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EXISTENCE THEOREM FOR COMMON SOLUTION OF DIFFERENTIAL EQUATIONS IN LOCALLY CONVEX SPACES

BY

B.C. DHAGE

1.Introduction. H a d z i c [3,4], A g a s e [1] and Y u a s a [9] have used the measure of noncompactness for proving the existence theorems for differential equations in locally convex spaces. The main tool used in proving these theorems is the fixed point theorem of S a d o v s k i i [8] type, which involves the use of measure of noncompactness in locally convex spaces. In the present paper we deal with the common solution of the differential equations in locally convex spaces via the measure of noncompactness. First of all we shall prove a common fixed point theorem for a pair of condensing mappings on ordered locally convex spaces and then apply it to prove the common solution of differential equations in locally convex spaces.

Let E denote the complete Hausdorff locally convex space (in shortly CHLCS) with the topology generated by a family of semi-norms $P = \{p_\alpha : \alpha \in I\}$, where I denotes the index set. Let $\mathcal{U}$ be the complete system of basis neighbourhoods of the origin of E consisting of the closed convex and bounded subsets of E. Then Y u a s a [9] has defined the measure of noncompactness of a bounded set B in E as a set $Q(B)$ in $\mathcal{U}$ given by

$$(1.1)\qquad Q(B) = \{U \in \mathcal{U} \mid \text{For given } \epsilon > 0, \text{there exists a precompact set } K_\epsilon \text{ such that } B \subset K_\epsilon + \epsilon U\}$$

If $\epsilon = 1$ in the definition (1.1) of the measure of noncompactness Q, then we get the measure of noncompactness $M(B)$ of a bounded set B due to Reich [7], namely

$$(1.2)\qquad M(B) = \{U \in \mathcal{U} \mid \text{there exists a precompact set } K \text{ such that } B \subset K + U\}$$

It is clear that $M(B) \subset Q(B)$ and the measure of noncompactness Q is more general than M. Therefore, in the subsequent discussion of the paper we shall deal with the measure of noncompactness Q due to Y u a s a [9]. Below we state a few properties of Q which we need in the sequel.

Lemma 1.1.

(a) $Q(B) = \mathcal{U} \iff B$ *is precompact,*

(b) $A \subset B \Longrightarrow Q(A) \supset Q(B)$,

(c) $Q(A \cup B) = Q(A) \cap Q(B)$,

(d) $Q(co\ A) = Q(\overline{co}\ A) = Q(A)$, *where* $co\ A$ *and* $\overline{co}\ A$ *denote the convex and closed convex hull of* A,

(e) $Q(A + B) = Q(A)$, *if* B *is compact,*

(f) $Q(aA) \supseteq Q(bA)$, *for* $0 \leq a \leq b$.

Proof. The proof is given in Y u a s a [9].

Definition 1.1. *A mapping* $T : E \to E$ *is said to be* Q*-condensing if for any nonprecompact (bounded) subset* B *of* E, $Q(TB) \supsetneq Q(B)$.

We also need certain monotonicity concepts of the mappings on CHLCS E. Let $\leq$ be the order relation in CHLCS E defined by the closed cone K in E, i.e. for $x, y \in E$, $x \leq y$ iff $y - x \in K$. In this case we also write $y \geq x$. However the order relation $\leq$ may be defined in any manner as we do not require any property of the cone K in the sequel. A mapping $T : E \to E$ is said to be isotone increasing if for each $x, y \in E$ with $x \leq y$, we have $Tx \leq Ty$. Again two mappings $S, T : E \to E$ are said to be weakly isotone increasing if $Sx \leq TSx$ and $Tx \leq STx$ for all $x \in E$. Similarly two mappings $S, T : E \to E$ are said to be weakle isotone decreasing if $Sx \geq TSx$ and $Tx \geq STx$ for all $x \in E$. Now we are in a position to formulate the main common fixed point theorem for a pair of mappings on CHLCS E.

