumption, we have $n-1$ globally linearly independent d–covector fields $Z^\alpha_i, \alpha = 1, \dots, n-1$, on M^{*n}, then d–covector Z^n_i given by

$$Z^n_k = \varepsilon_{hi\dots jk} Z^1_r g^{rh} Z^2_s g^{si} \dots Z^{n-1}_t g^{tj}$$

is orthogonal to $Z^\alpha_i, \alpha = 1, \dots, n-1$. Hence the proof is complete.

From Proposition 3.1 we obtain

Proposition 3.2. *Every 2–dimensional generalized Hamilton space M^{*2} is parallelizable.*

If M^{*n} has a positive definite fundamental tensor g^{ij}, then we can apply the Gram–Schmidt method for orthonormalization of the d-covectors $\{Z^1_i, \dots, Z^n_i\}$ by means of g^{ij}. So we get

Theorem 3.1. *If a parallelizable generalized Hamilton space M^{*n} has a positive definite fundamental tensor g^{ij}, then there exists an orthonormal d–coframe $\mathcal{R} = \{(x,p) : e^1_i, \dots, e^n_i\}$, globally defined on M^{*n}.*

Obviously we have

$$g^{ij} e^\alpha_i e^\beta_j = \delta^{\alpha\beta}. \tag{3.1}$$

We denote the inverse matrix of $\|e^\alpha_i\|$ by $\|e^i_\alpha\|$, that is,

$$e^\alpha_i e^i_\beta = \delta^\alpha_\beta, \qquad e^\alpha_i e^j_\alpha = \delta^j_i. \tag{3.2}$$

The orthonormality condition (3.1) of e^α_i gives us

Proposition 3.3. *Let $H\overset{\circ}{\Gamma}\left(\overset{\circ}{N}\right) = \left(\overset{\circ}{N}_{ij}, \overset{\circ}{H}_j{}^i{}_k, \overset{\circ}{C}_j{}^{ik}\right)$ be a fixed d– connection. Then we have*

$$g^{ij}{}_{\overset{\circ}{|}k} + e^i_\alpha e^\alpha_{h\overset{\circ}{|}k} g^{hj} + e^j_\alpha e^\alpha_{h\overset{\circ}{|}k} g^{ih} = 0, \quad g^{ij}\overset{\circ}{|}{}^k + e^i_\alpha e^\alpha_h \overset{\circ}{|}{}^k g^{hj} + e^j_\alpha e^\alpha_h \overset{\circ}{|}{}^k g^{ih} = 0. \tag{3.3}$$

Proof. From (3.1) and (3.2) we get $g^{ij} e^\alpha_i = \delta^{\alpha\beta} e^j_\beta$ and $g^{ij} = \delta^{\alpha\beta} e^i_\alpha e^j_\beta$. Hence we get

$$\begin{aligned} g^{ij}{}_{\overset{\circ}{|}k} &= \delta^{\alpha\beta} e^i_{\alpha\overset{\circ}{|}k} e^j_\beta + \delta^{\alpha\beta} e^i_\alpha e^j_{\beta\overset{\circ}{|}k} = e^i_{\alpha\overset{\circ}{|}k} g^{hj} e^\alpha_h + e^j_{\beta\overset{\circ}{|}k} g^{ih} e^\beta_h = \\ &= -e^i_\alpha e^\alpha_{h\overset{\circ}{|}k} g^{hj} - e^j_\alpha e^\alpha_{h\overset{\circ}{|}k} g^{ih}. \end{aligned}$$

The second equation of (3.3)is obtain in the same way as above.

Theorem 3.2. *In a parallelizable generalized Hamilton space M^{*n} with a positive definite fundamental tensor g^{ij} we fix the d–connection $H\overset{\circ}{\Gamma}\left(\overset{\circ}{N}\right) = \left(\overset{\circ}{N}_{ij}, \overset{\circ}{H}_j{}^i{}_k, \overset{\circ}{C}_j{}^{ik}\right)$. Then any d–connection $H\Gamma(N) = (N_{ij}, H_j{}^i{}_k, C_j{}^{ik})$ with the properties:*

(B) e^α_i $(\alpha = 1, \dots, n)$ are parallel with respect to $H\Gamma(N)$, that is, $e^\alpha_{i|k} = 0$, $e^\alpha_i|^k = 0$,

