Hence we can have

$$(2.24) \qquad (D_W D_U R)(X,Y,Z,V) - (D_U D_W R)(X,Y,Z,V) = \\ = v(W,U)Q(X,Y,Z,V)$$

where $v(W,U) = (D_W v)(U) - (D_U v)(W)$. Therefore, in consequence of (2.14) we have

$$(2.25) \qquad v(W,U)Q(X,Y,Z,V) + v(Z,V)Q(X,Y,W,U) + \\ + v(X,Y)Q(Z,V,U,W) = 0$$

Hence since we have $Q(X,Y,Z,V) = Q(X,V,X,Y)$, (2.25) are of the form (2.15) in which the indices i,j,k are replaced by pairs, ZV, XY, WU. Since $r \neq 0$, Q are not all zero. Accordingly we have

$$v(W,U) = 0$$

or

$$(D_W v)(U) = (D_U v)(W)$$

which is the condition for v to be gradient.

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CURVATURE TYPE FUNCTIONS FOR PLANE CURVES

BY

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Introduction. In the paper we define a curvature type function which imitates the curvature of a plane curve. In the definition of the usual curvature we put vector function $s \to p(s)$ instead of the tangent vector. The curvature type functions have properties similar to the logarithm of a number (see formulas (6.1)-(9.1)). Let $k_{\Gamma,p}$ denote the curvature type function with respect to the vector-function $p(s)$ for a plane curve Γ.

The main result of the paper concerns equality $k_{\Gamma,p} = k_{\Gamma,q}$. Namely the last equality holds if and only if the angle γ between vectors $p(s)$ and $q(s)$ is constant. To show applications of the curvature type function we consider the class Q plane curves described in [1] which contains the class of all ovals [3]. We present a special curvature type function k_{Γ,p_n} associated with curves with constant n-width. Exactly we prove that curvature type function k_{Γ,p_n} is equal to the curvature k of the examined curve if and only if the curve has constant n-width. Next we give necessary and sufficient condition so that the relationship of the perimeters of the two described n-polygon and m-polygon was constant. In the end we give examples of curves for which the relation is constant.

1. On the curvature type functions. Let $\mathbb{R}$ and C denote the real line and the complex plane, respectively. Let a C-class regular plane curve Γ be represented by the following equation

$$(1.1) \qquad \Gamma : s \to z(s) = x(s) + iy(s) \in C$$

where s varies on an opened interval $(a,b) \subseteq \mathbb{R}$. With the curve Γ we associate a real function imitating the curvature for a plane curve. For this purpose we hook a vector-function

$$s \to p(s) = x_p(s) + iy_p(s)$$

at the point $z(s)$.

We assume that the vector-function $p(s)$ satisfies at the point $z(s)$ the following conditions:

(A) $|p(s)| > 0$ for all $s \in (a,b)$, where $|w|$ denotes the length of the vector w,

(B) There exist the derivatives of the quotients $\frac{y_p(s)}{x_p(s)}$ or $\frac{x_p(s)}{y_p(s)}$ for all $s \in (a,b)$, whenever $x_p(s) \neq 0$ or $y_p(s) \neq 0$.

Now we denote by ω the angle between the vectors $p(s)$ and $p(s+h)$ and we denote by 1 the length of the arc of Γ between the points $z(s)$ and $z(s+h)$. Then there exists the following bound

$$\lim_{h\to 0}\frac{\omega}{1}. \tag{2.1}$$

Indeed

$$\lim_{h\to 0}\frac{\omega}{1} = \lim_{h\to 0}\frac{\omega}{\sin\omega}\frac{\sin\omega}{1} = \lim_{h\to 0}\frac{\sin\omega}{\omega} =$$

$$= \lim_{h\to 0}\frac{[p(s),p(s+h)]}{|p(s)||p(s+h)|\int_s^{s+h}|z'(t)|dt} =$$

$$= \frac{1}{|z'(s)||p(s)|^2}\lim_{h\to 0}\frac{[p(s),p(s+h)]}{h}$$

where $[z,w] = \operatorname{Im} \bar{z}w$ is the determinant of the vectors z and w. But

$$\lim_{h\to 0}\frac{[p(s),p(s+h)]}{h} = \lim_{h\to 0}\frac{x_p(s)y_p(s+h) - x_p(s+h)y_p(s)}{h}.$$

