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MEROMORPHIC FUNCTION FIELDS ON SUBSETS OF OPEN RIEMANN SURFACES

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It is well known that the conformal structure of an open Riemann surface is determined by the algebraic structure of the ring of analytic functions or the field of meromorphic functions on it. B e r s [1] has shown that if X, Y are plane domains and $H(X), H(Y)$ are the rings of analytic functions on X, Y, respectively, then any ring isomorphism of $H(X)$ with $H(Y)$ is induced by a conformal (or anti-conformal) equivalence of Y with X.This result has been generalized to open Riemann surfaces by R u d i n [8], R o y d e n [7], and N a k a i [5]. I s s ' s a [3] has shown that it holds for reduced Stein spaces. The case of plane sets X, Y which need not be open is examined in S u [9] and M i n d a [4]. Su and Minda have shown that the above fact under the assumption that the given isomorphism preserves complex constants unchanged. In this article, we will show that if R_1 and R_2 are two open Riemann surfaces and X, Y are connected subsets of R_1, R_2 respectively, each containing at least one interior point such that there is an abstract field isomorphism between the fields $M(X)$ and $M(Y)$ of all meromorphic functions on X and Y, respectively, then X and Y are conformally equivalent.

Definition. *Let R_1 and R_2 be two open Riemann surfaces and X, Y be connected subsets of R_1, R_2 respectively, each containing at least one interior point. A function $S : X \to R_2$ is said to be analytic if for each point $p \in X$ there is a neighborhood U_p of p and an analytic function $F_p : U_p \to R_2$ such that F_p and S coincide on $U_p \cap X$.*

This definition is equivalent to assuming that there is a single open set $U \supset X$ and an analytic function $F : U \to R_2$ such that $F \mid X = S$ [4].

Let $A(X, Y)$ denote the set of all analytic functions $S : X \to R_2$ with $S(X) \subset Y$. For $Y = R_2 = C_\infty$ a function in $A(X,C)$ is called holomorphic and we write $H(X)$. In case $Y = R_2 = C_\infty$, the extended complex plane or Riemann sphere, each function in $A(X, C_\infty)$ is called *meromorphic* and we set $M(X)$. To say that $S : X \to R_2$ is anti-analytic means that $\overline{\phi_\alpha \circ S} : X \to C$ is analytic for any local coordinate ϕ_α on $R_2 \cdot S : X \to Y$ is said to be a conformal mapping if S is analytic, one-to-one, and onto. X and Y are said to be *conformally equivalent* if there is a one-to-one conformal mapping from X onto Y.

It is well known that $H(X)$ forms a ring under pointwise addition and multiplication. In fact, $H(X)$ is an algebra over the complex numbers C. Since X is a connected subset of R_1, $M(X)$ is a field. The sum and product of two elements of $M(X)$ are defined on all of X with the possible exception of a discrete subset of X and have unique meromorphic to all of X. Moreover, $M(X)$ is the quotient field of the ring $H(X)$.

Let R be an open Riemann surface and X a connected subset of R containing at least one interior point and $p \in X$, $f \in M(X)$. In terms of a local parameter ϕ_α in a parametric neighborhood of p with $\phi_\alpha(p) = 0$, f has a local expression

$$(f \circ \phi_\alpha)(z) = \sum_{k=-\infty}^{\infty} c_z z^k$$

in a neighborhood of $\phi_\alpha(p) = 0$. We denote by $\partial_f(p)$ the *order* of f at p,

$$\partial_f(p) = \inf\{k : c_k \neq 0\}$$

which is an integer-valued function on X unless $f \equiv 0$, and does not depend on the choice of ϕ_α. We adopt the convention $\partial_0(p) = +\infty$. Evidently, f is anlytic at $p \in X$ if and only if $\partial_f(p) \geq 0$, and $f \in H(X)$ if and only if $\partial_f(p) \geq 0$ for all $p \in X$. Furthermore, if $f . g \in M(X)$, then $\partial_{fg} = \partial_f + \partial_g$, $\partial_{f-g} \geq \min\{\partial_f, \partial_g\}$ and $Z(f) = \{p \in X : f(p) = 0\} = \{p \in X : \partial_f(p) > 0\}$.