Theorem 1.1. *Let* S *and* T *be two continuous mappings on a closed, convex and bounded subset* B *of the CHLCS* E *into itself such that*

(a) S *and* T *are* Q*-condensing,*

(b) S *and* T *are either weakly isotone increasing or decreasing.*

Then S *and* T *have a common fixed point.*

Proof. The proof is similar to that of Banach space case as given in D h a g e [2]. But for the sake of the completeness we give the details of the proof. Consder the sequence $\{x_n\}$ in B defined by

$$\bar{x} = x_0,\ x_{2n+1} = Sx_{2n},\ x_{2n+2} = Tx_{2n+1},\quad n \geq 0 \tag{1.3}$$

Suppose that S and T are weakly isotone increasing, then by definition of the sequence $\{x_n\}$, we get

$$x_0 \leq x_1 \leq x_2 \leq \ldots \leq x_n \leq \ldots. \tag{1.4}$$

Let $A = \{x_0, x_1, \ldots, x_n, \ldots\}$. We prove that A is precompact. Suppose not, then

$$\begin{aligned} A &= \{x_0\} \cup \{x_1, x_3, \ldots, x_{2n+1}, \ldots\} \cup \{x_2, x_4, \ldots, x_{2n}, \ldots\} = \\ &= \{x_0\} \cup S(A_1) \cup T(A_2) \end{aligned}$$

where $A_1 = \{x_1, x_3, \ldots, x_{2n+1}, \ldots\} \subset A$ and $A_2 = \{x_2, x_4, \ldots, x_{2n}, \ldots\} \subset \subset A$.

Now by property (c) of Lemma 1.1, we get

$$\begin{aligned} Q(A) &= Q(\{x_0\} \cup S(A_1) \cup T(A_2)) \\ &= Q(\{x_0\}) \cap Q(SA_1) \cap Q(TA_2) \\ &\supsetneq \mathcal{U} \cap Q(A_1) \cap Q(A_2) \\ &= Q(A) \end{aligned}$$

which is a contradiction. This shows that A is a precompact and $\bar{A}$ is compact. Therefore, the sequence $\{x_n\}$ which is monotone increasing has a unique limit point x^* in B such that $\lim_n x_{2n} = x^*$. Then every subsequence of the sequence $\{x_n\}$ converges to the same limit point x^*. By continuity of S and T, we get

$$\begin{aligned} x^* &= \lim_n x_{2n+1} = \lim_n Sx_{2n} = S \lim_n x_{2n} = Sx^*, \text{and} \\ x^* &= \lim_n x_{2n+2} = \lim_n Tx_{2n+1} = T \lim_n x_{2n+1} = Tx^*. \end{aligned}$$

Similarly, if S and T are weakly isotone increasing then it can be shown that the sequence $\{x_n\}$ defined by (1.3) is monotone decreasing and converges to a unique limit point which is again a common fixed point of S and T. This completes the proof.

2. Differential equations. Let E be a CHLCS with an order relation $\leq$ and let $J = [t_0, t_0 + a] \subset R$ be an interval for some $a > 0$. Let $C[J, E]$ be the space of continuous functions from J into E with the uniform topology. Then $C[J, E]$ is also a CHLCS. We define an order relation $\leq$ in $C[J, E]$ as follows. Let $x, y \in C[J, E]$. Then by $x \leq y$ we mean $x(t) \leq y(t)$ for all $t \in J$. Let $\mathcal{U}$ be the complete system of basis neighbourhoods of the origin of E, consisting of the closed convex neighbourhoods of the origin of $C[J, E]$. Let $\overline{\mathcal{U}} = \{\overline{U} \subset C[J, E] \mid U \in \mathcal{U}\}$. Then $\overline{\mathcal{U}}$ forms a basis neighbourhoods of the origin of $C[J, E]$ consisting of the closed convex neighbourhoods of the origin of $C[J, E]$. We identify $\overline{U} \in \overline{\mathcal{U}}$ with $U \in \mathcal{U}$ and characterize the measure of noncompactness of a bounded set B of $C[J, E]$ in terms of the measure of noncompactness in E. Thus whenever we calculate the measure of noncompactness in $C[J,E]$ with respect to a fixed $\overline{\mathcal{U}}$, we will understand that we also calculate the measure of noncompactness in E with respect to a fixed $\overline{\mathcal{U}}$ and identify $\overline{\mathcal{U}}$ with $\overline{\mathcal{U}}$.

