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THE H_∞ -PROBLEM WITH BOUNDARY DISTURBANCE.THE PARABOLIC CASE

BY

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1.Introduction and problem formulation. Consider the following input-output system

$$x'(t) = Ax(t) + B_1w(t) + B_2u(t)\ ,\ t \in R^+ = [0, +\infty) \tag{1.1}$$

$$z(t) = Cx(t) + Du(t)\ ,\ t \in R^+ \tag{1.2}$$

Here $x(t) \in X$ is the state of the system,$u(t) \in U$ is the control input,$w(t) \in W$ is the disturbance input and $z(t) \in Z$ is the controlled output.

The following preliminary assumptions will be maintained throughout the paper

(i) X,U,W,Z are real Hilbert spaces.We shall denote by $\langle\cdot,\cdot\rangle,\langle\cdot,\cdot\rangle_U$, $\langle\cdot,\cdot\rangle_W$, $\langle\cdot,\cdot\rangle_Z$ their scalar products and by $|\cdot|,|\cdot|_U,|\cdot|_W,|\cdot|_Z$ the corresponding norms;

(ii) A is the infinitesimal generator of an analytic semigroup e^{At} on X,$t > 0$,$\lambda_0 \in \rho(A), \lambda_0 > 0, A - \lambda_0$ is the infinitesimal generator of an exponentially stable semigroup;

(iii) $B_1 = (\lambda_0 - A)H_1$,where $H_1 \in L(W, D((\lambda_0 - A)^\alpha)), \alpha \in (0,1)$ (equivalentely $(\lambda_0 - A)^\alpha H_1 = H_2 \in L(W, X)$),

$$B_2 \in L(U, X).$$

By assumption (iii) follows (see [3]) that for every $T > 0$,system (1.1) with initial condition $x(0) = x_0 \in X$ and inputs $u \in L^2(0,T;U), w \in$

$L^2(0,T;W)$ has a mild solution $x \in L^2(0,T;X) \cap C([0,T];D(A^*)')$ given by

$$(1.3)\quad x(t) = e^{At}x_0 + (\lambda_0 - A)\int_0^t e^{A(t-s)}H_1 w(s)ds + \int_0^t e^{A(t-s)}B_2 u(s)ds$$

(the mild solution not necessary belongs to C([0,T];X)).

This is the abstract formulation of a large class of control systems including the heat equation with Dirichlet and Neumann boundary disturbance and structurally damped plates with point disturbance or boundary disturbance (see e.g. [7]). For instance the input-output system

$$(1.4)\qquad \begin{cases} x_t - \Delta x = u \ in \ \Omega \times R^+ \\ x = w \ in \ \partial\Omega \times R^+, \end{cases}$$

where $u \in L^2(R^+; L^2(\Omega)), w \in L^2(R^+; L^2(\partial\Omega))$ can be written in the form (1.1),where $X = L^2(\Omega), U = L^2(\Omega), W = L^2(\partial\Omega)$ and

$$Ah = \Delta h \ , \ D(A) = H^2(\Omega) \cap H_0^1(\Omega),$$

$B_2 u = u, B_1 w = -A\bar{D}w$,where $\bar{D}$ (the Dirichlet map) is defined by

$$\Delta \bar{D}w = 0 \ in \ \Omega, \bar{D}w = w \ in \ \partial\Omega.$$

Here $\alpha \in (0, \frac{1}{4})$ (see e.g. [7]).

For a given feedback controller $F \in L(X,U)$,denote by $S_F : L^2(R^+;W) \to L^2(R^+;Z)$ the operator

$$(1.5)\qquad z = S_F w = (C + DF)\tilde{x}(t),$$

where $\tilde{x}$ is the solution to

$$(1.6)\qquad \tilde{x}(t) = (\lambda_0 - A)\int_0^t e^{A(t-s)}H_1 w(s)ds + \int_0^t e^{A(t-s)}B_2 F\tilde{x}(s)ds$$

(the existence of a unique solution can be proved via a fixed point result).

The standard H$_\infty$-control problem is the following (see also [1],[5]): given $\gamma > 0$,find $F \in L(X,U)$ which stabilizes the system (1.1) and makes $S_F \in L(L^2(R^+;W), L^2(R^+;Z))$ with $\|S_F\| < \gamma$ (where $\|\cdot\|$ denotes the induced operator norm).

The main result of this paper gives necessary and sufficient conditions for existence of a suboptimal solution F to this H$_\infty$-control problem in terms of an algebraic Riccati equation.For $B_1 \in L(W,X)$ and B_2 unbounded

operator (more exactely $B_2 \in L(U, D(A^*)')$ and for any $0 < T < +\infty$,the operator $B_2^* e^{A^* t}$ admits a continuous extension from X to $L^2(0,T;U)$) such a result has been established in a different way by V.Barbu in [1].For $B_1 \in L(W,X)$ (which is a particular case) a similar result has been proved by B.van Keulen,M.Peters and R.Curtain in [5].

The extension of [5] to this more general class of systems arises some difficult problems which can be solved using arguments related on some recent papers [3],[6],[7] concerning the synthesis of quadratic optimal control problems of parabolic type with unbounded coefficients and infinite horizon.

Remark. If $F \in L(X,U)$ and A+B_2F is generating an exponentially stable semigroup,then for every $w \in L^2(R^+;W)$ we have that $\tilde{x} \in L^2(R^+;X)$ (where $\tilde{x}$ is the solution of (1.6)).Moreover, the map $w \mapsto \tilde{x}$ is continuous from $L^2(R^+;W)$ to $L^2(R^+;X)$ (see the Appendix).