Because $|p(s)| > 0$, so $x_p(s) \neq 0$ or $y_p(s) \neq 0$ for all $s \in (a,b)$. Therefore

$$\lim_{h\to 0}\frac{[p(s),p(s+h)]}{h} = \begin{cases} x_p^2(s)\frac{d}{ds}\left(\frac{y_p(s)}{x_p(s)}\right), if \;\; x_p(s) \neq 0 \\ y_p^2(s)\frac{d}{ds}\left(\frac{x_p(s)}{y_p(s)}\right), if \;\; y_p(s) \neq 0 \end{cases}$$

Obviously if $x_p(s)y_p(s) \neq 0$, then the both formulas are equal. Finally we obtain the following expression

$$\lim_{h\to 0}\frac{\omega}{1} = \begin{cases} \frac{x_p^2(s)}{|z'(s)||p(s)|^2}\frac{d}{ds}\left(\frac{y_p(s)}{x_p(s)}\right), if \;\; x_p(s) \neq 0 \\ \frac{y_p^2(s)}{|z'(s)||p(s)|^2}\frac{d}{ds}\left(\frac{x_p(s)}{y_p(s)}\right), if \;\; y_p(s) \neq 0 \end{cases} \tag{3.1}$$

In particular, if the correspondence $s \to p(s)$ is C^1-class plane curve, then

$$\lim_{h\to 0}\frac{\omega}{1} = \frac{[p,p']}{|z'||p|^2} \tag{4.1}$$

Definition 1. *The bound (2.1) we will call the curvature type function of Γ and it we will denote $(a,b) \ni s \to k_{\Gamma,p}(s)$.*

In the case where $s \to p(s)$ is C^1-class plane curve we have:

$$k_{\Gamma,p}(s) = \frac{[p(s),p'(s)]}{|p(s)||z'(s)|^2} \tag{5.1}$$

In particular if Γ is C^2-class regular plane curve we have

$$k_{\Gamma,z'} = \frac{[z',z'']}{|z'|^3}. \tag{6.1}$$

This is the usual curvature of Γ.

Let $\mathbb{M}$ denote the set of all C^1-class regular plane curves $s \to z(s)$ for which the tangent vector $z'(s)$ satisfies condition (B), i.e. there exist derivatives of the quotients

$$\frac{y'(s)}{x'(s)} \quad or \quad \frac{x'(s)}{y'(s)},$$

whenever $x'(s) \neq 0$ or $y'(s) \neq 0$.

Then for each curve $\Gamma \in \mathbb{M}$ there exists curvature type function $k_{\Gamma,z'}(s)$ and it is the extension of the curvature on the class $\mathbb{M}$ [2]. Obviously

$$k_{\Gamma,z'}(s) = \begin{cases} \frac{(x'(s))^2}{|z'(s)|^3}\frac{d}{ds}\left(\frac{y'(s)}{x'(s)}\right), if \;\; x'(s) \neq 0 \\ \frac{(y'(s))^2}{|z'(s)|^3}\frac{d}{ds}\left(\frac{x'(s)}{y'(s)}\right), if \;\; y'(s) \neq 0 \end{cases} \tag{5.1}$$

The curvature type function has the following properties

$$k_{\Gamma,p_1p_2} = k_{\Gamma,p_1} + k_{\Gamma,p_2} \tag{6.1}$$

$$k_{\Gamma,\frac{1}{p}} = -k_{\Gamma,p} \tag{7.1}$$

$$k_{\Gamma,\frac{p}{q}} = k_{\Gamma,p} - k_{\Gamma,q} \tag{8.1}$$

Remark 1. Let p,q,r denote C^1-class plane curves. Then the quotient of two curvature type functions $k_{\Gamma,p}$ and $k_{\Gamma,q}$ gives the function

$$(L_q(p))(s) = \frac{k_{\Gamma,p}(s)}{k_{\Gamma,q}(s)}$$

which exactly imitates of the logarithm, i.e.

$$L_q(p_1 p_2) = L_q(p_1) + L_q(p_2)$$
$$L_q\left(\frac{p_1}{p_2}\right) = L_q(p_1) - L_q(p_2)$$
$$L_q(p) = \frac{L_r(p)}{L_r(q)}.$$

The function $L_q(p)$ can be obtained as the following bound

$$L_q(p) = \lim \frac{\omega_1(h)}{\omega_2(h)},$$

where $\omega_1(h)$ denotes the angle between vectors $p(s)$ and $p(s+h)$ however $\omega_2(h)$ is the angle between $q(s)$ and $q(s+h)$.