(C) $H\Gamma(N)$ is metric with respect to g^{ij}(i.e., $g^{ij}{}_{|k} = 0$, $g^{ij}|^k = 0$), is given by

$$\begin{aligned} N_{ij} &= \overset{\circ}{N}_{ij} + X_{ij}, \qquad H_j{}^i{}_k = \overset{\circ}{H}_j{}^i{}_k + e^i_\alpha e^\alpha_{j\overset{\circ}{|}k} + X_{mk}\left(\overset{\circ}{C}_j{}^{im} + e^i_\alpha e^\alpha_j \overset{\circ}{|}{}^m\right), \\ C_j{}^{ik} &= \overset{\circ}{C}_j{}^{ik} + e^i_\alpha e^\alpha_j \overset{\circ}{|}{}^k, \end{aligned} \tag{3.4}$$

Proof. From Remark 3.1 we must check only the metric property (C) By a straight forward calculation we have

$$\begin{aligned} g^{ij}{}_{|k} = g^{ij}{}_{\overset{\circ}{|}k} &+ e^i_\alpha e^\alpha_{h\overset{\circ}{|}k} g^{hj} + e^j_\alpha e^\alpha_{h\overset{\circ}{|}k} g^{ih} + X_{mk}\Big(g^{ij}\overset{\circ}{|}{}^m + \\ &+ e^i_\alpha e^\alpha_h \overset{\circ}{|}{}^m g^{hj} + e^j_\alpha e^\alpha_h \overset{\circ}{|}{}^m g^{ih}\Big), \end{aligned}$$

$$g^{ij}|^k = g^{ij}\overset{\circ}{|}{}^k + e^i_\alpha e^\alpha_h \overset{\circ}{|}{}^k g^{hj} + e^j_\alpha e^\alpha_h \overset{\circ}{|}{}^k g^{ih}.$$

Hence, Proposition 3.3 shows us $g^{ij}_{|k} = 0$ and $g^{ij}|^k = 0$.

If we fix the non–linear connection N we have

Theorem 3.3. *In a parallelizable generalized Hamilton space $M^{*n}$with a positive definite fundamental tensor g^{ij} we fix the d–connection $H\overset{\circ}{\Gamma}(N) = \left(\overset{\circ}{H}_j{}^i{}_k, \overset{\circ}{C}_j{}^{ik}\right)$. Then the d– connection $H\Gamma(N) = (H_j{}^i{}_k, C_j^{ik})$ with the properties (B) and (C) is given by*

$$H_j{}^i{}_k = \overset{\circ}{H}_j{}^i{}_k + e^i_\alpha e^\alpha_{j\overset{\circ}{|}k}, \qquad C_j{}^{ik} = \overset{\circ}{C}_j{}^{ik} + e^i_\alpha e^\alpha_j \overset{\circ}{|}{}^k. \tag{3.5}$$

§4. Linear connection on T^*M ([2,4]). Let M be an n–dimensional differentiable manifold and T^*M its cotangent bundle. We consider a fixed non–linear connection N. Then we have the distributions N and V and $\{\delta_i, \dot\partial^i\}$ is a local basis on $\mathcal{X}(T^*M)$ adapted to the distributions N and V.

On the total space T^*M we consider the 2-form:

$$\theta = \delta p_i \wedge dx^i. \tag{4.1}$$

Then θ is globally defined on T^*M and is an almost symplectic structure on T^*M associated to the non-linear connection N.

The tensor field $\widetilde{\theta}$ of θ has the following components in the adapted basis $\{\delta_i, \dot{\partial}^i\}$:

$$\widetilde{\theta}(\delta_i, \delta_j) = 0 \quad \widetilde{\theta}\left(\delta_i, \dot{\partial}^j\right) = -\delta_i^j, \quad \widetilde{\theta}\left(\dot{\partial}^j, \delta_i\right) \delta_i^j, \quad \widetilde{\theta}\left(\dot{\partial}^i, \dot{\partial}^j\right) = 0. \tag{4.2}$$

Definition 4.1. A linear connection ∇ on the total space T^*M of the cotangent bundle of M is called an N-connection of Hamilton type if

1°. ∇ preserves by parallelism each of the distributions N and V.

