ANALELE STIINTIFICE ALE UNIVERSITATII "AL.I.CUZA" IASI
Tomul XXXIX, s.I.a, Matematica, 1993, f.3.

ON CONHARMONIC* BI-SYMMETRIC KÄHLER MANIFOLD

BY

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1. Introduction. Considering a $2n$-dimensional Kähler manifold with a vector valued tensor function F and a Riemannian metric g which satisfy the following conditions

(1.1) $$\bar{\bar{X}} = -I_{2n}, \quad \text{where} \quad F(X) = \bar{X}$$

(1.2) $$g(\bar{X}, \bar{Y}) = g(X, Y)$$

(1.3) $$'F(X, Y) = g(\bar{X}, Y)$$

(1.4) $$(D_X F)Y = 0$$

where D is the Riemannian connection.

Let R, S and r stand for the Riemannian curvature tensor, Ricci tensor, and the scalar curvature tensor respectively. If we define

(1.5) $$(a)'H(Y, Z) = -\frac{1}{2} C_3^1 \overline{R(Y, Z)},$$
$$(b)^* H(Y, Z) = -'H(Y, \bar{Z})$$

then we have

(1.6) $$(a)'H(Y, Z) = S(Y, \bar{Z}),$$
$$(b)^* H(Y, Z) = -'H(Y, \bar{Z}) = S(Y, Z).$$

A Kähler manifold is said to be bi-symmetric if it satisfies

$$(D_U D_W R)(X,Y,Z)=0 \tag{1.7}$$

In this case

$$\begin{aligned}&(a)(D_U D_W S)(Y,Z)=0\\&(b)(D_U D_W R)Y=0\end{aligned} \tag{1.8}$$

and

$$D_U D_W r=0. \tag{1.9}$$

On a Kähler manifold the projective curvature tensor W, conformal curvature tensor C, Conharmonic curvature tensor L, Concircular curvature tensor V, H-projective curvature tensor P, H-conformal curvature tensor B, H-conharmonic curvature tensor T, and the H-concircular curvature tensor K are given respectively by

$$W(X,Y,Z)=R(X,Y,Z)-\frac{1}{2n-1}[S(Y,Z)X-S(X,Z)Y] \tag{1.10}$$

$$\begin{aligned}C(X,Y,Z)=R(X,Y,Z)-&\frac{1}{2(n-1)}[S(Y,Z)X-S(X,Z)Y-\\&-g(X,Z)RY+g(Y,A)RX]+\frac{r}{2(n-1)(2n-1)}\cdot\\&\cdot[g(Y,Z)X-g(X,Z)Y]\end{aligned} \tag{1.11}$$

$$\begin{aligned}L(X,Y,Z)=R(X,Y,Z)-&\frac{1}{2(n-1)}[S(Y,Z)X-S(X,Z)Y-\\&-g(X,Z)RY+g(Y,Z)RX]\end{aligned} \tag{1.12}$$

$$\begin{aligned}V(X,Y,Z)=R(X,Y,Z)-&\frac{r}{2n(2n-1)}[g(Y,Z)X-\\&-g(X,Z)Y]\end{aligned} \tag{1.13}$$

$$\begin{aligned}P(X,Y,Z)=R(X,Y,Z)+&\frac{1}{2(n+1)}[S(X,Z)Y-S(Y,Z)X+\\&+S(Y,\bar Z)\bar X-S(X,\bar Z)\bar Y-2S(X,\bar Y)\bar Z]\end{aligned} \tag{1.14}$$

$$\begin{aligned}B(X,Y,Z)=R(X,Y,Z)+&\frac{1}{2(n+2)}[S(X,Z)Y-\\&-S(Y,Z)X+g(X,Z)RY-g(Y,Z)RX-S(X,\bar Z)\bar Y+\\&+S(Y,\bar Z)\bar X+'F(X,Z)R\bar Y-'F(Y,Z)R\bar X+\\&+2'F(X,Y)R\bar Z-2S(X,\bar Y)\bar Z]-\\&-\frac{r}{4(n+1)(n+2)}[g(Y,Z)X-g(X,Z)Y+\\&+'F(X,Z)\bar Y-'F(Y,Z)\bar X+'F(X,Y)\bar Z]\end{aligned} \tag{1.15}$$