Using these facts we may conclude that if there exists such an F,then there exists a γ large enough such that the H$_\infty$-problem has solution.

2.THE MAIN RESULT Beside (i)-(iii) we shall assume here that

(iv) The pair (C,A) is exponentially detectable,i.e. there exists $K \in L(Z,X)$ such that A+KC generates an exponentially stable semigroup;

(v) D^* [C D] = [0 I]

Theorem 2.1 Suppose that $\gamma > 0$ and assumptions (i)-(v) hold.If there exists an $F \in L(X,U)$ such that A+B_2F generates an exponentially stable semigroup and $\|S_F\| < \gamma$,then there exists $P \in L(X,X)$ with $B_1^*P \in L(X,W)$, $P = P^* \geq 0$ satisfying

$$(2.1)\qquad \begin{aligned} &\langle Ax, Py\rangle + \langle Px, Ay\rangle - \langle P(B_2B_2^* - \gamma^{-2}B_1B_1^*)Px, y\rangle + \\ &+ \langle C^*Cx, y\rangle = 0, \ \forall x, y \in D(A) \end{aligned}$$

and such that $A - (B_2B_2^* - \gamma^{-2}B_1B_1^*)P$ generates an exponentially stable semigroup.

Conversely,if there exists $P \in L(X,X)$ with $B_1^*P \in L(X,W), P = P^* \geq 0$ satisfying (2.1) and such that $A - (B_2B_2^* - \gamma^{-2}B_1B_1^*)P$ generates an exponentially stable semigroup, then the state feedback $F = -B_2^*P$ is exponentially stabilizing and $\|S_F\| \leq \gamma$.

Now coming back to (1.4) we note that (iv) and (v) are satisfied for the controlled output

$$z(t) = (\tilde{x}(t), u(t)) \in Z = L^2(\Omega) \times L^2(\Omega).$$

Proof of Theorem 2.1 a) Suppose that there exists an exponentially stabilizing feedback F such that $\|S_F\| < \gamma$ for some $\gamma > 0$.

Consider $\tilde{K} : L^2(R^+;U) \times L^2(R^+;W) \to R$ defined by

$$\tilde{K}(u,w) = \int_0^\infty (|Cx|_Z^2 + |u|_U^2 - \gamma^2 |w|_W^2)dt \tag{3.1}$$

where x is the mild solution of (1.1) with $x(0) = x_0$ (i.e. x is given by (1.3)).

We shall study for the begining the minimization problem

$$(P_1) \qquad \inf_{u \in L^2(R^+;U)} \tilde{K}(u,w)$$

where $w \in L^2(R^+;W)$ is arbitrary but fixed. In a standard way it is possible to prove that (P_1) has a unique optimal control $\bar{u} = \Gamma w$ ($\bar{x}$ is the mild solution corresponding to $\bar{u}$).

Lemma 3.1 The pair $(\bar{u}, \bar{x})$ which satisfies (1.3),$\bar{u} \in L^2(R^+;U), C\bar{x} \in L^2(R^+;Z)$,is optimal for (P_1) if and only if there exists $p \in C(R^+;X) \cap L^2(R^+;X)$ such that

$$p' = -A^* p + C^* C\bar{x} \ , \ t \in R^+ \tag{3.2}$$

$$B_2^* p(t) = \bar{u}(t) \ a.e. \ t > 0 \tag{3.3}$$

Equation (3.2) should be understood in the following sense

$$p(t) = e^{A^*(T-t)} p(T) - \int_t^T e^{A^*(s-t)} C^* C\bar{x}(s)ds \tag{3.4}$$

Proof of Lemma 3.1 It has been proved in [5] that there exists $P_0 \in L(X,X), P_0 = P^* \geq 0$ such that the feedback law $u = -B_2^* P_0 x$ is stabilizing the system (1.1).Let $p \in C(R^+;X) \cap L^2(R^+;X)$ be a mild solution of

$$p' = -(A - B_2 B_2^* P_0)^* p - P_0 B_2 \bar{u} + C^* C\bar{x} \ , \ t \in R^+ \tag{3.5}$$

i.e.

$$p(t) = -\int_t^\infty e^{(A - B_2 B_2^* P_0)^*(s-t)} (C^* C\bar{x}(s) - P_0 B_2 \bar{u}(s))ds \ , \tag{3.6}$$

(Since $A - B_2 B_2^* P_0$ is the generator of an exponentially stable semigroup and $C\bar{x} \in L^2(R^+;Z), \bar{u} \in L^2(R^+;U)$ it follows that $p \in L^2(R^+;X)$)

Consider $v \in L^2(R^+;U)$ such that the mild solution of

$$\begin{cases} y = Ay + B_2 v \ , \ t \in R^+ \\ y(0) = 0 \end{cases} \tag{3.7}$$

belongs to $L^2(R^+;X)$.Since

$$\int_0^\infty (|C\bar{x}|_Z^2 + |\bar{u}|_U^2)dt \leq \int_0^\infty (|Cx^{\bar{u}+\varepsilon v}|_Z^2 + |\bar{u} + \varepsilon v|_U^2)dt \ , \ \forall \varepsilon > 0 \tag{3.8}$$