2°. $\widetilde{\theta}$ is covariant constant with respect to ∇.

Theorem 4.1. *A linear connection ∇ on the cotangent bundle T^*M is an N-connection of Hamilton type if and only if in the adapted basis $\{\delta_i, \dot{\partial}^i\}$, it is of the form:*

$$\begin{aligned} \nabla_{\delta_k}\delta_j &= H_j{}^i{}_k \delta_i, & \nabla_{\delta_k}\dot{\partial}^i &= -H_j{}^i{}_k \dot{\partial}^j, \\ \nabla_{\dot{\partial}^k}\delta_i &= C_i{}^{jk}\delta_j, & \nabla_{\dot{\partial}^k}\dot{\partial}^j &= -C_i{}^{jk}\dot{\partial}^i, \end{aligned} \tag{4.3}$$

where $(H_j{}^i{}_k, C_i{}^{jk})$ are the coefficients of a d-connection $H\Gamma(N)$ on M.

It follows that we have a bijective mapping between the set of the d-connections $\{H\Gamma(N)\}$ on M and the set of the N- connections of Hamilton type $\{\nabla\}$ on T^*M. For this reason, the N-connection of Hamilton type ∇ which corresponds to the d-connection $H\Gamma(N)$ is called the lift to T^*M of the d-connection $H\Gamma(N)$.

The torsion T and curvature R of the lift ∇ of the d- connection $H\Gamma(N)$ in the adapted basis $\{\delta_i, \dot{\partial}^i\}$ are expressed by

$$\begin{aligned} T(\delta_k, \delta_j) = T_j{}^i{}_k\delta_i + R_{ijk}\dot{\partial}^i, \quad T\left(\dot{\partial}^k, \delta_i\right) = C_i{}^{jk}\delta_j + P^k_{ji}\dot{\partial}^j, \\ T\left(\dot{\partial}^k, \dot{\partial}^j\right) = S_i{}^{jk}\dot{\partial}^i, \end{aligned} \tag{4.4}$$

$$\begin{aligned} R(\delta_h, \delta_k)\delta_j &= R_j{}^i{}_{kh}\delta_i, & R(\delta_h, \delta_k)\dot{\partial}^i &= -R_j{}^i{}_{kh}\dot{\partial}^j, \\ R\left(\dot{\partial}^h, \delta_k\right)\delta_j &= P_j{}^i{}_k{}^h\delta_i, & R\left(\dot{\partial}^h, \delta_k\right)\dot{\partial}^i &= -P_j{}^i{}_k{}^h\dot{\partial}^j, \\ R\left(\dot{\partial}^h, \dot{\partial}^k\right)\delta_j &= S_j{}^{ikh}\delta_i, & R\left(\dot{\partial}^h, \dot{\partial}^k\right)\dot{\partial}^i &= -S_j{}^{ikh}\dot{\partial}^j. \end{aligned} \tag{4.5}$$

Next we consider a generalized Hamilton space $M^{*n} = (M, g^{ij}(x,p))$ and a fixed non-linear connection N on T^*M. The fundamental tensor $g^{ij}(x,p)$ determine on the total space T^*M the tensor field:

$$J(x,p) = g_{ij}(x,p)\,\dot{\partial}^i \otimes dx^j - g^{ij}(x,p)\,\delta_i \otimes \delta p_j. \tag{4.6}$$

Then J is a tensor field of type (1,1) globally defined on T^*M and is an almost complex structure on T^*M. We have

$$J(\delta_i) = g_{ij}\dot{\partial}^j, \qquad J\left(\dot{\partial}^i\right) = -g^{ij}\delta_j, \qquad J \circ J = -I, \tag{4.7}$$

where I is the identity.

The fundamental tensor $g^{ij}(x,p)$ determines the Riemanian metric on T^*M:

$$G(x,p) = g_{ij}(x,p)\,dx^i \otimes dx^j + g^{ij}(x,p)\,\delta p_i \otimes \delta p_j. \tag{4.8}$$

The metric $G(x,p)$ will be called the N-lift of the metrical structure $g^{ij}(x,p)$ of M^{*n}.