$$\begin{aligned}T(X,Y,Z)=R(X,Y,Z)-&\frac{1}{2(n+2)}[S(Y,Z)X-S(X,Z)Y+\\&+S(\bar Y,Z)\bar X-S(\bar X,Z)\bar Y+\\&+2S(\bar X,Y)\bar Z+g(Y,Z)RX-g(X,Z)RY+\\&+'F(Y,Z)R\bar X-'F(X,Z)R\bar Y-2'F(X,Y)R\bar Z]\end{aligned} \tag{1.16}$$

$$\begin{aligned}K(X,Y,Z)=R(X,Y,Z)-&\frac{r}{4n(n+1)}[g(Y,Z)X-g(X,Z)Y+\\&+'F(Y,Z)\bar X-'F(X,Z)\bar Y-2'F(X,Y)\bar Z]\end{aligned} \tag{1.17}$$

If Q stands for any of the above curvature tensor, then it is said that the Kähler manifold is Q-bi-symmetric if it satisfies

$$(D_U D_W Q)(X,Y,Z)=0. \tag{1.18}$$

2. Conharmonic* bi-symmetric Kähler manifold. A Kähler manifold will be called Conharmonic* bi-symmetric if it satisfies

$$(D_U D_W T^*)(X,Y,Z)=0, \tag{2.1}$$

where T^* is the so called Conharmonic* curvature tensor which is given by

$$T^*(X,Y,Z)=R(X,Y,Z)+\frac{r}{2(n-1)(2n-1)}[g(Y,Z)X-g(X,Z)Y]. \tag{2.2}$$

Differentiating covariantly twice we get

$$\begin{aligned}(D_U D_W T^*)(X,Y,Z)=&(D_U D_W R)(X,Y,Z)+\\&+\frac{D_U D_W r}{2(n-1)(2n-1)}[g(Y,Z)X-g(X,Z)Y].\end{aligned} \tag{2.3}$$

If the manifold is Conharmonic* bi-symmetric we have

$$(2.4)\quad (D_U D_W R)(X,Y,Z) + \frac{D_U D_W r}{2(n-1)(2n-1)}[g(Y,Z)X - g(X,Z)Y] = 0.$$

Contracting this equation with respect to X we obtain

$$(2.5)\quad (D_U D_W S)(Y,Z) + \frac{D_U D_W r}{2(n-1)} g(Y,Z) = 0.$$

Theorem 2.1. *Every bi-symmetric Kähler manifold is Conharmonic* bi-symmetric.*

Theorem 2.2. *A Conharmonic* bi-symmetric Kähler manifold is bi-symmetric if and only if the first covariant derivative of the scalar curvature is constant.*

Theorem 2.3. *On a Conharmonic* bi-symmetric Kähler manifold we have (2.5).*

Theorem 2.4. *A Conharmonic* bi-symmetric Kähler manifold is Ricci-bi-symmetric if and only if the first covariant derivative of the scalar curvature is constant.*

Now barring Z in (2.4) and contracting with respect to X we get by using (1.3) and (1.6)a

$$(2.6)\quad (D_U D_W\, 'H)(Y,Z) = \frac{D_U D_W r}{2(n-1)}\, 'F(Y,Z)$$

Hence we have

Theorem 2.5. *On a Conharmonic* bi-symmetric Kähler manifold the second covariant derivative of $'H$ vanishes if and only if the first covariant derivative of the scalar curvature is constant.*

Theorem 2.6. *Every bi-recurrent Conharmonic* bi-symmetric Kähler manifold is flat.*

Proof. If the manifold is Conharmonic* bi-recurrent we have for a non zero recurrence vector v

$$(2.7)\quad (D_U D_W T^*)(X,Y,Z) = v(U,W)T^*(X,Y,Z)$$

Since the manifold is Conharmonic* bi-symmetric, and v non zero, therefore, we have the statement from (2.1).

Theorem 2.7. *If a Conharmonic* bi-recurrent Kähler manifold is Ricci-bi-recurrent we have*

$$(2.8)\quad (D_U D_W R)(X,Y,Z) + v(U,W)[T^*(X,Y,Z) - R(X,Y,Z)].$$

Proof. If the manifold is Ricci-bi-recurrent we have

$$(2.9)\quad (D_U D_W S)(Y,Z) = v(U,W)S(Y,Z)$$

From this we have

$$(2.10)\quad D_U D_W r = v(U,W)r$$

Now multiplying (2.2) by $v(U,W)$ we get

$$(2.11)\quad \begin{aligned} v(U,W)T^*(X,Y,Z) = v(U,W)R(X,Y,Z) + \frac{v(U,W)r}{2(n-1)(2n-1)}\cdot \\ \cdot [g(Y,Z)X - g(X,Z)Y]. \end{aligned}$$