Let $B \subset C[J,E]$ and let $B(t) = \{f(t) \mid f \in B\}$ and $B(J) = \bigcup_{t \in J} B(t)$.

Then we have the following lemma.

Lemma 2.1. *Let E be a CHLCS and let $J = [t_0, t_0 + a] \subset R$, $a > 0$ be an interval. Let $B \subset C[J,E]$ be a bounded equi-continuous set. Then we have*

$$\bigcap_{t \in J} Q(B(t)) = Q(B(J)) = Q(B).$$

Proof. The proof is given in Y u a s a [9].

Now we consider the nonlinear differential equations

$$x' = f(t,x)$$
$$x(t_0) = x_0 \in E \tag{2.1}$$

and

$$x' = g(t,x)$$
$$x(t_0) = x_0 \in E \tag{2.2}$$

where $f, g \in C[J \times A, E]$ and $A \subset E$ is open.

We require the following assumptions:

(A_1) There exists an open set $G \subset A$ such that $x_0 \in G \subset \bar{G} \subset A$ and $f(J \times \bar{G})$ and $g(J \times \bar{G})$ are bounded, or

(A_1') There exists a closed convex and bounded set F containing x_0 in A such that

$B_1 = \overline{co}(f(J \times F) \cup \{0\})$ is bounded and $x_0 + \alpha_1 B_1 \subset F$, and

$B_2 = \overline{co}(g(J \times F) \cup \{0\})$ is bounded and $x_0 + \alpha_2 B_2 \subset F$, for some $\alpha_1, \alpha_2 > 0$.

(A_2) For any bounded set $B \subset \bar{B} \subset A$, there exist the intervals $J' = [t_0, t_0 + a'] \subset J$ and $J'' = [t_0, t_0 + a''] \subset J$ and the constants $c_1, c_2 > 0$ such that for all (bounded) nonprecompact sets $S \subset B$,

$Q(c_1 f(J' \times S)) \supsetneq Q(S)$ and $Q(c_2 g(J' \times S)) \supsetneq Q(S)$

with respect to a fixed $\mathcal{U}$ (if B is precompact, we understand that these conditions are vacuously satisfied).

(A_3) $f(t,x) \le g\left(t, x_0 + \int_{t_0}^t f(s,x(s))ds\right)$ and

$g(t,x) \le f\left(t, x_0 + \int_{t_0}^t g(s,x(s))ds\right)$ for all $(t,x) \in J \times C[J \times A]$.

We also need the following useful lemma given in M a r t i n [6].

Lemma 2.2.

$$\int_{t_0}^t f(s,x(s))ds \in (t - t_0)\overline{co}(\{f(s,x(s)) \mid s \in [t_0,t]\}). \tag{2.3}$$

Theorem 2.1. *Assume the hypotheses (A_1)-(A_3). Then there exists a common solution of the differential equations* (2.1) *and* (2.2) *on $[t_0, t_0 + \alpha]$ for some α, $0 < \alpha \le a$.*

Proof. It is easy to see that the hypothesis (A_1) implies (A_1'). For let $B_1 = \overline{co}(f(J \times \bar{G}_1) \cup \{0\})$. By assumptions, B_1 is bounded. Choose a number α_1, $0 < \alpha_1 \le a$. such that $x_0 + \alpha_1 B_1 \subset G$. Let $F_1 = x_0 + \alpha_1 B_1$ and let $B' = \overline{co}(f(J \times F_1) \cup \{0\})$. Since $F_1 \subset G$, we have $B' \subset B_1$ and hence $x_0 + \alpha_1 B_1 \subset F_1$. Similarly, $x_0 + \alpha_2 B_2 \subset F_2$ for some α_2, $0 < \alpha_2 \le a$, where $B_2 = \overline{co}(g(J \times \bar{G}) \cup \{0\})$ and $F_2 = x_0 + \alpha_2 B_2 \subset G$. We set $F = F_1 \cup F_2$. Then clearly (A_1') holds.