Proposition 4.1. *The distributions N and V are orthogonal with respect to the metric $G(x,p)$.*

Theorem 4.2. 1°. *The pair (G, J) is an almost Hermitian structure on T^*M.*

2°. $\theta = \delta p_i \wedge dx^i$ is the associated almost symplectic structure to (G, J).

Theorem 4.3. *If $G(x,p)$ is a Riemannian metric on T^*M, N is the supplementary orthogonal distribution to the vertical distribution V with respect to G and (G, J) is the almost Hermitian structure on T^*M for which $\theta = \delta p_i \wedge dx^i$ is associated almost symlectic structure, then there exists a unique metric structure $g^{ij}(x,p)$ on M such that G is its N-lift.*

Let $H^{2n} = (T^*M, (G, J))$ be an almost Hermitian space determinated by the structure (G, J).

Theorem 4.3 shows that if the non-linear connection N is fixed, there exists a bijective correspondence between the set of the spaces M^{*n} and the set of the space H^{2n}. Therefore H^{2n} is called the N-almost Hermitian model of the generalized Hamilton space M^{*n}. Shortly we say "H^{2n} is the N-model of M^{*n}".

Definition 4.2. *A linear connection ∇ on T^*M is called an N-almost Hermitian connection if*

1°. ∇ is an N-connection of Hamilton type on T^*M,

2°. ∇ is a metric connection with respect to the N-lift $G(x,p)$ of the metric structure g^{ij} of M^{*n}.

Proposition 4.2. *If ∇ is an N–almost Hermitian connection on T^*M, then we have $\nabla_X J = 0$ for all $X \in \mathcal{X}(T^*M)$.*

Theorem 4.4. *There exists a bijective correspondence between the set $\{\nabla\}$ of the N–almost Hermitian connections and the set $\{H\Gamma(N)\}$ of the metric d–connections with respect to the fundamental tensor field $g^{ij}(x,p)$.*

§5. Some geometrical meanings of parallelizable space. In this section we fix the non–linear connection N. Then we have two distributions N and V such that $T(T^*M) = N \oplus V$, where $T(T^*M)$ is tangent bundle of T^*M. We call the vector belonging to N and V the horizontal and vertical vector, respectively.

Now let $M^{*n} = (M, g^{ij}(x,p))$ be a generalized Hamilton space. From the fundamental tensor $g^{ij}(x,p)$ we have the almost complex structure J and the Riemannian metric G on T^*M. We denote the N–almost Hermitian model of M^{*n} by $H^{2n} = (T^*M, (G, J))$.

Definition 5.1. *An N–almost Hermitian model $H^{2n} = (T^*M, (G,J))$ of the generalized Hamilton space $M^{*n} = (M, g^{ij}(x,p))$ is parallelizable if T^*M is parallelizable.*

Proposition 5.1. *If M^{*n} is parallelizable, then there exist n globally linearly independent vertical vector fields on H^{2n}.*

Proof. From Definitions 2.1 and 3.1 we have n globally linearly independent d–covector fields $Z_i^\alpha, \alpha = 1, \ldots, n$. Then the vectors Z_V^α defined by $Z_V^\alpha = Z_i^\alpha \dot{\partial}^i$ $(\alpha = 1, \ldots, n)$ are n globally linearly independent vertical fields on H^{2n}.

Proposition 5.2. *If H^{2n} has n globally linearly independent vertical vector fields, then H^{2n} is parallelizable.*

Proof. By assumption we have n globally linearly independent vertical vector fields $Z_V^1, \ldots, Z_V^n$. Then we define the horizontal vector fields Z_α^H by $Z_\alpha^H = J(Z_V^\alpha)$ $(\alpha = 1, \ldots, n)$. Then we have $2n$ globally linearly independent vector fields $Z_V^1, \ldots, Z_V^n, Z_1^H, \ldots, Z_n^H$ on H^{2n}.