(where $x^{u+\varepsilon v}$ is given by (1.3) with $u = \bar{u} + \varepsilon v$) it follows that

$$\int_0^\infty (\langle C\bar{x}, Cy\rangle_Z + \langle \bar{u}, v\rangle_U)dt \geq 0 \tag{3.9}$$

Multiplying (3.6) by $B_2 v$ and integrating from 0 to $+\infty$ we obtain after an easy calculation that

$$-\int_0^\infty \langle p(t), B_2 v(t)\rangle dt = \int_0^\infty (\langle P_0 B_2 B_2^* p(t), y(t)\rangle - \langle P_0 B_2 \bar{u}, y(t)\rangle + \langle C\bar{x}(t), Cy(t)\rangle_Z)dt \tag{3.10}$$

(we have used that

$$y(t) = \int_0^t e^{(A - B_2 B_2^* P_0)(t-s)} (B_2 B_2^* P_0 y(s) + B_2 v(s))ds \ )$$

Using (3.9) and (3.10) we may conclude that

$$\int_0^\infty \langle \bar{u}(t) - B_2^* p(t), v(t) + B_2^* P_0 y(t)\rangle_U dt \geq 0 \tag{3.11}$$

If we take $v = u - B_2^* P_0 y$,where $u \in L^2(R^+;U)$ is arbitrary,then $y \in L^2(R^+;X)$ and

$$\int_0^\infty \langle \bar{u}(t) - B_2^* p(t), u(t)\rangle_U dt \geq 0 \ , \ \forall u \in L^2(R^+;U),$$

which implies (3.3).

Conversely the proof is trivial.

Consider $\Gamma : L^2(R^+;W) \to L^2(R^+;U)$ defined by $\Gamma w = \bar{u}$.We shall prove that Γ is an affine function,i.e.

$$\Gamma w = \Gamma_0 w + f_0 \ , \tag{3.12}$$

where $\Gamma_0 \in L(L^2(R^+;W), L^2(R^+;U))$ and $f_0 = arg\ inf \tilde{K}(u,0)$ (see also [1]).

Using Lemma 3.1 it easily follows that Γ is an afine function.More difficult is to prove that Γ_0 is a bounded operator.

Consider $\bar{x}_0$ the mild solution of (1.1),with $\bar{x}_0 = x_0$, corresponding to $u = f_0, w = 0$ and let $\bar{v}$ be the optimal control of (P_1) corresponding to the initial condition $x_0 = 0$ and to the input disturbance w (we denote by y the function given by (1.3) and corresponding to $x_0 = 0, u = \bar{v}$ and w).We get

$$\int_0^\infty (|C\bar{x}|_Z^2 + |\Gamma w|_U^2)dt \le \int_0^\infty (|C(\bar{x}_0 + \bar{y})|_Z^2 + |f_0 + \bar{v}|_U^2)dt =$$

$$= \int_0^\infty (|C\bar{x}_0|_Z^2 + |C\bar{y}|_Z^2 + 2\langle C\bar{x}_0, C\bar{y}\rangle_Z +$$

$$+ |f_0|_U^2 + |\bar{v}|_U^2 + 2\langle f_0, \bar{v}\rangle_U)dt \le 2\int_0^\infty (|C\bar{x}_0|_Z^2 + |f_0|_U^2)dt +$$

$$+ 2\int_0^\infty (|C\bar{y}|_Z^2 + |\bar{v}|_U^2)dt \le M + 2\gamma^2 \|w\|^2_{L^2(R^+;W)}$$

$\forall w \in L^2(R^+;W)$,where $M > 0$ is a constant independent of w.It means that

$$\|\Gamma w\|^2_{L^2(R^+;U)} \le c_1 + c_2\|w\|^2_{L^2(R^+;W)} \ , \ \forall w \in L^2(R^+;W),$$

where $c_1, c_2 > 0$ are constants independent of w and after an easy calculation involving the relation $\Gamma w = \Gamma_0 w + f_0$ we obtain the boundedness of Γ_0.

Let $\varphi : L^2(R^+;W) \to R$ defined by

$$\varphi(w) = -\tilde{K}(\Gamma w, w)$$

By (3.12) we may represent

$$\varphi(w) = \|\tilde{D}w\|^2_{L^2(R^+;W)} + ((w,f)) + \delta \ ,$$

where $\tilde{D} \in L(L^2(R^+;W), L^2(R^+;W))$, $f \in L^2(R^+;W)$ and $((\cdot,\cdot))$ is the inner product in $L^2(R^+;W)$, $\delta \in R$ ($((l,m)) = \int_0^{+\infty} \langle l(t), m(t)\rangle_W dt$).The existence of an $F \in L(X,U)$ such that $\|S_F\| < \gamma$ implies $\varphi(w) \ge \eta\|w\|^2_{L^2(R^+;W)} + \beta$, $\forall w \in L^2(R^+;W)$, where $\eta > 0$ and $\beta \in R$.

