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ON π-CLOSEDNESS: A UNIFIED THEORY

BY

M.N. MUKHERJEE and G.SENGUPTA[1]

1.Introduction. Some definitions and results. Here, as the title suggests, we are intending to discuss the notion of π-closed spaces along with their certain characterizations. Therefrom we switch over to π-closed topological spaces as a motivated particularization of the above, where π can read any one of the following: *quasi-Hausdorff*, S, s, N and *Hausdorff* (of course under the assumption of a separation axiom in the last case) and many others. So this work does serve as a unified theory agglomerating different types of the so-called *closedness* that are studied with keen interest and attention in general topology. It is, in fact, the immense importance of these concepts in topology that necessitated the origination of such a unified theory which facilitates their study and manipulation in the respective fields by binding them altogether.

Let X be a nonempty set and $\mathcal{P}(X)$ denote the power set of X. By a *space* we shall mean here a triplet $(X, \mathcal{P}, \theta)$ where X is the whole set, $\mathcal{P} \subseteq \mathcal{P}(X)$ such that $X, \emptyset \in \mathcal{P}$ and $\theta : \mathcal{P}(X) \to \mathcal{P}(X)$ is an operator which satisfies the following axiom: $\theta(X) = X$.

We define a second operator $\theta' : \mathcal{P}(X) \to \mathcal{P}(X)$ by $\theta'(A) = X - \theta(X - A)$, for all $A \in \mathcal{P}(X)$. Thus θ' becomes the dual operator of θ. We deduce that $\theta'(\emptyset) = \emptyset$. Also, we shall write $\mathcal{P}' = \{F \subseteq X : X - F \in \mathcal{P}\}$. Then $\mathcal{P}' \neq \emptyset$ as $X \in \mathcal{P}'$. Any member of $\mathcal{P}$ ($\mathcal{P}'$) will be called a $\mathcal{P}$-set (resp. $\mathcal{P}'$-set). Let $a \in X$. Then $P(a)$ ($P'(a)$) will denote a $\mathcal{P}$-set (resp. $\mathcal{P}'$-set) containing a. The class of all such $\mathcal{P}$-sets (resp. $\mathcal{P}'$-sets) containing a will be denoted by $\mathcal{P}(a)$ (resp. $\mathcal{P}'(a)$). Clearly $\mathcal{P}(a) \neq \emptyset$ and $\mathcal{P}'(a) \neq \emptyset$, as $X \in \mathcal{P}(a) \cap \mathcal{P}'(a)$. Let $A(\neq \emptyset) \subseteq X$. Any cover $\mathcal{U}$ of A (i.e., $A \subseteq \bigcup\mathcal{U}$) such

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that $\mathcal{U} \subseteq \mathcal{P}$ will be called a $\mathcal{P}$-cover of A. Next we resort to the following definitions.

Definition 1.1. *Let $A(\neq \emptyset) \subset X$. Then the $(\mathcal{P}-\theta)$-closure of A denoted by $< A >$, is defined as follows:*

$$< A > = \{x \in X : A \cap \theta(P(x)) \neq \emptyset \ \text{ for all } \ P(x) \in \mathcal{P}(x)\}.$$

Definition 1.2. *Let $A(\neq \emptyset) \subset X$. A filterbase $\mathcal{F}$ in A is said to*

(a) $\mathcal{P}$-accumulate at a point $a \in A$ if for every $P(a) \in \mathcal{P}(a)$ and for every $F \in \mathcal{F}$, $F \cap \theta(P(a)) \neq \emptyset$,

(b) $\mathcal{P}$-converge to a point $a \in A$ if for every $P(a) \in \mathcal{P}(a)$, there exists an $F \in \mathcal{F}$ such that $F \subseteq \theta(P(a))$.

Definition 1.3. *Let $y : (D, \geq) \to A$ be a net in a nonempty subset A of X, where (and also in all subsequent discussions) $(D, \geq)$ or simply D denotes a directed set. For each $m \in D$, let $T_m = \{n \in D : n \geq m\}$. The net y is said to*

(a) $\mathcal{P}$-accumulate at a point $a \in A$ if for every $P(a) \in \mathcal{P}(a)$ and for every $m \in D$, $y(T_m) \cap \theta(P(a)) \neq \emptyset$,

(b) $\mathcal{P}$-converge to a point $a \in A$ if for every $P(a) \in \mathcal{P}(a)$, there exists $m \in D$, such that $y(T_m) \subseteq \theta(P(a))$, i.e. y is eventually in $\theta(P(a))$, for every $P(a) \in \mathcal{P}(a)$.