Proposition 5.3. *If H^{2n} is parallelizable, then H^{2n} has n globally linearly independent vertical vector fields.*

Proof By assumption we have $2n$ globally linearly independent vector fields $X_a = X_a^i \delta_i + \dot{X}_i^a \dot{\partial}^i$ $(a = 1, \ldots, 2n)$. Hence $\det \left\| X_a^i \ \dot{X}_a^i \right\| \neq 0$. Therefore the rank of the matrix $\left\| \dot{X}_i^a \right\|$ is equal to n and we can take $\det \left\| \dot{X}_i^a \right\| \neq 0$ $\quad (a, i = 1, \ldots, n)$. Hence we obtain n globally linearly independent vertical vector fields X_V^a defined by $X_V^a = \dot{X}_i^a \dot{\partial}^i$ $\quad (a = 1, \ldots, n)$.

Theorem 5.1. *M^{*n} is parallelizable if and only H^{2n} is parallelizable.*

Proof. If M^{*n} is parallelizable, then H^{2n} is paralllizable from Propositions 5.1 and 5.2. Conversely, from Proposition 5.3 we have n globally linearly independent vertical vector fields $Z_V^\alpha = Z_i^\alpha(x,p)\dot{\partial}^i$ $(\alpha = 1, \ldots, n)$. Hence we have n globally linearly independent d–covector fields $Z_i^\alpha(x,p)$ $(\alpha = 1, \ldots, n)$ on M.

Proposition 5.4. *If M^{*n} with a positive definite fundamental tensor $g^{ij}(x,p)$ is parallelizable, then there exist n globally orthonormal vertical vector fields on H^{2n}.*

Proof. From Theorem 3.1 we have n globally orthonormal d–covector fields $e_i^1, \ldots, e_i^n$. Then the vectors e_V^α deefined by $e_V^\alpha = e_i^\alpha \dot{\partial}^i$ $(\alpha = 1, \ldots, n)$ are n globally orthonormal vertical vector fields on H^{2n}.

Proposition 5.5. *If H^{2n} has n globally orthonormal vertical vector fields, then H^{2n} has $2n$ globally orthonormal vector fields.*

Proof. By assumption we hav n globally orthonormal vertical vector fields $e_V^\alpha = e_i^\alpha \dot{\partial}^i$ $(\alpha = 1, \ldots, n)$. Using the non–linear connection N and the fundamental tensor $g^{ij}(x,p)$ we define the horizontal vector fields e_α^H by $e_\alpha^H = J(e_V^\alpha)$ $(\alpha = 1, \ldots, n)$. Hence we have $2n$ globally orthonormal vector fields $e_V^1, \ldots, e_V^n, e_1^H, \ldots, e_n^H$ on H^{2n}.

Hence we obtain

Theorem 5.2. *If M^{*n} with a positive definite fundamental tensor $g^{ij}(x,p)$ is parallelizable, then there exist $2n$ globally orthonormal vector fields on H^{2n}.*

Theorem 5.3. *If M^{*n} with a positive definite fundamental tensor $g^{ij}(x,p)$ is parallelizable, then*

1°. *There exists at least an N–almost Hermitian connection ∇ on H^{2n} with the properties:*

$$\nabla_X e_V^\alpha = 0, \qquad \nabla_X e_\alpha^H = 0 \qquad \text{for all } X \in \mathcal{X}(T^*M),$$

where $e_V^\alpha = e_i^\alpha \dot{\partial}^i, e_\alpha^H = J(e_V^\alpha)$ and $e_i^1, \ldots, e_i^n$ are globally orthonormal d–covector fields on M^{*n}.

2°. *The curvature tensor field R of ∇ vanishes.*

Proof. From Theorems 2.3 and 3.3 we have the d–connection $H\Gamma(N) = (H_j{}^i{}_k, C_j^{ik})$ with the properties:

$$(i) \qquad e_{i|k}^\alpha = 0, \qquad e_i^\alpha|^k = 0, \qquad g^{ij}{}_{|k} = 0, \qquad g^{ij}|^k = 0,$$

$$(ii) \qquad R_j{}^i{}_{kh} = 0, \qquad P_j{}^i{}_k{}^h = 0, \qquad S_j^{ikh} = 0.$$