Remark 2. The curvature type function $k_{\Gamma,p}$ is the extension curvature of the curve $w(s)$ expressed as follows

$$w(s) = \int_{s_0}^{s} \frac{p(t)}{|p(t)|}|z'(t)|dt \tag{9.1}$$

where s_0 is a fixed parameter from (a,b) and $s \in (a,b)$.

The curvature type function $k_{\Gamma,p}$ has interpretation similar to the curvature of a plane curve, i.e.:

Lemma 1. *The angle between vectors $p(s_1)$ and $p(s_2)$, $s_1, s_2 \in (a,b)$ is equal to the following integral:*

$$\int_{s_1}^{s_2} k_{\Gamma,p}(s)|z'(s)|ds.$$

Now we establish the main result concerns applications of a curvature type function.

Theorem 1. *Two curvature type functions satisfy the following equality*

$$k_{\Gamma,q}(s) = k_{\Gamma,p}(s) \mp \frac{1}{|z'(s)|}\frac{d}{ds}\gamma(s),$$

where $\gamma(s)$ denotes the angle between the vectors $p(s)$ and $q(s)$.

Theorem 2. *Two curvature type functions are equal, i.e.:*

$$k_{\Gamma,p}(s) = k_{\Gamma,q}(s)$$

for all $s \in (a,b)$ if and only if the angle between vectors $p(s)$ and $q(s)$ is constant.

Proof of Theorems 1 and 2. Denoting $\tan\alpha = \frac{y_p}{x_p} = a$ and $\tan\beta = \frac{y_q}{x_q} = b$ we have $|\gamma| = |\beta - \alpha|$. In the case if $\gamma = \beta - \alpha$ we obtain:

$$\tan\gamma = \frac{\tan\beta - \tan\alpha}{1 + \tan\alpha\tan\beta}.$$

Hence

$$\gamma' = \cos^2\gamma \frac{d}{ds}\tan\gamma = \cos^2\gamma \frac{(b'-a')(1+ab) - (a'b+ab')(b-a)}{(1+ab)^2} =$$
$$= \frac{\cos^2(\beta-\alpha)\left(\frac{b'}{\cos^2\alpha} - \frac{a'}{\cos^2\beta}\right)}{\frac{\cos^2(\beta-\alpha)}{\cos^2\alpha\cos^2\beta}} = b'\cos^2\beta - a'\cos^2\alpha =$$
$$= \left(\frac{y_q}{x_q}\right)'\frac{x_p^2}{x_q^2+y_q^2} - \left(\frac{y_p}{x_p}\right)'\frac{x_p^2}{x_p^2+y_p^2} = |z'|k_{\Gamma,q} - |z'|k_{\Gamma,p}.$$

This means that

$$k_{\Gamma,q} = k_{\Gamma,p} + \frac{1}{|z'|}\gamma'.$$

However, if $\gamma = \alpha - \beta$ this calculation gives the following relation

$$|z'|k_{\Gamma,q} = |z'|k_{\Gamma,p} - \gamma'.$$

This completes the proof.

In particular if the curve Γ is represented in the arc parametrization, then

$$k_{\Gamma,q} = k_{\Gamma,p} \mp \gamma'. \tag{10.1}$$

This means that the difference between two curvature type functions is equal to the derivative of the angle between the vectors $p(s)$ and $q(s)$.

Remark 3. Applying the function $L_q(p)$ we obtain the counterpart of Th.2 in the form: *The angle between vectors $p(s)$ and $q(s)$ is constant if and only if $L_q(p) = 1$ for each $s \in (a,b)$.*

Remark 4. If the angle between vectors $p(s)$ and $q(s)$ has the extreme in the point $s = s_0$, then $k_{\Gamma,p}(s_0) = k_{\Gamma,q}(s_0)$. Because $\tan\gamma(s_0) = \frac{\beta}{\alpha}$ and $\left(\frac{\beta}{\alpha}\right)'(s_0) = 0$.

2. Curvature type functions in the class Q. In this section we consider plane curves belonging to the class Q described in [1]. The class of all ovals is included in the class Q. Let C_L be the set of all positive continuous periodic functions defined on $\mathbb{R} = (-\infty, +\infty)$ with the period $L > 0$.