Definition 1.4. *A nonempty subset A of X will be called π-closed relative to X or simply a π-set if for every $\mathcal{P}$-cover $\{P_\alpha : \alpha \in I\}$ of A (I being any index set), there exists a finite subfamily $\{P_{\alpha_i} : i = 1, 2, \ldots, n\}$ (say) of the above family such that $A \subseteq \bigcup_{i=1}^{n} \theta(P_{\alpha_i})$; if in addition, $A = X$ then X is called a π-closed space.*

We now state and prove a sequence of lemmas to use them in the sequel.

Lemma 1.5. *Let $A(\neq \emptyset) \subset X$ and let $\mathcal{F}$ be a filterbase in A and $\mathcal{M}$ a maximal filterbase in A. Then*

(a) $\mathcal{F}$ $\mathcal{P}$-accumulates at $a \in A$ if $\mathcal{F}$ $\mathcal{P}$-converges to a,

(b) $\mathcal{M}$ $\mathcal{P}$-accumulates at $a \in A$ iff $\mathcal{M}$ $\mathcal{P}$-converges to a.

Proof. (a) Since $\mathcal{F}$ $\mathcal{P}$-converges to $a \in A$, for every $P(a) \in \mathcal{P}(a)$, there exists $F \in \mathcal{F}$ such that $F \subseteq \theta(P(a))$. If $G \in \mathcal{F}$, then $F \cap G \neq \emptyset$, so that $G \cap \theta(P(a)) \neq \emptyset$, i.e. $\mathcal{F}$ $\mathcal{P}$-accumulates at a.

(b) We first assume that $\mathcal{M}$ $\mathcal{P}$-accumulates at $a \in A$. Let $P(a) \in \mathcal{P}(a)$. Then for every $M \in \mathcal{M}$, $M \cap \theta(P(a)) \neq \emptyset$, so that

(1) $$M \cap [\theta(P(a)) \cap A] \neq \emptyset$$

We claim that $\theta(P(a)) \cap A \ (\neq \emptyset) \in \mathcal{M}$. If possible, let $\theta(P(a)) \cap A \notin \mathcal{M}$. Let $\mathcal{G} = \mathcal{M} \cup \{\theta(P(a)) \cap A\}$. Then by virtue of (1), it is easy to check that $\mathcal{G}$ has finite intersection property (henceforth abbreviated as f.i.p.). Let $\mathcal{F} = \left\{\bigcap_{i=1}^{n} G_i : n \in N,\ G_i \in \mathcal{G}\right\}$, where N denotes the set of all natural numbers. Then clearly $\mathcal{F}$ is a filterbase in A such that $\mathcal{F} \supseteq \mathcal{G} \supseteq \mathcal{M}$. Since $\mathcal{M}$ is a maximal filterbase in A we have $\mathcal{F} = \mathcal{M} = \mathcal{G}$. Thus $\theta(P(a)) \cap A \in \mathcal{G} = \mathcal{M}$. Thus for every $P(a) \in \mathcal{P}(a)$, there exists $\theta(P(a)) \cap A \in \mathcal{M}$ such that $\theta(P(a)) \cap A \subseteq \theta(P(a))$. Hence $\mathcal{M}$ $\mathcal{P}$-converges to a. The proof of the converse part follows from (a).

Lemma 1.6. *Let $A(\neq \emptyset) \subset X$ and $y : (D, \geq) \to A$ be a universal net in A. If y $\mathcal{P}$-accumulates at $a (\in A)$, then y $\mathcal{P}$-converges to a.*

Proof. Let $P(a) \in \mathcal{P}(a)$. Since y $\mathcal{P}$-accumulates at a, for every $m \in D$ there exists $n \in D$ with $n \geq m$ such that $y(n) \in \theta(P(a))$, i.e.,

(1) $$y(n) \in \theta(P(a)) \cap A.$$

Since y is a universal net in A, it is eventually in either $\theta(P(a)) \cap A$ or $A - \theta(P(a)) \cap A$. But the second case is ruled out in view of (1). Hence there exists $m_1 \in D$ such that $y(n) \in \theta(P(a)) \cap A \subseteq A$, for all $n \in D$ with $n \geq m_1$. Hence the result follows.