The theorem of W e i e r s t r a s s - F l o r a c k [2] for open Riemann surfaces states that for every sequence of distinct points $\{p_n\}_{n\in N}$ of R, which has no cluster points in R, and for every sequence of positive integers $\{\alpha_n\}_{n\in N}$, there exists a holomorphic function f on R which vanishes at each p_n with order α_n and at no other points. According to this theorem, for every point p in R, there exists a holomorphic function f on R which has only a simple zero at p, and for two distinct points p, q in R, there exists a holomorphic function f on R such that $f(p) \neq f(q)$.

A *divisor* ∂ on R is a mapping $\partial : R \to Z$ (the set of integers) such that $\{p \in R : \partial(p) \neq 0\}$ is discrete in R. According to the theorem of Weierstrass-Florack for open Riemann surfaces, for any divisor ∂ on R there exists a unique $f \in M(R) - \{0\}$ such that $\partial = \partial_f$ on R. For any f in $M(X)$ the mapping $p \to \partial_f(p)$ is a divisor since $\{p \in X : \partial_f(p) \neq 0\}$ is the set of zeros and poles of f in X which is discrete.

A *valuation* ν on $M(X)$ is a mapping of $M(X)^* = M(X) - \{0\}$ onto Z such that $\nu(fg) = \nu(f) + \nu(g)$, $\nu(f + g) \geq \min\{\nu(f), \nu(g)\}$ with the convention that $\nu(0) = +\infty$, and $\nu(C^*) = \{0\}$. where $C^* = C - \{0\}$. For any fixed p in X, $f \to \partial_f(p)$ is a valuation, called the *point valuation* at p. I s s ' s a [13] shows that any valuation ν on $M(R)$ is essentially a point valuation, i.e. there exists a unique point p in R such that $\nu(f) = \partial_f(p)$ for every f in $M(R)$. M i n d a [4] has shown that it holds for the case of X which is a connected subset of R containing more than one point.

Theorem. *Let R_1 and R_2 be two open Riemann surfaces and X, Y be connected subsets of R_1, R_2, respectively, each containing at least one interior point. If $\Phi : M(X) \to M(Y)$ is an abstract field isomorphism (not C-algebra), then there exists a unique map $S : Y \to X$ such that either*

(i) *S is conformal and $\Phi(f) = f \circ S$ for all $f \in M(X)$, or*

(ii) *S is anti-conformal and $\Phi(f) = \overline{f \circ S}$ for all $f \in M(X)$.*

The proof of this theorem is based on following lemmas. The methods of proof used here are closer to the work of M. N a k a i and L. S a r i o [6]. Henceforth, we will denote by C_r, the complex rational number field, where any complex number whose both real and imaginary parts are real rations numbers, is called **complex rational number.**

Lemma 1. *Let $\Phi : M(X) \to M(Y)$ be an abstract field isomorphism, then $\Phi(C) = \Phi^{-1}(C) = C$.*

Proof. since, $\Phi(1) = 1$ and $\Phi(i)^2 = \Phi(i^2) = \Phi(-1) = -1$, so either $\Phi(i) = i$ or $\Phi(i) = -i$. In the first case, we have $\Phi(r) = r$ for all $r \in C_r$ and in the second case $\Phi(r) = \bar{r}$ for all $r \in C_r$. We assume that $\Phi(i) = i$, the proof for $\Phi(i) = -i$ is similar. We only have to show that $\lambda \in C$ implies $\Phi(\lambda) \in C$ since the same for Φ^{-1} yields $\Phi(C) = C$. Assume, to the contrary, that $\Phi(\lambda) \notin C$, so that $\lambda \in C - C_r$. Since $\Phi(\lambda)$ is a nonconstant meromorphic function, we can find a subdomain U in R_2 such that $\Phi(\lambda)\ (U)$ is a nonempty open set in C. Since C_r is dense in C there exist a point $q \in U$ such that $\Phi(\lambda)\ (q) = r$ belongs to $C_r \cap \Phi(\lambda)(U)$. So $(\Phi(\lambda) - r)(q) = 0$. Let $\partial_{\Phi(\lambda)-r}(q) = n \in N$, the set of positive integers. For any $m \in N$ we can find an m^{th} root $\mu_m \in C$ of $\lambda - r \neq 0$ i.e., $\mu_m^m = \lambda - r$. Then

$[\Phi(\mu_m)]^m = \Phi(\mu_m^m) = \Phi(\lambda - r) = \Phi(\lambda) - r$, and

$m\partial_{\Phi(\mu_m)}(q) = \partial_{[\Phi(\mu_m)]^m}(q) = \partial_{\Phi(\lambda)-r}(q) = n.$

This means that any $m \in N$ is a factor of n, i.e. n is divisible by every integer and so must be zero, a contradiction. We have seen that $\Phi(C) = C$.