In a standard way we can prove the existence of an unique $w^* \in L^2(R^+;W)$ such that

$$w^* = arg\ inf\{\varphi(w); w \in L^2(R^+;W)\}$$

Lemma 3.2

$$w^*(t) = -\gamma^{-2}B_1^* p(t) \ a.e.\ t > 0 \tag{3.13}$$

Proof of Lemma 3.2 The following inequality

$$\int_0^\infty (|Cx^*|_Z^2 + |\Gamma w^*|_U^2 - \gamma^2|w^*|_W^2)dt$$

$$\ge\ge \int_0^\infty (|Cx^{w^*+\varepsilon w}|_Z^2 + |\Gamma(w^* + \varepsilon w)|_U^2 - \gamma^2|w^* + \varepsilon w|_W^2)dt \ ,$$

$\forall \varepsilon > 0$ (where x^* *is* the mild solution of (1.1) with $x^*(0) = x_0$ corresponding to the inputs $u = \Gamma w^*$ and $w = w^*$ and $x^{w^*+\varepsilon w}$ is the mild solution of (1.1) with $x(0) = x_0$ corresponding to the inputs $u = \Gamma\ (w^* + \varepsilon w)$ and $w^* + \varepsilon w$) implies that

$$\int_0^\infty (\langle Cx^*, Cy\rangle_Z + \langle \Gamma_0 w^* + f_0, \Gamma_0 w\rangle_U - \gamma^2\langle w^*, w\rangle_W)dt \ge 0, \tag{3.14}$$

$\forall w \in L^2(R^+;W)$,with $w(t) = 0$ $a.e.\ t \ge T_1$ ($T_1 \in R^+$) where y is the mild solution of

$$\begin{cases} y' = Ay + B_2\Gamma_0 w + B_1 w \ , \ t > 0 \\ y(0) = 0 \end{cases} \tag{3.15}$$

y satisfies $Cy \in L^2(R^+;Z)$.

We shall prove that $B_1^* p \in L^2(0,T_1;W)$.It will be enough to prove that $(\lambda_0 - A^*)^{1-\alpha} p \in L^2(0,T_1;X)$ (because $B_1^* = H_2^*(\lambda_0 - A^*)^{1-\alpha}$).We notice that

$$p(t) = e^{A^*(T-t)}p(T) - \int_t^T e^{A^*(s-t)}C^*Cx^*(s)ds,$$

where $T > T_1$ is arbitrary but fixed.This yields

$$(\lambda_0 - A^*)^{1-\alpha}p(t) = (\lambda_0 - A^*)^{-\alpha}e^{A^*(T_1-t)}(\lambda_0 - A^*)e^{A^*(T-T_1)}p(T) -$$

$$- \int_t^T (\lambda_0 - A^*)^{1-\alpha}e^{A^*(s-t)}C^*Cx^*(s)ds$$

($(\lambda_0-A^*)^{-\alpha}e^{A^*(T_1-\cdot)}(\lambda_0-A^*)e^{A^*(T-T_1)}p(T)$ is continuous from $[0,T_1]$ to X because $e^{A^*(T-T_1)}p(T)\in D(A^*)$ $(T>T_1)\Rightarrow(\lambda_0-A^*)e^{A^*(T-T_1)}p(T)\in X$ and $(\lambda_0-A^*)^{-\alpha}L(X,X)$).

Let's prove now that $\int_t^T(\lambda_0-A^*)^{1-\alpha}e^{A^*(s-t)}C^*Cx^*(s)ds$ belongs to $L^2(0,T_1;X)$.

Indeed,the analyticity of e^{At} implies the existence of a constant $M_1>0$ such that

$$\|(\lambda_0-A^*)^{1-\alpha}e^{A^*t}\|_{L(X)}\le\frac{M_1}{t^{1-\alpha}}\ ,\forall t\in(0,T].$$

We may conclude that

$$\int_0^{T_1}|\int_t^T(\lambda_0-A^*)^{1-\alpha}e^{A^*(s-t)}C^*Cx^*(s)ds|^2dt\le$$
$$\le M_1\int_0^{T_1}(\int_t^T\frac{|C^*Cx^*(s)|}{(s-t)^{1-\alpha}}ds)^2dt\le$$
$$\le M_1\int_0^{T_1}\int_t^T\frac{h(s)^2}{(s-t)^{1-\alpha}}ds\int_t^T\frac{1}{(s-t)^{1-\alpha}}dsdt\le$$

(We have denoted by h(t)=$|C^*Cx^*(t)|$)

$$\le M_2+M_3\int_0^{T_1}\int_t^{T_1}\frac{h(s)^2}{(s-t)^{1-\alpha}}dsdt=$$
$$M_2+M_3\int_0^{T_1}h(s)^2\int_0^s\frac{1}{(s-t)^{1-\alpha}}dtds\le$$
$$\le M_2+M_4\|h\|^2_{L^2(0,T_1)},$$

where $M_1,M_2,M_3,M_4>0$ are constants independent of h.

Since $\Gamma_0w\in L^2(R^+;U)$,we get

$$\int_0^\infty\langle p(t),B_2\Gamma_0w(t)\rangle dt+\int_0^\infty\langle B_1^*p(t),w(t)\rangle_W dt=$$
$$=-\int_0^\infty\int_t^\infty\langle e^{A^*(s-t)}C^*Cx^*(s),B_2\Gamma_0w(t)\rangle dsdt-$$
$$-\int_0^\infty\int_t^\infty\langle B_1^*e^{A^*(s-t)}C^*Cx^*(s),w(t)\rangle_W dsdt=$$
$$=-\int_0^\infty\int_0^s\langle Cx^*(s),Ce^{A(s-t)}(B_2\Gamma_0w(t)+B_1w(t))\rangle_Z dtds=$$
$$-\int_0^\infty\langle Cx^*(s),Cy(s)\rangle_Z ds$$