Lemma 1.7. *Let $A(\neq \emptyset) \subset X$ and let $y : D \to A$ be a net. Then $\{y(T_m) : m \in D\} = \mathcal{F}$ (say) is the filterbase in A associated with y. If $\mathcal{F}$ $\mathcal{P}$-accumulates at $a \in A$, then y $\mathcal{P}$-accumulates at a.*

Proof. Straightforward and omitted.

Lemma 1.8. *Let $\mathcal{F}$ be a filterbase in a nonempty subset A of X. Let $D = \{(x, F) \in X \times \mathcal{F} : x \in F\}$. For every $(x, F), (y, G) \in D$ define $(x, F) \geq (y, G)$ iff $F \subseteq G$. Then $\geq$ directs D and the net $y : D \to A$ associated with $\mathcal{F}$ is defined by $y(x, F) = x$, for all $(x, F) \in D$. If y $\mathcal{P}$-accumulates at $a \in A$, then $\mathcal{F}$ $\mathcal{P}$-accumulates at a.*

Proof. Straightforward and omitted.

Lemma 1.9. *Let $A(\neq \emptyset) \subset X$, $y : D \to A$ be a net and let $a \in A$. Then y $\mathcal{P}$-accumulates at a if there exists a subnet of y which $\mathcal{P}$-converges to a.*

Proof. Straightforward and omitted.

We conclude this section by observing the following fact, the proof of which is straightforward by the application of Zorn's lemma.

Lemma 1.10. *Let $A(\neq \emptyset) \subset X$ and let $\mathcal{F}$ be any filterbase in X which meets A. Then $\mathcal{F}$ is contained in a maximal filterbase in X which also meets A.*

2. Characterizations of π-closedness. This section is meant for giving several characterizations of π-closedness through covers, filterbases, nets and other classes of sets as follows.

Theorem 2.1. *Let $A(\neq \emptyset) \subset X$ and let $(D, \geq)$ be, as usual, a directed set. Then the following are equivalent:*

(a) *A is π-closed relative to X.*

(b) *For every family of nonempty sets $\{F_\alpha : \alpha \in I\}$ with $\left(\bigcap_{\alpha \in I} < F_\alpha >\right) \cap A = \emptyset$, there exists a finite subfamily $\{F_{\alpha_i} : i = 1, 2, \ldots, n\}$ (say) such that $\left(\bigcap_{i=1}^{n} F_{\alpha_i}\right) \cap A = \phi$.*

(c) *For every family $\{F_\alpha : \alpha \in I\}$ of nonempty $\mathcal{P}'$-sets with $\left(\bigcap_{\alpha \in I} F_\alpha\right) \cap A = \emptyset$, there exists a finite subfamily $\{F_{\alpha_i} : i = 1, 2, \ldots, n\}$ (say) such that $\left(\bigcap_{i=1}^{n} \theta'(F_{\alpha_i})\right) \cap A = \phi$.*

(d) *Every filterbase in A $\mathcal{P}$-accumulates at some point in A.*

(e) *Every filterbase in X which meets A, $\mathcal{P}$-accumulates at some point in A.*

(f) *Every maximal filterbase in A $\mathcal{P}$-converges to some point in A.*

(g) *Every maximal filterbase in X which meets A, $\mathcal{P}$-converges to some point in A.*

(h) *Every net $y : D \to A$ $\mathcal{P}$-accumulates at some point in A.*

(i) *Every net $y : D \to A$ has a subnet which $\mathcal{P}$-converges to some point in A.*

(j) *Every universal net $y : D \to A$ $\mathcal{P}$-converges to some point in A.*

Proof. We note that it suffices to prove the folowing sequence of implications: (a) $\Longrightarrow$ (c) $\Longrightarrow$ (a) $\Longrightarrow$ (d) $\Longrightarrow$ (e) $\Longrightarrow$ (f) $\Longrightarrow$ (g) $\Longrightarrow$ (d) $\Longrightarrow$ (h) $\Longrightarrow$ (j) $\Longrightarrow$ (i) $\Longrightarrow$ (h) $\Longrightarrow$ (d) $\Longrightarrow$ (b) $\Longrightarrow$ (a) and so we proceed accordingly.

(a) $\Leftrightarrow$ (c): The proof is obvious in the light of De Morgan's laws.

