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Tomul XXXIX, s.I.a, Matematica, 1993, f.3.

ON WEIGHTED DEFICIENCIES OF MEROMORPHIC FUNCTIONS - II

BY

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Let $f(z)$ be a meromorphic function in the finite plane $|z| < \infty$. If the order of $f(z)$ is infinite, then the so called It exceptional set appears in the second fundamental theorem of Nevanlinna. Therefore in contrast to the case of finite order we have many difficulties and troubles in the investigation of value distribution of meromorphic functions of infinite order. To avoid some of them L.F. H e a t h [3] introduced some new notions weighted characteristic function and deficiencies etc. to the Nevanlinna's theory of meromorphic functions. We use the symbols $T(r, f)$, $m(r, a)$, $N(r, a)$, $N(r, f)$, $\delta(a, f)$, $\delta(r, f)$ etc. of the Nevanlinna's theory of meromorphic functions [2].

L.F. H e a t h [3] defines the weighted characteristic functions and deficiencies etc. as follows.

Let E be the class of functions α such that

(i) there is $A > 0$ such that $\alpha : [A, \infty) \to [1, \infty)$

(ii) $\frac{\alpha(t)}{t}$ is non decreasing on $[A, \infty)$.

(iii) there is $k > 1$ such that $\alpha(t + 1) \leq k\alpha(t)$ for all $t \in [A, \infty)$.

(iv) $\int_A^\infty \frac{\log t}{\alpha(t)} dt$ is convergent.

For $\alpha \in E$ define $\psi_\alpha(r) = \int_A^r \frac{\psi(t)}{\alpha(t)} dt$ where ψ is one of the functions $m(r, a)$, $N(r, a)$, $T(r, a)$ etc. of Nevanlinna's theory.

Define

$$\lambda_\alpha = \lim_{r\to\infty} \inf \frac{\log T_\alpha(r,f)}{\log r},$$

$$\delta_\alpha(a,f) = \lim_{r\to\infty} \inf \frac{m_\alpha(r,a)}{T_\alpha(r,f)},$$

$$\Delta_\alpha(a,f) = \lim_{r\to\infty} \sup \frac{m_\alpha(r,a)}{T_\alpha(r,f)},$$

$$\theta_\alpha(a,f) = \lim_{r\to\infty} \inf \frac{N_\alpha(r,a) - \bar{N}_\alpha(r,a)}{T_\alpha(r,f)},$$

$$H_\alpha(a,f) = 1 - \lim_{r\to\infty} \sup \frac{\bar{N}_\alpha(r,a)}{T_\alpha(r,f)}.$$

$\alpha \in E$ is said to be admissible for a meromorphic function f if $T_\alpha(r,f)$ is unbounded.

With these definitions he proves that the set $D = \{a \in \bar{C} \mid H_\alpha(a,f) > > 0\}$ is countable and

$$\sum_{a\in D}\{\delta_\alpha(a) + \theta_\alpha(a)\} \le \sum_{a\in D} H_\alpha(a,f) \le 2.$$

in [4] we have proved some basic properties using the notations of above definitions.

in this paper we are going to prove some other results using the notations of above definitions.

Theorem 1. *If α is admissible for f and if $\sum_{a\in\bar{C}} H_\alpha(a,f) = 2$, then*

$$\lim_{r\to\infty} \frac{\bar{N}_\alpha(r,a,f)}{T_\alpha(r,f)} = 1 - H_\alpha(a,f) \text{ for all } a \in \bar{C}.$$

Proof. Let $a \in \bar{C}$ and let $\{a_i\}_{i=1}^{\infty}$ be an infinite sequence of distinct elements of $\bar{C}$ which includes every $b \in \bar{C}$ satisfying $b \ne a$ and $H_\alpha(b,f) \ne 0$. Then

$$\sum_{i=1}^{\infty} H_\alpha(a_i,f) = \sum_{b\in\bar{C}} H_\alpha(b,f) = 2 - H_\alpha(a,f).$$

Let q be any integer ≥ 3. From L.F. H e a t h [3] we have

$$(q-2)T_\alpha(r,f) \le \sum_{i=1}^{q-1} \bar{N}_\alpha(r,a_i,f) + \bar{N}_\alpha(r,a,f) + o(T_\alpha(r,f))$$

Hence,

$$(q-2) \le \lim_{r\to\infty} \inf \frac{\bar{N}_\alpha(r,a,f)}{T_\alpha(r,f)} + \sum_{i=1}^{q-1}\{1 - H_\alpha(a_i,f)\}.$$

Thus,

$$\lim_{r\to\infty} \inf \frac{\bar{N}_\alpha(r,a,f)}{T_\alpha(r,f)} \ge \sum_{i=1}^{q-1} H_\alpha(a_i,f) - 1.$$

Since this holds for all $q \ge 3$, letting $q \to \infty$, we get

$$\liminf_{r} \frac{\bar{N}_\alpha(r,a,f)}{T_\alpha(r,f)} \ge \sum_{i=1}^{\infty} H_\alpha(a_i,f) - 1 = 1 - H_\alpha(a,f).$$

However by definition of $H_\alpha(a,f)$

$$\limsup_{r\to\infty} \frac{\bar{N}_\alpha(r,a,f)}{T_\alpha(r,f)} = 1 - H_\alpha(a,f).$$

Hence,

$$\lim_{r\to\infty} \frac{\bar{N}_\alpha(r,a,f)}{T_\alpha(r,f)} = 1 - H_\alpha(a,f).$$

Corollary. It If α is admissible for f and if $\sum_{a\in\bar{C}} \delta_\alpha(a,f) = 2$ then $\delta_\alpha(a) = \Delta_\alpha(a)$.