From (3.14) and the last relation follows that

$$-\int_0^\infty\langle\Gamma_0^*B_2^*p(t),w(t)\rangle_W dt-\int_0^\infty\langle B_1^*p(t),w(t)\rangle_W dt+$$
$$+\int_0^\infty\langle\Gamma_0w^*(t)+f_0(t),\Gamma_0w(t)\rangle_U dt-\gamma^2\int_0^\infty\langle w^*(t),w(t)\rangle_W dt=$$
$$=\int_0^\infty\langle B_1^*p(t)+\gamma^2w^*(t),w(t)\rangle_W dt\ge0\ ,\ \forall w\in L^2(R^+;W)$$

with $w(t)=0$ $a.e.$ $t\ge T_1$ (because $B_2^*p(t)=\Gamma_0w^*(t)+f_0(t)$),which implies that $B_1^*p(t)=-\gamma^2w^*(t)$ $a.e.$ $t\in[0,T_1]$ and since T_1 is arbitrary we may conclude that

$$w^*(t)=-\gamma^{-2}B_1^*p(t)\ a.e.\ t\in R^+.$$

We set

$$Px_0=-p(0)$$

Applying Yakubovich's lemma (lemma 3.4 in [5]) we infer the existence of an unique $R\in L(X,L^2(R^+;W))$ such that

$$w^*=Rx_0 \tag{3.16}$$

Relation (3.2) implies that P is a linear operator.It is also easy to prove that f_0 is a linear and continuous function of argument x_0. Using (3.12) we may conclude that

$$\int_0^\infty(|Cx^*|_Z^2+|u^*|_U^2)dt\le2\gamma^2\|w^*\|^2_{L^2(R^+;W)}+$$
$$+2\int_0^\infty(|Ce^{(A-B_2B_2^*P_0)t}x_0|_Z^2+|B_2^*P_0e^{(A-B_2B_2^*P_0)t}x_0|_U^2)dt\le d|x_0|^2,$$

where $d>0$ is a constant independent of x_0.

The last relation and (3.6) imply that

$$|p(0)|\le d|x_0|$$

and $P\in L(X,X)$. Since (u^*,w^*) is also the optimal solution to problem

$$\sup_{w\in L^2(R_t^+;W)}\ \inf_{L^2(R_t^+;U)}\ \tilde K(u,w)$$

(where $R_t^+=[t,+\infty)$) we infer that

$$Px^*(t)=-p(t)\ ,\ \forall t\ge0.$$

Let denote by $S_P(t) : X \to X$,the family of operators

$$S_P(t)x_0 = x^*(t) \ , \ \forall t \geq 0$$

It is readily seen that $S_P(t)$ is a C_0-semigroup in X and $A_P = A - B_2B_2^*P + \gamma^{-2}B_1B_1^*P$ is the generator of this semigroup (see Proposition 1 in [3]).

Using now the equality

$$Px_0 = e^{A^*T}Pe^{A_PT}x_0 + \int_0^\infty e^{A^*t}C^*Ce^{A_Pt}x_0 dt$$

we conclude that

$$B_1^*P = H_1^*(\lambda_0 - A^*)P = H_1^*(\lambda_0 - A^*)^\alpha(\lambda_0 - A^*)^{1-\alpha}P$$

This yields

$$|(\lambda_0 - A^*)^{1-\alpha}Px_0| \leq \|(\lambda_0 - A^*)^{1-\alpha}e^{A^*T}\||Pe^{A_PT}x_0|+$$
$$+\int_0^T \|(\lambda_0 - A^*)^{1-\alpha}e^{A^*t}\||C^*Ce^{A_Pt}x_0|dt \leq$$
$$\leq c_1|x_0| + c_2\int_0^T \frac{1}{t^{1-\alpha}}|x_0|dt \leq d|x_0| \ , \ \forall x_0 \in X$$

(c_1, c_2, d are positive constants independent of x_0).

The last relations show that $B_1^*P \in L(X, W)$.As in [4],p.330-331 it is possible to prove that

$$\langle Px_0, z_0\rangle = \int_0^\infty \langle C^*Ce^{A_P\tau}x_0, e^{A_P\tau}z_0\rangle d\tau + \int_0^\infty \langle u^*(\tau, x_0), u^*(\tau, z_0)\rangle_U d\tau -$$
$$-\gamma^2\int_0^\infty \langle w^*(\tau, x_0), w^*(\tau, z_0)\rangle_W dt,$$

where $u^*(\cdot, y_0)$ and $w^*(\cdot, y_0)$ are the optimal inputs corresponding to the initial condition $x(0) = y_0$ (anyway the proof of this fact is trivial).

The last relation shows that $P = P^* \geq 0$.Since $P = P^*$ we obtain that

$$Px = \int_0^\infty e^{A_P^*t}C^*Ce^{At}xdt =$$
$$= \int_t^\infty e^{A_P^*(t-s)}C^*Ce^{A(s-t)}xds \ , \ \forall x \in D(A)$$

Multiplying by $y \in D(A)$ and derivating with respect to t we get

$$0 = \langle C^*Cx, y\rangle + \int_t^\infty \langle C^*Ce^{A(t-s)}Ax, e^{A_P(s-t)}y\rangle ds+$$
$$+\int_t^\infty \langle C^*Ce^{A(s-t)}x, A_Pe^{A_P(s-t)}y\rangle ds =$$
$$= \langle Cx, Cy\rangle_Z + \langle Ax, Py\rangle + \int_t^\infty \langle (A - B_2B_2^*P+$$
$$+\gamma^{-2}B_1B_1^*P)e^{A_P^*(s-t)}C^*Ce^{A(t-s)}x, y\rangle ds =$$
$$= \langle Cx, Cy\rangle_Z + \langle Ax, Py\rangle + \langle Px, Ay\rangle - \langle B_2^*Px, B_2^*Py\rangle_U+$$
$$+\gamma^{-2}\langle B_1^*Px, B_1^*Py\rangle_W$$

That means P is a solution of (2.1) (see also [6],p.161).

b) Suppose that there exists $P \in L(X, X)$ with $B_1^*P \in L(X, W)$, $P = P^* \geq 0$ satisfying (2.1) and such that $A_P = A - (B_2B_2^* - \gamma^{-2}B_1B_1^*)P$ is the generator of an exponentially stable semigroup $T_1(t)$.