Let us fix a function $k \in C_L$ such that

$$\int_0^L k(s)ds = 2\pi. \tag{1.2}$$

Now we consider a curve $\Gamma \in Q$ represented by the formula ((1.1) [1])

$$\Gamma : z(s) = r_{f,k}(s) = \int_0^s f(t)k(t)e^{iK(t)}dt, \tag{2.2}$$

where $K(t) = \int_0^t k(u)du$. The extension curvature $k_{\Gamma,z'}(s) = \frac{1}{f(s)}$. Let K^{-1} denote the inverse function for K. Then we define

$$\varphi_n(s) = K^{-1}\left(K(s) + \frac{2\pi}{n}\right). \tag{3.2}$$

Then we recall [1] the following sequence vector-functions:

$$s \to p_n(s) = z(s) + \epsilon^{n-1}z(\varphi_n(s)) + \ldots + \epsilon z(\varphi_n^{n-1}(s)),$$

where $\epsilon \in \mathbb{C}$ and $\epsilon^n = 1$, $\epsilon \neq 1$ and $\varphi_n^v(s) = \underbrace{\varphi_n(\varphi_n(\ldots(\varphi_n(s))\ldots))}_{v-times.}$ Now we examine the curvature type functions:

$$k_{\Gamma,p_n}(s), \quad n = 2, 3, \ldots$$

Lemma 2. *For each curve $\Gamma \in Q$ the curvature type functions $k_{\Gamma,p}(s)$, $n = 2, 3, \ldots$ are expressed by the following formula:*

$$k_{\Gamma,p_n}(s) = \frac{k(s)(f(s) + f(\varphi_n(s)) + \ldots + f(\varphi_n^{n-1}(s)))\omega_n(s)}{k(s)f(s)(\omega_n^2(s) + \Omega_n^2(s))},$$

where $\omega_n(s) = [p_n(s), T]$, $\Omega_n(s) = [p_n, N]$ and $T = e^{iK(s)}$, $N = iT$.

Proof. Obviously

$$p_n'(s) = k(s)(f(s) + f(\varphi_n(s)) + \ldots + f(\varphi_n^{n-1}(s)))e^{iK(s)}.$$

Hence by (4.1) we obtain

$$k_{\Gamma,p_n}(s) = \frac{k(s)(f(s) + f(\varphi_n(s)) + \ldots + f(\varphi_n^{n-1}(s)))\omega_n(s)}{k(s)f(s)(\omega_n^2 + \Omega_n^2)},$$

because $p_n = \Omega_n T - \omega_n N$. This completes the proof.

The functions $\omega_n(s)$, $n = 2, 3, \ldots$ are equal to n-width of the curve Γ associated with the direction of vector T (formula (3.1) [1]).

In particular a curve with constant 2-width is a curve with constant width.

Now we give two applications of the curvature type function k_{Γ,p_n}.

Theorem 3. *Every curve $\Gamma \in Q$ has constant n-width if and only if the equality $k_{\Gamma,p_n}(s) = k_{\Gamma,z'}(s) = \frac{1}{f(s)}$ is satisfied for all $s \in \mathbb{R}$.*

Proof. Let a curve $\Gamma \in Q$ and let for the curve the equality

$$k_{\Gamma,p_n}(s) = k_{\Gamma,z'}(s) \quad for \quad s \in \mathbb{R} \tag{$*$}$$

be satisfied. Then by Th.2 the angle γ between vectors p_n and z' is constant. This means that

$$\tan\gamma = -\frac{\omega_n}{\Omega_n}.$$

Hence we have $\Omega_n = \lambda\omega_n$, where λ is a constant. Inserting the relation $\Omega_n = \lambda\omega_n$ in the equality $(*)$ we obtain

$$k(s)(f(s) + f(\varphi_n(s)) + \ldots + f(\varphi_n^{(n-1)}(s))) = (1 + \lambda^2)k(s)\omega_n(s).$$

Integrating the last equality we obtain

$$nM = (1 + \lambda^2)nM,$$

where $M = \int_0^L k(s)f(s)ds$ is the perimeter of Γ. It implies that $\lambda = 0$. Thus $\Omega_n = 0$ and $\omega_n' = k\Omega_n = 0$. So n-width $\omega_n(s)$ is a constant function. This completes the proof.