Substituting from (2.11) in (2.4) by using (2.10) we get (2.8). Now from (1.10) and (2.2) we can have

$$(2.12)\quad \begin{aligned} T^*(X,Y,Z) = W(X,Y,Z) + \frac{1}{2n-1}[S(Y,Z)X - S(X,Z)Y] + \\ + \frac{r}{2(n-1)(2n-1)}[g(Y,Z)X - g(X,Z)Y] \end{aligned}$$

Differentiating covariantly twice we get

$$(2.13)\quad \begin{aligned} (D_U D_V T^*)(X,Y,Z) = (D_U D_V W)(X,Y,Z) + \frac{1}{2n-1}\cdot \\ \cdot [(D_U D_V S)(Y,Z)X - (D_U D_V S)(X,Z)Y] + \\ + \frac{D_U D_V r}{2(n-1)(2n-1)}[g(Y,Z)X - g(X,Z)Y]. \end{aligned}$$

Hence is consequence of (1.8),(1.9),(1.18) and (2.1) we can have

Theorem 2.8. *If on a Kähler manifold any two of the following hold, then third also holds*

a) It is Conharmonic bi-symmetric manifold*
b) It is projective bi-symmetric manifold
c) It is Ricci-bi-symmetric manifold.

Theorem 2.9. *If on a Kähler manifold any two of the following hold, then third also holds*

a) It is Conharmonic bi-symmetric manifold*
b) It is conformal bi-symmetric manifold
c) It is Ricci-bi-symmetric manifold.

Theorem 2.10. *If on a Kähler manifold any two of the following hold, then third also holds*

a) It is Conharmonic bi-symmetric manifold*

b) It is Conharmonic bi-symmetric manifold

c) It is Ricci-bi-symmetric manifold.

Theorem 2.11. *If on a Kähler manifold any two of the following hold, then third also holds*

a) It is Conharmonic bi-symmetric manifold*

b) It is concircular bi-symmetric manifold

c) It is of constant first covariant derivative of the scalar curvature.

Theorem 2.12. *If on a Kähler manifold any two of the following hold, then third also holds*

a) It is Conharmonic bi-symmetric manifold*

b) It is H-projective bi-symmetric manifold

c) It is Ricci-bi-symmetric manifold.

Theorem 2.13. *If on a Kähler manifold any two of the following hold, then third also holds*

a) It is Conharmonic bi-symmetric manifold*

b) It is H-conformal bi-symmetric manifold

c) It is Ricci-bi-symmetric manifold.

Theorem 2.14. *If on a Kähler manifold any two of the following hold, then third also holds*

a) It is Conharmonic bi-symmetric manifold*

b) It is H-Conharmonic bi-symmetric manifold

c) It is Ricci-bi-symmetric manifold.

Theorem 2.15. *If on a Kähler manifold any two of the following hold, then third also holds*

a) It is Conharmonic bi-symmetric manifold*

b) It is H-concircular bi-symmetric manifold

c) It is of constant first covariant derivative of the scalar curvature.

Now before proving the next theorem let us state two well known results of Walker

Lemma 2.1. *A Riemannian space satisfies the following relation*

$$\begin{aligned}&(D_W D_U R)(X,Y,Z) - (D_U D_W R)(X,Y,Z,V) + \\ &+ (D_Z D_V R)(X,Y,W,U) - (D_V D_Z R)(X,Y,W,U) + \\ &+ (D_X D_Y R)(Z,V,U,W) - (D_Y D_X R)(Z,V,U,W) = 0\end{aligned} \tag{2.14}$$

Lemma 2.2. *If a_{ij}, b_k are numbers satisfying*

$$a_{ij} = a_{ji}, \quad a_{ij}b_k + a_{jk}b_i + a_{ki}b_j = 0 \tag{2.15}$$

for $i, j, k = 1, \ldots, N$, then either a_{ij} are all zero, or all the b_k are zero.