(a) $\Longrightarrow$ (d): Let A be a π-set and if possible, let there exists a filterbase $\mathcal{F}$ in A such that $\mathcal{F}$ does not $\mathcal{P}$-accumulates at any point of A. Then for each $a \in A$, there exist $P(a) \in \mathcal{P}(a)$ and $F_a \in \mathcal{F}$ such that

$$F_a \cap \theta(P(a)) = \emptyset \tag{1}$$

Now $\{P(a) : a \in A\}$ being a $\mathcal{P}$-cover of A, by (a) there exists a finite number

of points $a_1, \ldots, a_n \in A$ such that

$$A \subseteq \bigcup_{i=1}^{n} \theta(P(a_i)) \tag{2}$$

From (1), $F_a \subseteq X - \theta(P(a))$, for all $a \in A$. Hence $\bigcap_{i=1}^{n} F_{a_i} \subseteq \bigcap_{i=1}^{n} [X - \theta(P(a_i))] = X - \bigcup_{i=1}^{n} \theta(P(a_i)) \subseteq X - A$ (by (2)), i.e., $\left(\bigcap_{i=1}^{n} F_{a_i}\right) \cap A = \emptyset$ so that $\bigcap_{i=1}^{n} F_{a_i} = \emptyset$. The contradiction proves (d).

(d) $\Longrightarrow$ (e): Let $\mathcal{F}$ be any filterbase in X which meets A. Then $\mathcal{G} = \{F \cap A : F \in \mathcal{F}\}$ is clearly a filterbase in A. Hence by (d), $\mathcal{G}$ $\mathcal{P}$-accumulates at some point $a \in A$, i.e., for every $P(a) \in \mathcal{P}(a)$, $(F \cap A) \cap \theta(P(a)) \neq \emptyset$, and consequently, $F \cap \theta(P(a)) \neq \emptyset$ for every $F \in \mathcal{F}$. Hence $\mathcal{F}$ $\mathcal{P}$-accumulates at $a \in A$.

(e) $\Longrightarrow$ (f): Let $\mathcal{M}$ be a maximal filterbase in A. Then clearly $\mathcal{M}$ is a filterbase in X which meets A so that by (e), $\mathcal{M}$ $\mathcal{P}$-accumulates at some point a of A. Hence by Lemma 1.5 (b), $\mathcal{M}$ $\mathcal{P}$-converges to $a(\in A)$.

(f) $\Longrightarrow$ (g): Let $\mathcal{M}$ be any maximal filterbase in X which meets A. Then $\mathcal{N} = \{M \cap A : M \in \mathcal{M}\}$ is clearly a filterbase in A. Hence there exists a maximal filterbase $\mathcal{N}^*$ in A such that $\mathcal{N}^* \supseteq \mathcal{N}$. By (f), $\mathcal{N}^*$ $\mathcal{P}$-converges to some point a of A. It then follows from Lemma 1.5 that $\mathcal{N}^*$ $\mathcal{P}$-accumulates at a. Now for every $P(a) \in \mathcal{P}(a)$ and every $N \in \mathcal{N}$, $N \cap \theta(P(a)) \neq \emptyset$, i.e., for every $P(a) \in \mathcal{P}(a)$ and for every $M \in \mathcal{M}$, $M \cap [\theta(P(a)) \cap A] \neq \emptyset$. Now, as in the proof of Lemma 1.5 (b), we observe similarly that $(\emptyset \neq) \theta(P(a)) \cap A \in \mathcal{M}$. Hence for every $P(a) \in \mathcal{P}(a)$, there exists $\theta(P(a)) \cap A \in \mathcal{M}$ such that $\theta(P(a)) \cap A \subseteq \theta(P(a))$ so that (g) follows.

(g) $\Longrightarrow$ (d): Let $\mathcal{F}$ be any filterbase in A. By Lemma 1.10, $\mathcal{F}$ is contained in a maximal filterbase $\mathcal{F}^*$ (say) in X which meets A. By (g), $\mathcal{F}^*$ $\mathcal{P}$-converges to some $a \in A$ so that by Lemma 1.5 (taking $A = X$), $\mathcal{F}^*$ $\mathcal{P}$-accumulates at a. Now, for every $P(a) \in \mathcal{P}(a)$ and for every $F \in \mathcal{F}$, $F \cap \theta(P(a)) \neq \emptyset$. Hence $\mathcal{F}$ $\mathcal{P}$-accumulates at $a \in A$.