Consider $x_0 \in D(A) = D(A_3)$,where $A_3 = A - B_2B_2^*P$.We have

$$\frac{d}{dt}\langle T_3(t)x_0, PT_3(t)x_0\rangle = \langle A_3T_3(t)x_0, PT_3(t)x_0\rangle+$$

(A_3 is the generator of the analytic semigroup $T_3(t)$-see [8])

$$+\langle PT_3(t)x_0, A_3T_3(t)x_0\rangle =$$

(using (2.1))

$$= -\langle (C^*C + P(B_2B_2^* + \gamma^{-2}B_1B_1^*)P)T_3(t)x_0, T_3(t)x_0\rangle \leq$$
$$\leq |CT_3(t)x_0|_Z^2 - |B_2^*PT_3(t)x_0|_U^2$$

Integrating from 0 to T we get

$$\langle T_3(t)x_0, PT_3(t)x_0\rangle - \langle x_0, Px_0\rangle \leq$$
$$\leq -\int_0^T |CT_3(t)x_0|_Z^2dt - \int_0^T |B_2^*PT_3(t)x_0|_U^2dt$$

and since $P \geq 0$ it follows that

$$\int_0^T |CT_3(t)x_0|_Z^2dt + \int_0^T |B_2^*PT_3(t)x_0|_U^2dt \leq \langle x_0, Px_0\rangle$$

Now T was arbitrary and since D(A) is dense in X we see that $CT_3(\cdot)x_0 \in L^2(R^+;Z)$ and $B_2^*PT_3(\cdot)x_0 \in L^2(R^+;U), \forall x_0 \in X$. In addition $T_3(t)x_0$ is also the mild solution of

$$\begin{cases} x'(t) = (A+KC)x(t) - B_2B_2^*PT_3(t)x_0 - KCT_3(t)x_0, t \geq 0 \\ x(0) = x_0 \end{cases}$$

(i.e. $T_3(t)x_0 = e^{(A+KC)t}x_0 - \int_0^t e^{(A+KC)(t-s)}(B_2B_2^*PT_3(s)x_0 + KCT_3(s)x_0)ds$) and A+KC is the generator of an exponentially stable semigroup.These facts allow us to conclude that $T_3(\cdot)x_0 \in L^2(R^+;X), \forall x_0 \in X$, which implies that A_3 is the generator of an exponentially stable semigroup (see [2] or [8]).

Consider now $x_0 = 0$ and x the mild solution of

$$\begin{cases} x' = Ax - B_2B_2^*Px + B_1w \ , \ t \geq 0 \\ x(0) = 0 \end{cases}$$

i.e.

$$x(t) = \int_0^t T_3(t-s)B_1w(s)ds,$$

which denotes the unique solution $x \in L^2(0,T;X), \forall T > 0$ of the problem

$$x(t) = -\int_0^t e^{A(t-s)}B_2B_2^*Px(s)ds + \int_0^t e^{A(t-s)}B_1w(s)ds$$

(here $w \in L^2(R^+;W)$).Denote by

$$h(t) = \int_0^t e^{A(t-s)}B_1w(s)ds$$

$h \in L^2(0,T;X), \forall T > 0$ (this fact has been proved for example in [3]).

Let J:C([0,T];X)→ C([0,T];X) be defined by

$$(J\tilde{\varphi})(t) = -\int_0^t e^{A(t-s)}B_2B_2^*P(\tilde{\varphi}+h)(s)ds$$

and

$$\|f\|_{[0,T]} = sup_{t\in[0,T]}e^{-\beta t}|f(t)|$$

a norm in C([0,T];X) which is equivalent to the usual norm on this space (β will be precised later).We get

$$e^{-\beta t}(J\tilde{\varphi} - J\psi)(t) = -e^{-\beta t}\int_0^t e^{A(t-s)}e^{\beta s}e^{-\beta s}B_2B_2^*P(\tilde{\varphi}-\psi)(s)ds \Rightarrow$$

$$e^{-\beta t}|(J\tilde{\varphi} - J\psi)(t)| \leq e^{-\beta t}\int_0^t e^{\lambda_0(t-s)}e^{\beta s}\|B_2B_2^*P\|\|\tilde{\varphi}-\psi\|_{[0,T]}ds =$$

$$= \|B_2B_2^*P\|\|\tilde{\varphi}-\psi\|_{[0,T]}e^{-(\beta-\lambda_0)t}\frac{1}{\beta-\lambda_0}(e^{(\beta-\lambda_0)t}-1)$$

$$\Rightarrow \|J\tilde{\varphi} - J\psi\|_{[0,T]} \leq L\|\tilde{\varphi}-\psi\|_{[0,T]}, with\ 0 < L < 1,$$

for any $\beta > \|B_2B_2^*P\| + \lambda_0$ arbitrary but fixed.