Lemma 2. *Let $\Phi : M(X) \to M(Y)$ be an abstract field isomorphism. Then, there exists a one-to-one and onto mapping $S : Y \to X$ such* *that* $\Phi(f)(q) = \Phi(f(S(q)))$ *for all* $f \in M(X)$, $q \in Y$ *by the convention* $\Phi(\infty) = \infty$.

Proof. Given an arbitrary point $q \in Y$, we consider $\nu(f) = \partial_{\Phi(f)}(q)$ for every $f \in M(X)$. It is clear that $f \to \nu(f)$ is a valuation on $M(X)$ and therefore, by the point valuation theorem, there exists a unique point $p \in X$ such that $\nu(f) = \partial_f(p)$ for every $f \in M(X)$. Since p is determined by q, we denote it by $S(q)$. Now, if $f(S(q)) \neq \infty$ for an $f \in M(X)$, $q \in X$, then $f - f(S(q))$ vanishes at p, i.e. $\partial_{f-f(S(q))}(S(q)) > 0$. Therefore,

$$\partial_{\Phi(f)-\Phi(f(S(q)))}(q) = \partial_{\Phi(f-f(S(q)))}(q) = \partial_{f-f(S(q))}(S(q)) > 0.$$

Hence $\Phi(f) - \Phi(f(S(q)))$ vanishes at p and we have $\Phi(f)(q) = \Phi(f(S(q)))$. If $f(S(q)) = \infty$, then $\Phi(f(S(q))) = \infty$ and $\partial_{\Phi(f)}(q) = \partial_f(S(q)) < 0$, and so $\Phi(f)(q) = \infty$. Therefore $\Phi(f)(q) = \Phi(f(S(q)))$. It is clear that this equality holds for $f \equiv 0$. It remains to show that S is one-to-one and onto. Assume there are points $q_1, q_2 \in Y$ with $q_1 \neq q_2$ and $S(q_1) = S(q_2)$. By the Florack's theorem [2], there exists a $g \in M(R_2)$ such that $g(q_1) \neq g(q_2)$. Let $f = \Phi^{-1}$ $(g \mid Y)$, then $f \in M(X)$ and $\Phi(f)(q_1) \neq \Phi(f)(q_2)$. Since $\Phi(f)(q_1) = \Phi(f(S(q_1))) = \Phi(f(S(q_2))) = \Phi(f)(S(q_2))$, we have arrived at an obvious contradiction. Furthermore, from the definition of valuation ν defined on $M(X)$, for each $p \in X$ there exists a $q \in Y$ such that $\nu(f) = \partial_{\Phi(f)}(q) = \partial_f(p)$, where $f \in M(X)$ and $S(q) = p$. This implies that S is onto.

Lemma 3. *Let* $\Phi : M(X) \to M(Y)$ *be an abstract field isomorphism. If we define the mapping* $S : Y \to X$ *by* $\Phi(f)(q) = \Phi(f(S(q)))$ *for all* $f \in M(X)$ *and* $q \in Y$, *then* $\partial_{\Phi(f)}(q) = \partial_f(S(q))$.