Theorem 2.16. *If a Conharmonic* bi-symmetric Riemannian space of non zero scalar curvature is Ricci-recurrent, then the recurrence tensor is a gradient.*

Proof. Let us put

$$T^*(X,Y,Z,V) = R(X,Y,Z,V) - Q(X,Y,Z,V) \tag{2.16}$$

where

$$\begin{aligned}Q(X,Y,Z,V) = \frac{r}{2(n-1)(2n-1)}&[g(X,Z)g(X,V) - \\ &- g(Y,Z)g(X,V)].\end{aligned} \tag{2.17}$$

If the manifold is Ricci-recurrent we have

$$(D_U S)(Y,Z) = v(U)S(Y,Z) \tag{2.18}$$

From which we have

$$D_U r = v(U)r \tag{2.19}$$

Hence from (2.17) we have

$$(D_U Q)(X,Y,Z,V) = v(U)Q(X,Y,Z,V) \tag{2.20}$$

Now from (2.16) we have by using (2.20)

$$(D_U T^*)(X,Y,Z,V) = (D_U R)(X,Y,Z,V) - v(U)Q(X,Y,Z,V) \tag{2.21}$$

Differentiating covariantly we get by using (2.20)

$$\begin{aligned}(D_W D_U T^*)(X,Y,Z,V) = (D_W D_U R)&(X,Y,Z,V) - \\ &- (D_W v)(U)Q(X,Y,Z,V) - \\ &- v(U)v(W)Q(X,Y,Z,V)\end{aligned} \tag{2.22}$$

If the manifold is Conharmonic* bi-symmetric we have

$$\begin{aligned}(D_W D_U R)(X,Y,Z,V) = (D_W v)&(U)Q(X,Y,Z,V) - \\ &- v(U)v(W)Q(X,Y,Z,V)\end{aligned} \tag{2.23}$$

Hence we can have

$$(2.24)\qquad (D_W D_U R)(X,Y,Z,V) - (D_U D_W R)(X,Y,Z,V) = = v(W,U)Q(X,Y,Z,V)$$

where $v(W,U) = (D_W v)(U) - (D_U v)(W)$. Therefore, in consequence of (2.14) we have

$$(2.25)\qquad v(W,U)Q(X,Y,Z,V) + v(Z,V)Q(X,Y,W,U) + + v(X,Y)Q(Z,V,U,W) = 0$$

Hence since we have $Q(X,Y,Z,V) = Q(X,V,X,Y)$, (2.25) are of the form (2.15) in which the indices i,j,k are replaced by pairs, ZV, XY, WU. Since $v \neq 0$, Q are not all zero. Accordingly we have

$$v(W,U) = 0$$

or

$$(D_W v)(U) = (D_U v)(W)$$

which is the condition for v to be gradient.

Acknowledgement. I am grateful to my supervisor Dr. (Mrs.) M. Majumdar (Tarfdar) for her help.

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Received:5.V.1993

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Tomul XXXIX, s.I.a, Matematica, 1993, f.3.

CURVATURE TYPE FUNCTIONS FOR PLANE CURVES

BY

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Introduction. In the paper we define a curvature type function which imitates the curvature of a plane curve. In the definition of the usual curvature we put vector function $s \to p(s)$ instead of the tangent vector. The curvature type functions have properties similar to the logarithm of a number (see formulas (6.1)-(9.1)). Let $k_{\Gamma,p}$ denote the curvature type function with respect to the vector-function $p(s)$ for a plane curve Γ.

The main result of the paper concerns equality $k_{\Gamma,p} = k_{\Gamma,q}$. Namely the last equality holds if and only if the angle γ between vectors $p(s)$ and $q(s)$ is constant. To show applications of the curvature type function we consider the class Q plane curves described in [1] which contains the class of all ovals [3]. We present a special curvature type function k_{Γ,p_n} associated with curves with constant n-width. Exactly we prove that curvature type function k_{Γ,p_n} is equal to the curvature k of the examined curve if and only if the curve has constant n-width. Next we give necessary and sufficient condition so that the relationship of the perimeters of the two described n-polygon and m-polygon was constant. In the end we give examples of curves for which the relation is constant.

1. On the curvature type functions. Let $\mathbb{R}$ and $\mathbb{C}$ denote the real line and the complex plane, respectively. Let a C-class regular plane curve Γ be represented by the following equation

$$(1.1)\qquad \Gamma : s \to z(s) = x(s) + iy(s) \in \mathbb{C}$$

where s varies on an opened interval $(a,b) \subseteq \mathbb{R}$. With the curve Γ we associate a real function imitating the curvature for a plane curve. For this purpose we hook a vector-function

$$s \to p(s) = x_p(s) + iy_p(s)$$