(d) $\Longrightarrow$ (h): Let $y : D \to A$ be any net in A. We consider the filterbase $\mathcal{F} = \{y(T_m) : m \in D\}$ associated with y. By (d), $\mathcal{F}$ $\mathcal{P}$ - accumulates at some $a \in A$. Hence by Lemma 1.7, y $\mathcal{P}$- accumulates at $a(\in A)$.

(h) $\Longrightarrow$ (j): Follows from Lemma 1.6.

(j) $\Longrightarrow$ (i): Let $y : D \to A$ be any net. We know that every net has a subnet which is universal (see [10]). Hence there exists a subnet $: E \to A$

(where $(E, \geq)$ is a directed set) of y such that z is a universal net. By (j), z $\mathcal{P}$-converges to some point a of A. Hence (i) follows.

(i) $\Longrightarrow$ (h): The implication is obvious from Lemma 1.9.

(h) $\Longrightarrow$ (d): Let $\mathcal{F}$ be any filterbase in A. Then the net $y : D \to A$ associated with the filterbase $\mathcal{P}$-accumulates at some point $a \in A$. Hence by Lemma 1.8, $\mathcal{F}$ $\mathcal{P}$-accumulates at $a(\in A)$.

(d) $\Longrightarrow$ (b): If possible, let $\mathcal{F} = \{F_\alpha : \alpha \in I\}$ be a family of nonempty sets with $\left(\bigcap_{\alpha \in I} < F_\alpha >\right) \cap A = \emptyset$ such that for every finite subfamily $\{F_{\alpha_i} : i = 1, 2, \ldots, n\}$ of the above family, $\left(\bigcap_{\alpha \in I}^{n} F_{\alpha_i}\right) \cap A \neq \emptyset$. Then clearly $\mathcal{G} = \left\{\left(\bigcap_{i=1}^{n} F_{\alpha_i}\right) \cap A : n \in I\!N,\ F_{\alpha_i} \in \mathcal{F}\right\}$ is a filterbase in A. By (d), $\mathcal{G}$ $\mathcal{P}$-accumulates at some point a of A. Thus for every $P(a) \in \mathcal{P}(a)$ and for every $G \in \mathcal{G}$, $G \cap \theta(P(a)) \neq \emptyset$. In particular, we have for all $\alpha \in I$, $(F_\alpha \cap A) \cap \theta(P(a)) \neq \emptyset$, i.e., $F_\alpha \cap \theta(P(a)) \neq \emptyset$, so that $a \in < F_\alpha >$. Hence $a \in \bigcap_{\alpha \in I} < F_\alpha >$. Consequently, $\left[\bigcap_{\alpha \in I} < F_\alpha >\right] \cap A \neq \emptyset$ and we arrive at a contradiction. Hence (b) follows.

(b) $\Longrightarrow$ (a): Let $\{P_\alpha : \alpha \in I\}$ be any $\mathcal{P}$-cover of A. Then $X - A \supseteq X - \bigcup_{\alpha \in I} P_\alpha = \bigcap_{\alpha \in I} (X - P_\alpha)$ so that

$$(1) \qquad \left[\bigcap_{\alpha \in I} (X - P_\alpha)\right] \cap A = \emptyset$$

We consider the family of sets $\mathcal{F} = \{X - \theta(P_\alpha) : \alpha \in I\}$. Now if $X - \theta(P_{\alpha_0}) = \emptyset$ for some $\alpha_0 \in I$, then $A \subseteq X = \theta(P_{\alpha_0})$ and (a) follows. So we assume that $X - \theta(P_\alpha) \neq \emptyset$, for all $\alpha \in I$. We assert that $\left[\bigcap_{\alpha \in I} < X - \theta\left(P_\alpha\right) >\right] \cap A = \emptyset$. Let $\alpha \in I$. We first show that $< X - \theta(P_\alpha) > \subseteq X - P_\alpha$. In fact, let $x \in < X - \theta(P_\alpha) >$ (assuming it to be nonempty). Then for every $P(x) \in \mathcal{P}(x)$,