We set $I_n = n(n-A)^{-1}$.Consider x_n the mild solution of

$$(3.17) \qquad \begin{cases} x_n' = Ax_n - B_2B_2^*Px_n + (\lambda_0 - A)^{1-\alpha}I_nH_2w \ , \ t \geq 0 \\ x_n(0) = 0 \end{cases}$$

($(\lambda_0 - A)^{1-\alpha}I_nH_2$ is a bounded operator).

Suppose that w is continuously differentiable.We have $x_n(t) \in D(A)$ (see [3]-lemma 1) and

$$\begin{aligned} &\frac{d}{dt}\langle x_n(t), Px_n(t)\rangle = \langle A_3x_n(t), Px_n(t)\rangle + \\ &+ \langle(\lambda_0 - A)^{1-\alpha}I_nH_2w(t), Px_n(t)\rangle + \langle Px_n(t), A_3x_n(t)\rangle + \\ &+ \langle Px_n(t), (\lambda_0 - A)^{1-\alpha}I_nH_2w(t)\rangle = \\ &+ -\langle Cx_n(t), Cx_n(t)\rangle_Z - \langle B_2^*Px_n(t), B_2^*Px_n(t)\rangle_U - \\ (3.18) \quad &- \gamma^2\langle w(t) - \gamma^{-2}B_1^*Px_n(t), w(t) - \gamma^{-2}B_1^*Px_n(t)\rangle_W + \\ &+ \langle Px_n(t), (\lambda_0 - A)^{1-\alpha}I_nH_2w(t)\rangle + \gamma^2\langle w(t), w(t)\rangle_W + \\ &+ \langle(\lambda_0 - A)^{1-\alpha}I_nH_2w(t), Px_n(t)\rangle - \\ &- \langle PB_1w(t), x_n(t)\rangle - \langle B_1^*Px_n(t), w(t)\rangle_W \leq \\ &\leq -|z_n(t)|_Z^2 + \gamma^2|w(t)|_W^2 + \\ &+ 2\langle I_nH_2w(t), P_1x_n(t)\rangle - 2\langle H_2w(t), P_1x_n(t)\rangle, \end{aligned}$$

where $z_n = (C - DB_2^*P)x_n$ and $P = (\lambda_0 - A^*)^{\alpha-1}P_1$,with $P_1 \in L(X,X)$ (we already have proved that $(\lambda_0 - A^*)^{1-\alpha}P$ is a bounded operator).

Using the form of x_n we obtain that

$$\|x_n - x\|_{[0,T]} \leq L\|x_n - x\|_{[0,T]} + sup_{[0,T]}e^{-\beta t}|\cdot$$

$$\cdot\int_0^t e^{A(t-s)}(\lambda_0 - A)^{1-\alpha}[I_nH_2w(s) - H_2w(s)]ds$$

$(L \in (0,1))$.

The second term in the right hand side of the last inequality is convergent to 0 for $n \to +\infty$;so taking the limit for $n \to \infty$ we obtain $x_n \to x$ in C([0,T];X). Integrating in (3.18) from 0 to T and passing to the limit we obtain

$$\int_0^T |z(t)|_Z^2 dt \leq \gamma^2 \int_0^T |w(t)|_W^2 dt, \ \forall T > 0$$

$$\Rightarrow \|z\|^2_{L^2(R^+;Z)} \leq \gamma^2 \|w\|^2_{L^2(R^+;W)},$$

where $z = (C - DB_2^* P)x$.

Consider now $w \in L^2(R^+;W)$ and $w_n \in C^1(R^+;W) \cap L^2(R^+;W)$ such that $w_n \to w$ in $L^2(0,T;W)$, $\forall 0 < T < +\infty$.Let y_n be the mild solution of

$$\text{(3.19)} \qquad \begin{cases} y_n' = Ay_n - B_2 B_2^* P y_n + B_1 w_n \ , \ t \geq 0 \\ y_n(0) = 0 \end{cases}$$

i.e.

$$y_n(t) = -\int_0^t e^{A(t-s)} B_2 B_2^* P y_n(s) ds + \int_0^t e^{A(t-s)} B_1 w_n(s) ds$$

Using Young's inequality we obtain that $h_n(t) = \int_0^t e^{A(t-s)} B_1 w_n(s) ds$ is convergent for $n \to +\infty$ to $h(t) = \int_0^t e^{A(t-s)} B_1 w(s) ds$ in $L^2(0,T;X)$, $\forall T > 0$.

If we denote by $\tilde{y}_n = y - h_n$ and $\tilde{x} = x - h$ we may infer that

$$(\tilde{y}_n - \tilde{x})(t) = -\int_0^t e^{A(t-s)} B_2 B_2^* P (\tilde{y}_n - \tilde{x})(s) ds +$$
$$+ \int_0^t e^{A(t-s)} B_2 B_2^* P (h_n - h)(s) ds$$

and since the second term in the last equality is convergent to 0 in C([0,T];X),$\forall T > 0$ for $n \to +\infty$,we get

$$\|\tilde{y}_n - \tilde{x}\|_{[0,T]} \leq L \|y_n - x\|_{[0,T]} +$$
$$+ sup_{[0,T]} e^{-\beta t} \left| \int_0^t e^{A(t-s)} B_2 B_2^* P (h_n - h)(s) ds \right|$$

$(L \in (0,1))$.This last inequality implies that $\tilde{y}_n \to \tilde{x}$ in C([0,T];X) and $y_n \to x$ in $L^2(0,T;X)$.