Proof. If $\Phi(f)(q) = \Phi(f(S(q))) = 0$, then $f(S(q)) = 0$ since Φ is an isomorphism. Hence $\partial_{\Phi(f)}(q) > 0$ if and only if $\partial_f(S(q)) > 0$. If $\Phi(f)(q) = \Phi(f(S(q))) \neq 0$, then $f(S(q)) \neq 0$. Hence $\partial_{\Phi(f)}(q) = 0$ if and only if $\partial_f(S(q)) \neq 0$. Now, by the Florack's theorem, there exists an $f_p \in M(R_1)$ such that $\partial_{f_p}(p) = 1$ with $p = S(q)$, where $p \in X \subset R_1$. Then $f_p \mid X \in M(X)$ and $\partial_{f_p|X}(p) = 1$. Suppose $\partial_{\Phi(f)}(q) > 0$ so that $\partial_f(S(q)) > 0$, for an $f \in M(X)$. Then $\partial_{f/f_p|X}(S(q)) \geq 0$ and thus $\partial_{\Phi(f/f_p|X)}(q) \geq 0$, or $\partial_{\Phi(f)}(q) - \partial_{\Phi(f_p|X)}(q) \geq 0$. Hence $\partial_{\Phi(f)}(q) \geq \partial_{\Phi(f_p|X)}(q) > 0$. This means that $\partial_{\Phi(f_p|X)}(q) = 1$. In view of this we conclude that $\partial_{\Phi(f)}(q) = \partial_f(S(q))$ for all $f \in M(X)$, $q \in Y$.

By the Lemma 2 and Lemma 3, we see that the field isomorphism $\Phi : M(X) \to M(Y)$ induces a ring isomorphism $\Phi : H(X) \to H(Y)$, i.e. if $f \in H(X)$, then $\Phi(f) \in H(Y)$. Let $T = S^{-1} : X \to Y$ be the inverse mapping of S. Then T is also a one-to-one and onto mapping induced by $\Phi^{-1} : M(Y) \to M(X)$ and we have $\Phi^{-1}(g)(p) = \Phi^{-1}(g(T(p)))$ for all $g \in M(Y)$, $p \in X$.

Lemma 4. *Let* $\Phi : M(X) \to M(Y)$ *be an abstract field isomorphism and* $S : Y \to X$ *be as in Lemma 2. If* $K \subset Y$ *is a precompact set (i.e. the closure* $\bar{K}$ *is compact) in* Y, *then* $S(K)$ *is a precompact set in* X.

Proof. Suppose, to the contrary, that K is precompact in Y and yet $S(K)$ is not in X. Then there exists a discrete sequence $\{p_n\}_{n\in N}$ in $S(K) \subset X$. By the Florack's theorem there exists a mapping on R_1 such that $\partial(p_n) = 1$ $(n \in N)$ and $\partial(p) = 0$ $(p \in R_1 - \{p_n\})$. Hence ∂ is a divisor on R_1. Let $\partial^* = \partial \mid X$. Then ∂^* is a divisor on X. Let $q_n = S^{-1}(p_n)$ $(n \in N)$. Since S is a bijective mapping, $\partial^* \circ S : Y \to Z$ is a divisor on Y. But $\{q \in Y : \partial^* \circ S(q) \neq 0\} = \{q_n\} \subset K$ and $\{q_n\}_{n\in N}$ is not discrete since $\bar{K}$ is compact. This is a contradiction. Similar argument shows that the mapping $S^{-1} = T$ has the same property.

Lemma 5. *The field isomorphism* $\Phi : M(X) \to M(Y)$ *is continuous on* $\mathbb{C}$ *with respect to the usual plane topology in* $\mathbb{C}$.

Proof. To show that F is continuous we only have to derive a contradiction from the assumption to the contrary, i.e., of the existence of a sequence $\{\lambda_n\}_{n\in N}$ in $\mathbb{C}$ with $\lambda_n \to \lambda \in \mathbb{C}$ and $\Phi(\lambda_n) \to \infty$. Fix an interior point $p \in X \subset R_1$ and choose an f_p in $H(R_1)$ with $\partial_{f_p}(p) = 1$. Then $f_{p|X} \in H(X)$ and $\partial_{f_p|X}(p) = 1$. Let $f = f_p \mid X + \lambda$. Since f is a nonconstant analytic function on X, f maps a neighborhood of p in X homeomorphically to a neighborhood of λ in $\mathbb{C}$. Then we can find a sequence $\{p_n\}_{n\in N} \subset X$ such that $p_n \to p$ and $f(p_n) = \lambda_n$ for every n in N with $n \geq n_0$. In view of the precompactness of $\{T(p_n)\}_{n\in N}$ $(T = S^{-1})$ in Y, we may assume that $T(p_n) \to q \in Y$, by choosing a subsequence if necessary. By Lemma 2,

$$\Phi(f)(T(p_n)) = \Phi(f(S(T(p_n)))) = \Phi(f(p_n)) = \Phi(\lambda_n)$$

and on letting $n \to \infty$, we obtain $\Phi(f)(q) = +\infty$. Hence by lemma 3 and $f \in H(X)$, $0 > \partial_{\Phi(f)}(q) = \partial_f(S(q)) \geq 0$, a contradiction.