Now by condition (A_2), there exist the intervals $J' = [t_0, t_0 + a'] \subset J$ and $J'' = [t_0, t_0 + a''] \subset J$, and the constants $c_1 > 0$, $c_2 > 0$, such that for all (bounded) nonprecompact sets $B \subseteq F$,

$Q(c_1 f(J' \times B)) \supsetneq Q(B)$ and $Q(c_2 g(J'' \times B)) \supsetneq Q(B)$

Let $\alpha = \min\{\alpha_1, \alpha_2, a', a'', c_1, c_2\}$, and let $I = [t_0, t_0 + \alpha] \subset J$. Since $0 < \alpha \le c_1$ and $0 < \alpha \le c_2$, by Lemma 1.1 (f), we have

$Q(\alpha f(I \times B)) \supset Q(c_1 f(I \times B))$ and $Q(\alpha g(I \times B)) \supset Q(c_2 g(I \times B))$.

Moreover, since $I \subset J'$ and $I \subset J''$, we have,

$$f(I \times B) \subset f(J' \times B) \;\; and \;\; g(I \times B) \subset g(J'' \times B).$$

Hence

$$c_1 f(I \times B) \subset c_1 f(J' \times B) \text{ and } c_2 g(I \times B) \subset c_2 g(J'' \times B).$$

It follows from Lemma 1.1 (b), that

$Q(c_1 f(I \times B)) \supset Q(c_1 f(J' \times B))$ and $Q(c_2 g(I \times B)) \supset Q(c_2 g(J'' \times B))$.

For all (bounded) nonprecompact sets $B \subseteq F$,

$$Q(\alpha f(I \times B)) \supsetneq Q(B) \text{ and } Q(\alpha g(I \times B)) \supsetneq Q(B).$$

Define a set H in $C[I,F]$ by

$$H = \{x \in C[I,F] \mid x(t_0) = x_0,\ x(t) - x(t') \in (t - t')B_0\},$$

where $B_0 = B_1 \cup B_2$. Clearly H is a non empty, closed, convex and equicontinuous set in $C[I, E]$. Now the differential equation (2.1) and (2.2) are equivalent to the integral equations,

$$x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))ds, \quad t \in I, \text{ and} \tag{2.4}$$

$$x(t) = x_0 + \int_{t_0}^{t} g(s, x(s))ds, \quad t \in I, \tag{2.5}$$

respectively, where the integrals are in Riemann's sense.

Consider two operators S and T on H defined by

$$Sx(t) = x_0 + \int_{t_0}^{t} f(s, x(s))ds, \quad t \in I, \text{ and} \tag{2.6}$$

$$Tx(t) = x_0 + \int_{t_0}^{t} g(s, x(s))ds, \quad t \in I. \tag{2.7}$$

Clearly S and T map H in to itself.

Now the problem of common solution of the differential equations (2.1) and (2.2) is just reduced to the problem of finding common fixed points of the mappings S and T on H. Using the arguments similar to Y u a s a [9], it can be shown that S and T are continuous and Q-condensing on H. We only show here that the mappings S and T are weakly isotone increasing on H. (We use lemmas 2.1 and 2.2 in the above arguments.) Now by assumption (A_3) we obtain,

$$\begin{aligned} Sx(t) &= x_0 + \int_{t_0}^{t} f(s, x(s))ds, \\ &\leq x_0 + \int_{t_0}^{t} g(s, x_0 + \int_{t_0}^{s} f(\omega, x(\omega))d\omega)ds, \\ &= x_0 + \int_{t_0}^{t} g(s, Sx(s))ds, \\ &= T(Sx(t)), \end{aligned}$$

for all $t \in I$, showing that $S(x) \leq T(S(x))$, $x \in H$. Similarly $T(x) \leq S(T(x))$, for all $x \in H$. Thus S and T are weakly isotone increasing on H. Now an application of Theorem 1.1 yields that the mappings S and T have a common fixed point, i.e., there is a point $x^* \in H$ such that $Sx^* = x^*$ and $Tx^* = x^*$. This further implies that the differential equations (2.1) and (2.2) have a common solution on $I = [t_0, t_0 + \alpha] \subset J$, for $\alpha > 0$. This completes the proof.

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Gurukul Colony
Ahmedpur - 413 515
(Maharashtra)
INDIA