$$(2) \qquad [X - \theta(P_\alpha)] \cap \theta(P(x)) \neq \emptyset$$

Hence $x \notin P_\alpha$ (for otherwise, $P_\alpha \in \mathcal{P}(x)$, but $[X - \theta(P_\alpha)] \cap \theta(P_\alpha) = \emptyset$ contradicting (2)), i.e. $x \in X - P_\alpha$. Thus $< X - \theta(P_\alpha) > \subseteq X - P_\alpha$, for all $\alpha \in I$. Hence $\left[\bigcap_{\alpha \in I} < X - \theta(P_\alpha) >\right] \cap A \subseteq \left[\bigcap_{\alpha \in I} (X - P_\alpha)\right] \cap A = \emptyset$ (by (1)) so that $\left[\bigcap_{\alpha \in I} < X - \theta(P_\alpha) >\right] \cap A = \emptyset$. By (b), there exists a finite subfamily $\{X - \theta(P_{\alpha_i}) : i = 1, 2, \ldots, n\}$ (say) of $\mathcal{F}$ such that $\left[\bigcap_{i=1}^{n} (X - \theta(P_{\alpha_i}))\right] \cap A = \emptyset$. Then $A \subseteq X - \bigcap_{i=1}^{n} (X - \theta(P_{\alpha_i})) = \bigcup_{i=1}^{n} \theta(P_{\alpha_i})$, which proves (a). This completes the proof of the theorem.

In particular, if we take A to be equal to the entire space X in the above theorem, then we get the following characterization of π-closed spaces.

Theorem 2.2. *In a space X, the following are equivalent:*

(a) *X is π-closed.*

(b) *For every family of nonempty sets $\{F_\alpha : \alpha \in I\}$ with $\bigcap_{\alpha \in I} < F_\alpha > = \emptyset$, there exists a finite subfamily $\{F_{\alpha_i} : i = 1, 2, \ldots, n\}$ such that $\bigcap_{i=1}^{n} F_{\alpha_i} = \emptyset$.*

(c) *For every family $\{F_\alpha : \alpha \in I\}$ of nonempty $\mathcal{P}'$-sets with $\bigcap_{\alpha \in I} F_\alpha = \emptyset$, there exists a finite subfamily $\{F_{\alpha_i} : i = 1, 2, \ldots, n\}$ such that $\bigcap_{i=1}^{n} \theta'(F_{\alpha_i}) = \emptyset$.*

(d) *Every filterbase in X $\mathcal{P}$-accumulates at some point in X.*

(e) *Every maximal filterbase in X $\mathcal{P}$-converges to some point in X.*

(f) *Every net $y : D \to X$ $\mathcal{P}$-accumulates at some point in X.*

(g) *Every net $y : D \to X$ has a subnet which $\mathcal{P}$-converges to some point in X.*

(h) *Every universal net $y : D \to A$ $\mathcal{P}$-converges to some point in X.*

3. Particular cases. We now consider a topology τ on X and consider our space to be a quadruple $(X, \tau, \mathcal{P}, \theta)$. Henceforth, we shall be concerned with π-closedness of topological spaces and instead of speaking about *π − closed spaces*, we shall, from now on, be speaking about *π − closed topological spaces*.

In this connection, we will read the pair $\mathcal{P}, \theta$ to be *open, closure, semi-open, closure, semi-open, semi-closure, regular open, interior closure* to get the corresponding characterizations of QHC spaces ([3],[8],[15]), $\mathcal{S}$-closed spaces ([7],[18]), s-closed spaces ([12]), and N-closed spaces (which are popularly known as nearly compact spaces [17]) respectively from Theorem 2.2, and also the characterizations of sets QHC relative to X, S-closed relative to X, s-closed relative to X and N-closed relative to X ([5]) respectively are obtained from Theorem 2.1. Moreover, when the space satisfies the Hausdorff separation axiom, we replace the role of $\mathcal{P}, \theta$ by *open, closure* to achieve the corresponding characterizations of H-closed spaces ([1],[9],[19]) and of sets H-closed relative to X from Theorems 2.2 and 2.1 respectively. What is the more striking feature of these two theorems is that since $\mathcal{P}$ is any subset of $\mathcal{P}(X)$ with $X, \emptyset \in \mathcal{P}$ and $\theta : \mathcal{P}(X) \to \mathcal{P}(X)$ is any operator with $\theta(X) = X$, we can get, from the above two theorems, the characterizations of all the different forms of compactness (call them A-closedness, B-closedness, C-closedness or so) that arise out of all the possible combinations of $\mathcal{P}$ and θ. In the following table we list down some such cases:

$\mathcal{P}$ is the class	**$\theta : \mathcal{P}(X) \to \mathcal{P}(X)$ is the operator**
1.$\mathcal{P}(X)$	1.Identity
2.of all open sets	2.Closure
3.of all semi-open sets [11]	3.Interior
4.of all pre-open sets [13]	4.Semi-closure [6]
5.of all regular open sets [16]	5.Semi-interior [6]
6.of all semi-pre-open sets [2]	6.& Pre-closure [2]
7.of all α-sets [14]	7.Pre-interior [2]
8.of all clopen sets	8.Semi-pre-closure [2]
9.of all regular semi-open sets [4]	9.Semi-pre-interior [2]
$2', 3', 4', 5', 6', 7'$	10.Interior-closure
	11.Closure-interior
	12.Interior closure interior
	13.Closure interior closure
	2α& 3 α
	10s,&11s, 10p, 11p, 10sp

where:

(i) 2 ′,3 ′,. . .,7 ′ denote the cases where one should read 'closed' for 'open' in 2,3,. . .,7 respectively.

(ii) 2α, 3α stand for the cases where one should mean closure and interior with respect to the α-topology τ^α [14] in (2) and (3) respectively.

(iii) 10s, 11s denote the cases where one should replace *interior* and *closure* by *semi-interior* and *semi-closure* respectively in 10 and 11; 10p, 11p denote the cases where one should replace *interior* and *closure* by *pre-interior* and *pre-closure* respectively in 10 and 11; and 10sp denotes the case where one should replace *interior* and *closure* by *semi-pre-interior* and *semi-pre-closure* respectively in 10.

Thus our theory unifies the three hundred types of compactnesses (though some of them are identical, e.g. if we denote by (i, j) the compactness obtained by considering the ith case for $\mathcal{P}$ and the jth case for θ, we get $(2, 10) \equiv (5, 1) \equiv (5, 10)$ and the corresponding spaces are known as nearly compact spaces [17]) by giving a good number of characterizations that can be fruitfully used in future studies (some of them have already been introduced). This justifies the theory being called a *unified theory*.

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Department of Pure Mathematics
University of Calcutta
35, Ballygunge Circular Road
Calcutta - 700 019
INDIA

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NEW RESULTS ON THE OPERATOR DIFFERENTIAL EQUATIONS

BY

I.E. SHARKAWI

1.Some notions and notations. The class C of continuous complex valued functions of a non-negative real variable forms a commutative algebra without zero divisors where the product is defined as the finite convolution $fg = \int_0^t f(t-\tau)g(\tau)d\tau$, the sum and scalar product are defined in the usual way. The quotient field of this algebra is the operator field M of M i k u s i n s k i [1]. The elements of M are $\frac{f}{g}$ *the convolution quotient*, where f and g are in C, $g \neq 0$. Some special elements from M : s; the differential operator, $\ell = \{1\} = \frac{1}{s}$; the integral operator and 1; the unit element. In M we have an equivalence relation: $\frac{f}{g} = \frac{h}{k} \iff fk = gh$ (in C), so an element $\frac{f}{g}$ in M is a representative of an equivalence class. We denote by f of $\{f(t)\}$ the representation of $f(t) \in C_{[0,\infty)}$ while $x(\lambda) = q\{x(\lambda,t)\}$ stands for operation function, where $q \in M$ and $\{x(\lambda,t)\}$ is a parametric function. $M(\lambda)$ is the set of all mappings of $0 \le \lambda < \infty$ into M. Finally $D(T) : 0 \le \lambda \le \Lambda < \infty$, $0 \le t \le T < \infty$ is the domain R^2. After this introduction we define some subspaces which are needed for later developments.

2.Subspaces $J(D), C_s(\lambda), C_s(\lambda), A_q, A_s$ and A_ε.

$J(D)$: is defined as the set of all complex-valued functions $f(\lambda,t)$, such that (i) for every $T < \infty$, $f(\lambda,t)$ is defined, bounded and measurable over $D(T)$, (ii) for every $\lambda \in [0,\Lambda]$, $f(\lambda,t) \in C$. $J(\lambda)$ is the algebra which lies in $M(\lambda)$, its elements are of the form $f(\lambda) = q\{f(\lambda,t)\}$, $q \in M$ and $f(\lambda,t) \in J(D)$.

$C_s(\lambda)$: is a commutative algebra which lies in $M(\lambda)$, its elements are of the form $f(\lambda) = s^\beta\{f(\lambda,t)\}$, $\beta > 0$ and $f(\lambda,t) \in J(D)$ [5].

$C_s(\lambda)$: is a vector space that contains $C_s(\lambda)$ [5].