It means

$$\|z\|^2_{L^2(R^+;Z)} \leq \gamma^2 \|w\|^2_{L^2(R^+;W)}, \forall w \in L^2(R^+;W)$$

Hence $\|S\| \leq \gamma$.

Remark. We have proved only that $\|S\| \leq \gamma$.It would be interesting to know if it is possible to prove that $\|S\|$ is strictly less than γ.

Appendix If $B \in L(X,X)$ and A+B is the generator of an exponentially stable semigroup, then the map $w \mapsto \tilde{x}$,where $\tilde{x}$ is the mild solution of (A.1)

$$\text{(A.1)} \qquad \begin{cases} \tilde{x}' = (A+B)\tilde{x} + B_1 w \ , \ t \geq 0 \\ \tilde{x}(0) = 0 \end{cases}$$

is continuous from $L^2(R^+;W)$ to $L^2(R^+;X)$.

Proof.

Consider y the mild solution of (A.2)

$$\text{(A.2)} \qquad \begin{cases} y' = (A - \lambda_0)y + B_1 w \ , \ t \geq 0 \\ y(0) = 0 \end{cases}$$

(i.e. $y(t) = \int_0^t e^{(A-\lambda_0)(t-s)} (\lambda_0 - A)^{1-\alpha} H_2 w(s) ds$)

We shall prove first that $y \in L^2(R^+;X)$.Indeed

$$\int_0^\infty \left| \int_0^t e^{(A-\lambda_0)(t-s)} (\lambda_0 - A)^{1-\alpha} H_2 w(s) ds \right|^2 dt \leq$$
$$\leq \int_0^\infty \left(\int_0^t |e^{(A-\lambda_0)(t-s)} (\lambda_0 - A)^{1-\alpha} H_2 w(s)| ds \right)^2 dt \leq$$
$$\leq d \int_0^\infty \left(\int_0^t \frac{e^{-\nu(t-s)}}{(t-s)^{1-\alpha}} |w(s)|_W ds \right)^2 dt \leq$$

for a suitable $\nu > 0$,where $d > 0$ is a constant independent of w (because $e^{(A-\lambda_0)t}$ is analytic and exponentially stable-see also [3],p.614)

$$\leq d \int_0^\infty \left(\int_0^t \frac{e^{-\nu(t-s)}}{(t-s)^{1-\alpha}} ds \right) \left(\int_0^t \frac{e^{-\nu(t-s)}}{(t-s)^{1-\alpha}} |w(s)|_W^2 ds \right) dt =$$

(from Hölder's inequality)

$$= d \int_0^\infty \left(\int_0^t \frac{e^{-\nu\tau}}{\tau^{1-\alpha}} d\tau \right) \left(\int_0^t \frac{e^{-\nu\tau}}{\tau^{1-\alpha}} |w(s)|_W^2 ds \right) dt \leq$$
$$\leq dM \int_0^\infty \int_s^\infty \frac{e^{-\nu(t-s)}}{(t-s)^{1-\alpha}} |w(s)|_W^2 dt ds =$$

(here $M = \int_0^\infty \frac{e^{-\nu\tau}}{\tau^{1-\alpha}} d\tau \in R^+$)

$$= dM \int_0^\infty |w(s)|_W^2 \int_0^\infty \frac{e^{-\nu\tau}}{\tau^{1-\alpha}} d\tau ds = dM^2 \|w\|^2_{L^2(R^+;W)}$$

Moreover,we have proved that the map $w \mapsto y$ is continuous from $L^2(R^+;W)$ to $L^2(R^+;X)$. $\tilde{x} - y$ is the mild solution for (A.3)

$$(A.3) \qquad \begin{cases} (\tilde{x} - y)' = (A+B)(\tilde{x} - y) + By + \lambda_0 y \ , \ t \geq 0 \\ (\tilde{x} - y)(0) = 0, \end{cases}$$

i.e.

$$(\tilde{x} - y)(t) = \int_0^t e^{(A+B)(t-s)}(By(s) + \lambda_0 y(s))ds$$

Since A+B is the generator of an exponentially stable semigroup,it easily follows that the function $\tilde{w} \mapsto \int_0^t e^{(A+B)(t-s)}\tilde{w}(s)ds$ is continuous from $L^2(R^+;X)$ to $L^2(R^+;X)$ (i.e.$\exists \eta > 0$ such that $\| \int_0^t e^{(A+B)(t-s)} \tilde{w}(s)ds\|_{L^2(R^+;X)} \leq \eta \|\tilde{w}\|_{L^2(R^+;W)}, \forall \tilde{w} \in L^2(R^+;X)$)

It means that

$$\|\tilde{x} - y\|_{L^2(R^+;X)} \leq \eta \|B + \lambda_0\| \|y\|_{L^2(R^+;X)}$$

We conclude that there exists $\bar{M} > 0$ such that

$$\|\tilde{x} - y\|_{L^2(R^+;X)} \leq \eta \|B + \lambda_0\| \|y\|_{L^2(R^+;X)}$$

and since the map $w \mapsto y$ is continuous from $L^2(R^+;W)$ to $L^2(R^+;X)$,we may infer that the map $w \mapsto \tilde{x}$ is continuous from $L^2(R^+;W)$ to $L^2(R^+;X)$.

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