By Theorem 4.4 the lift ∇ of this d–connection $H\Gamma(N)$ is the N–almost Hermitian connection on T^*M. From (4.3) we get

$$\nabla_{\delta_k} e_V^\alpha = \nabla_{\delta_k}\left(e_i^\alpha \dot{\partial}^i\right) = (\delta_k e_i^\alpha)\dot{\partial}^i - e_i^\alpha H_j{}^i{}_k \dot{\partial}^j = e_{i|k}^\alpha \dot{\partial}^i = 0,$$

$$\nabla_{\dot{\partial}^k} e_V^\alpha = \nabla_{\dot{\partial}^k}\left(e_i^\alpha \dot{\partial}^i\right) = \left(\dot{\partial}^k e_i^\alpha\right)\dot{\partial}^i - e_i^\alpha C_j^{ik} \dot{\partial}^j = e_i^\alpha|^k \dot{\partial}^i = 0,$$

which shows us $\nabla_X e_V^\alpha = 0$. Using Proposition 4.2, we have $\nabla_X e_\alpha^H = \nabla_X J(e_V^\alpha) = J(\nabla_X e_V^\alpha) = 0$. From (4.5) and (ii) the curvature tensor field R of ∇ vanishes.

Conversely, we have

Theorem 5.4. *If M is a C^ω-manifold and M^{*n} has a positive definite fundamental tensor $g^{ij}(x,p)$ and H^{2n} has an N-almost Hermitian connection ∇ without curvature, then in every point $x \in M^{*n}$ we can find an open set $U_x \subset M^{*n}, x \in U_x$ and there exist n orthonormal d-covector fields on $M^{*n}|_{U_x}$.*

Proof. Let us consider the system of partial differential equations $\nabla_X Z = 0$. This system is completely integrable by virtue of the fact that the curvature R of ∇ is zero. That means, there exists an open set U_x in every point $x \in M^{*n}$ and exist $2n$ linearly independent vector fields $Z_1, \ldots, Z_{2n}$ on $\pi^{*-1}(U_x) \subset T^*M$. We can decompose the vectors $Z_1, \ldots, Z_{2n}$ to the horizontal and vertical parts in the form: $Z_1 = Z_1^H + Z_V^1, \ldots, Z_{2n} = Z_{2n}^H + Z_V^{2n}$. We can take on $\pi^{*-1}(U_x)$ n linearly independent vertical vector fields from the vertical vector fields $\{Z_V^1, \ldots, Z_V^{2n}\}$. If we orthonormalize the above n vertical vectors by means of the fundamental tensor g^{ij}, then they determine n orthonormal d-covector fields on $M^{*n}|_{U_x}$.

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A VARIATIONAL CHARACTERIZATION OF THE EUCLIDEAN n-SPHERE

BY

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Let M be a closed orientable C^∞ n-manifold and $f : M \to E^{n+1}$ be a C^∞ immersion into Euclidean $(n+1)$-space, such that the origin O of coordinates does not lie in $f(M)$. Set

$$\tau_M(f) = \int_M |f|^{-n} dV,$$

where $|\ |$ is the Euclidean norm, and dV is the volume element of the induced Riemannian metric on M. If $I(M, E^{n+1})$ is the set of all such immersions, then τ_M is a real functional on this set.

Denote by ν the Gauss normal field of f, and consider the normal deformation

$$\overline{f} = f + t\varphi\nu,$$

where $\varphi : M \to R$ is a C^∞ real function, and the real parameter t varies in a small neighbourhood of zero. Following the conventions of [2], w denote by δ the operator $(\partial/\partial t)_{t=0}$, and call f a critical point of τ_M if

$$\delta\tau_M(f) = 0$$

for every choise of φ.

Theorem. *The functional τ_M has a critical point if and only if M is diffeomorphic to an Euclidean sphere. In fact, every such critical immersion is an embedding of M as an Euclidean sphere with centre 0.*

Proof. We have $\delta\tau_M(f) = \int_M (\delta\,|f|^{-n}\,dV + |f|^{-n}\,\delta dV)$. Setting $p = -f\cdot\nu$, since $|\overline{f}|^2 = |f|^2 + \varphi^2 t^2 - 2pt$, we obtain $\delta|f|^{-n} = np\varphi|f|^{-n-2}$. By a calculation very similar to those carried out in [2, p.127], we have $\delta dV = -n\varphi H dV$, where H is the first mean curvature, i.e. the arithmetic mean of the principal curvatures. Then

$$\delta\tau_M(f) = n\int_M \varphi|f|^{-n-2}(p - H|f|^2)dV.$$