Theorem 2. It If α is admissible for f with $\sum_{a\in\bar{C}} \delta_\alpha(a,f^{(\ell)}) = 2$ where $\ell \ge 2$ then

$$\sum_{a\in\bar{C}} \delta_\alpha(a,f) \le \delta_\alpha(0,f^{(\ell)}) \lim_{r\to\infty} \inf \frac{T_\alpha(r,f^{(\ell)})}{T_\alpha(r,f)}.$$

Proof. Since $\sum_{a\in\bar{C}} \delta_\alpha(a,f^{(\ell)}) = 2$. From Theorem 1 we have

$$\lim_{r\to\infty} \frac{N_\alpha(r,a,f^{(\ell)})}{T_\alpha(r,f^{(\ell)})} = 1 - \delta_\alpha(a,f^{(\ell)}).$$

in particular,

$$\lim_{r\to\infty} \frac{N_\alpha\left(r,\frac{1}{f^{(\ell)}}\right)}{T_\alpha\left(r,f^{(\ell)}\right)} = 1 - \delta_\alpha\left(0,f^{(\ell)}\right).$$

Let $\{a_i\}_{i=1}^{\infty}$ be a sequence of distinct elements of which includes every $a \in \bar{C}$ for which $\delta_\alpha(a, f) > 0$. Then for an arbitrary positive integer q, from lemma of [4] we have

$$\sum_{i=1}^{q} m_\alpha(r, a_i, f) + N_\alpha\left(r, \frac{1}{f^{(\ell)}}\right) \leq T_\alpha\left(r, f^{(\ell)}\right) + o(T_\alpha(r, f))$$

so

$$\sum_{i=1}^{q} \liminf_{r\to\infty} \frac{m_\alpha(r,a,f)}{T_\alpha(r,f)} + \lim_{r\to\infty} \frac{N_\alpha\left(r, \frac{1}{f^{(\ell)}}\right)}{T_\alpha\left(r, f^{(\ell)}\right)} \liminf_{r\to\infty} \frac{T_\alpha\left(r, f^{(\ell)}\right)}{T_\alpha(r,f)} \leq$$

$$\leq \liminf_{r\to\infty} \frac{T_\alpha\left(r, f^{(\ell)}\right)}{T_\alpha(r,f)}$$

That is

$$\sum_{i=1}^{q} \delta_\alpha(a_i, f) + \left\{1 - \delta_\alpha(o, f^{(\ell)})\right\} \cdot \liminf_{r\to\infty} \frac{T_\alpha\left(r, f^{(\ell)}\right)}{T_\alpha(r,f)} \leq$$

$$\leq \liminf_{r\to\infty} \frac{T_\alpha\left(r, f^{(\ell)}\right)}{T_\alpha(r,f)}$$

Letting $q \to \infty$, we obtain

$$\sum_{a \in \bar{C}} \delta_\alpha(a, f) \leq \delta_\alpha\left(o, f^{(\ell)}\right) \cdot \liminf_{r\to\infty} \frac{T_\alpha\left(r, f^{(\ell)}\right)}{T_\alpha(r,f)}$$

Theorem 3. It If α is admissible for f and assume that two of its values a and b are weighted deficient. If $\gamma = \max\{1 - \delta_\alpha(a), 1 - \delta_\alpha(b)\}$ then $\lambda_\alpha \geq 1$ if $\gamma = 0$

$$\lambda_\alpha \geq \frac{\log\left(\frac{1}{\gamma(2-\gamma)}\right)}{\log\left(1 + \frac{4}{\gamma(1-\gamma)}\right)} \quad if \quad \gamma \neq 0.$$

Proof. From an inequality proved by E d r e i and F u c h s [1]

$$T_\alpha(r, f) \leq \frac{4}{\sigma - 1} T(\sigma r, f) + \max\{N(\sigma r, o), N(\sigma r, f)\} + O(\log r)$$

for $\sigma > 1$ and $r > 2$. We can deduce

$$T_\alpha(r, f) \leq \frac{4}{\sigma(\sigma - 1)} T_\alpha(\sigma r, f) + \max\left\{\frac{N_\alpha(\sigma r, o)}{\sigma} \frac{N_\alpha(\sigma r, f)}{\sigma}\right\} + O(1)$$

We may suppose without loss of generality that $a = 0$ and $b = \infty$.

Let $\gamma < c' < c < 1$. in the above inequality choose $\sigma = 1 + \frac{4}{c(1-c)}$. Since $f(z)$ is clearly not a rational function we see $\lim_{r\to\infty} \frac{S_\alpha(r,f)}{T_\alpha(r,f)} = 0$. Hence $c' + \frac{S_\alpha(r)}{T_\alpha(r)} < 1$ for all sufficiently large r. Then we have for all sufficiently large r

$$T_\alpha(r, f) \leq T_\alpha(\sigma r, f) \frac{c(2-c)}{\sigma}.$$

Therefore, as in the proof of Theorem by E d r e i and F u c h s [1] we obtain

$$\liminf_{r\to\infty} \frac{\log T_\alpha(r,f)}{\log r} \geq \frac{\log \frac{1}{c(2-c)}}{\log \sigma}.$$

Now theorem follows by letting $c \to \gamma$.

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Received: 20.IX.1992

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