Theorem 4. *Let $\Gamma \in Q$. Then the ratio $\frac{\omega_m(s)}{\omega_n(s)}$ for a certain integer n, m is a constant function if and only if the equality $k_{\Gamma,p_m}(s) = k_{\Gamma,p_n}(s)$ holds for all $s \in \mathbb{R}$.*

Proof. Let $\frac{\omega_m(s)}{\omega_n(s)} = \mu$, where $\mu \in \mathbb{R}$. Then $\omega_m'(s) = \mu\omega_n'(s)$. But $\omega_n'(s) = k(s)\Omega_n(s)$ for $n = 2, 3, \ldots$, so $\Omega_m(s) = \mu\Omega_n(s)$. This means that $p_m(s) = \mu p_n(s)$ and $p_m'(s) = \mu p_n'(s)$ i.e.

$$\mu k(s)(f(s) + f(\varphi_n(s)) + \ldots + f(\varphi_n^{n-1}(s))) =$$

$$= k(s)(f(s) + f(\varphi_m(s)) + \ldots + f(\varphi_m^{m-1}(s))).$$

Thus (by integration) we have

$$nM = \mu m M \quad and \quad \mu = \frac{m}{n}.$$

Now we easily compute that $k_{\Gamma,p_n}(s) = k_{\Gamma,p_m}(s)$. Conversely, if $k_{\Gamma,p_n}(s) = k_{\Gamma,p_m}(s)$, then by Th.2 the angle $\gamma(s)$ between $p_n(s)$ and $p_m(s)$ is constant. On the other hand

$$k[p_n, p_m] = k[\Omega_n T - \omega_n N, \Omega_m - \omega_m N] = k(\omega_n \Omega_m - \omega_m \Omega_n) =$$

$$= \omega_n \omega_m' - \omega_m \omega_n' = \left(\frac{\omega_m}{\omega_n}\right)' \omega_n^2.$$

Therefore there exists $s_0 \in I\!R$ such that $k(s_0)[p_n(s_0), p_m(s_0)] = 0$. This means that $\gamma(s_0) = \pi$ and it follows that $\gamma(s) = \pi$ for all $s \in I\!R$. Simultaneously with the above we observe that $\left(\frac{\omega_m(s)}{\omega_n(s)}\right)' = 0$ for all $s \in I\!R$. Thus the ratio $\frac{\omega_m}{\omega_n}$ is constant. This completes the proof.

Corollary. *If the quotient $\frac{\omega_m}{\omega_n}$ is a constant function, then always $\frac{\omega_m(s)}{\omega_n(s)} = \frac{m}{n}$.*

Remark 5. Each convex polygon with n sides and equal interior angles is called n-polygon. Let $\Gamma \in Q$. Then tangents at points

$$z(s), z(\varphi_n(s)), \ldots, z(\varphi_n^{n-1}(s))$$

to the curve Γ determine the n-polygon described on Γ. Let Λ_n denote the perimeter of the n-polygon. By Lm.1 [1]

$$\Lambda_n(s) = 2\omega_n(s) \tan\frac{\pi}{n}.$$

Hence the equality of the curvature type functions: $k_{\Gamma,p_n} = k_{\Gamma,p_m}$ implies that

$$\frac{\Lambda_m}{\Lambda_n} = \frac{m \tan\frac{\pi}{m}}{n \tan\frac{\pi}{n}}.$$

This means that:

Theorem 5. *The curvature type functions k_{Γ,p_n}, k_{Γ,p_m} are equal if and only if the quotient of the perimters Λ_n and Λ_m (of the described n-polygon and m-polygon) is constant.*

In the end we give examples of curves for which $\frac{\Lambda_m}{\Lambda_n}$ is constant. Let a curve Γ be represented by formula (2.2). We consider the Fourier series for function f (see formula (1.8) [1]). Then the Fourier series for n-width $\omega_n(s)$ has the form

$$\omega_n(s) = \frac{nM}{2\pi} - n \sum_{\mu=1}^{\infty} \left[\frac{A_{n\mu}}{n^2\mu^2 - 1} \cos n\mu K(s) + \frac{B_{n\mu}}{n^2\mu^2 - 1} \sin n\mu K(s)\right].$$

Now we choose coefficients A_n and B_n for f such that $\frac{\omega_m(s)}{\omega_n(s)} = \frac{m}{n}$. In particular for $f(s) = \frac{M}{2\pi} + A_{nm} \cos nmK(s) + B_{nm} \sin nmK(s)$ we obtain curve Γ by (2.2) such that $\omega_n/\omega_m = n/m$.

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