Lemma 6. *If* $\Phi : M(X) \to M(Y)$ *be an abstract field isomorphism, then* $\Phi(\lambda) = \lambda$ *(or* $\bar{\lambda}$*) for all* $\lambda \in \mathbb{C}$.

Proof. For all λ in $\mathbb{C}$, we can find a sequence $\{\lambda_n\}_{n\in N}$ in $\mathbb{C}_r$ such that $\lambda_n \to \lambda$. As $\Phi(\lambda_n) = \lambda_n$ (or $\bar{\lambda}$) and Φ is continuous on $\mathbb{C}$ so we get $\Phi(\lambda) = \lambda$ (or $\bar{\lambda}$).

Lemma 7. *Let* $\Phi : M(X) \to M(Y)$ *be an abstract field isomorphism and* $S : Y \to X$ *be as in Lemma 2. Then,* $\Phi(f)(q) = f(S(q))$ *(or* $\Phi(f)(q) = \overline{f(S(q))}$*) for all* $f \in M(X)$, $q \in Y$.

Proof. Let $f(S(q)) = \lambda \in \mathbb{C}$. By Lemma 2 and 6,

$$\Phi(f)(q) = \Phi(f(S(q))) = \Phi(\lambda) = \lambda \text{ (or } \bar{\lambda})$$

Thus $\Phi(f)(q) = f(S(q))$ (or $\Phi(f(q)) = \overline{f(S(q))}$). If $f(S(q)) = \infty$, then $\Phi(f)(q) = \Phi(f(S(q))) = \Phi(\infty)$. Thus $\Phi(f)(q) = f(S(q))$.

Lemma 8. *The mapping S is continuous.*

Proof. It suffices to show that if $\{q_n\}_{n\in N}$ is a sequence of points in Y and $q_n \to q \in Y$, then $S(q_n) \to S(q)$ on X. Suppose not, then either we may choose a subsequence, again called $\{q_n\}_{n\in N}$, such that $S(q_n)$ converges to a point $p' \neq p = S(q)$ or else we may choose a subsequence such that $\{S(q_n)\}_{n\in N}$ has no limit point in R_2. In the first case, by the Florack's theorem there exists a function $f \in H(R_1)$ such that $f(p) = 0$ and $f(p') = 1$, then

$$1 = f(p') = \lim_{n\to\infty} f(S(q_n)) = \lim_{n\to\infty} \Phi(f \mid X)(q_n) =$$
$$= \Phi(f \mid X)(q) = f(S(q)) = f(p) = 0,$$

a contradiction. In the second case, let $f \in H(R_2)$ be 1 at each point $S(q_n)$ and 0 at p (by the Florack's theorem), then the same contradiction is obtained. Thus S must be continuous.

Lemma 9. *The mapping S is analytic.*

Proof. Fix $q \in Y$. By the Florack's theorem, there exists a function f in $H(R_1)$ which has only a simple zero at $p = S(q)$. Then f is univalent in an neighborhood U of p. Since S is continuous at q, there exists an open neighborhood V of q such that $S(V \cap Y) \subset U$. The fact that $\Phi(f \mid X) = (f \mid X) \circ S$ implies that $S \mid (V \cap Y) = (f \mid (U \cap X))^{-1} \circ \Phi(f \mid X) \mid (V \cap Y)$, i.e., $S = f^{-1} \circ \Phi(f \mid X)$ is valid in a neighborhood of q. This shows that S is analytic.

Lemma 10. *The mapping S is unique.*

Proof. Assume that $S' : Y \to X$ and $f \circ S' = \Phi(f) = f \circ S$ for all $f \in M(X)$. If $S' \neq S$, then there exists a point $q \in Y$ with $S'(q) \neq S(q)$. By the Florack's theorem, there exists an $f \in H(X)$ such that $f(S'(q)) \neq f(S(q))$. But this is impossible since $f(S'(q)) = \Phi(f)(q) = f(S(q))$. Therefore, S is unique.

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