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VECTOR EXTENSIONS OF THE VARIATIONAL EKELAND'S RESULT

BY

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0. Introduction. Let $(X,\mathcal{U})$ be a uniform space and $F : X \to R$, a bounded from below function. Denote,

$$X(F,\alpha) = \{x \in X; F(x) < \inf F(X) + \alpha\}\ ,\ \alpha > 0.$$

An important problem involving these elements is that of determining, in the neighbourhood of each point $X(F,\alpha)$, another point of the same, with "good" variational properties. In this direction, the following 1974 Ekeland's result [7] is basic to the considerations below.

Theorem 1. *Suppose $\mathcal{U}$ is induced by a metric $d : X^2 \to R_+$ with respect to which X is complete and F is lsc (in the usual sense). Then, for each $\varepsilon > 0, \lambda > 0$ and $y \in X(F,\varepsilon\lambda)$ there exists $z = z(\varepsilon,\lambda,y) \in X(F,\varepsilon\lambda)$ such that*
(a) $d(y,z) < \lambda$ *and* $F(y) - F(z) - \varepsilon d(y,z) \geq 0$,
(b) $F(z) - F(w) - \varepsilon d(z,w) < 0$ *for each* $w \in X, w \neq z$.

This principle found a number of interesting application to convex as well as nonconvex analysis; for a survey of these we refer to E k e - l a n d [8], P e n o t [20], and G e o r g i e v [9] among others. So, it must be not suprising that, especially in the last decade, a large number of extensions was proposed for this result. For example, in the "smooth" version of Theorem 1 due to B o r w e i n and P r e i s s [2], it is proved that a stronger conclusion than (b) is to retained from the same assumptions; namely
($\bar{b}$) $F(z) - F(w) - (\varepsilon/\lambda^{p-1})(G(w) - G(z)) \leq 0$, for each $w \in X$.
Here, $p \geq 1$ is arbitrary fixed and $G : X \to R$ (depending on the initial data

$(\varepsilon, \lambda, p, u))$ is introduced as

$$G(x) = \sum_n \mu_n(d(x, z_n))^p \ , \ x \in X,$$

for some sequence (μ_n) of positive numbers with $\Sigma_n \mu_n = 1$, and some sequence (z_n) of elements in X converging to z. At the same time, in the "monotone" variant of Theorem 1 obtained by the author [24] it has been demonstrated that, if the ambient space is $(\leq)$ -complete with respect to a certain self - closed ordering $\leq$ and F is decreasing (in the standard acception) then (a) + (b) above may be written as
(a') $y \leq z, d(y,z) < \lambda$, $F(z) - F(y) - \varepsilon d(y,z) \geq 0$,
(b') $F(z) - F(w) - \varepsilon d(z,w) < 0$ for all $w > z$.

But , in the following, we shall be interested in vector extensions of the variational Ekeland's principle. These, roughly speaking, are being produced by allowing vectorial values for the couple (F,d) and require two main ingredients:

1) an adequate notion of almost minimum point (i.e., a vectorial substitute of the sets $X(F,\alpha)$ above)

2) a maximality principle involving in a direct way the ambient uniform structure of X.

As a basic result in this area we mention the one obtained in 1986 by N e – m e t h [16]. To state it, let E be a locally convex Hausdorff space, ordered by a normal cone K, and $F : X \to E$, a mapping with

$$X(F,K) = \{x \in X; (F(x) - K) \cap F(X) \text{ has a lower bound } \} \neq \emptyset.$$

For any part U of E, let U^c denote its (absolute) complement in E (that is, $E \backslash U$).

Theorem 2. *Suppose the uniformity of X is generated by a (vectorial) metric $d : X^2 \to H$ (where H is a K -bound regular complete subcone of K) with respect to which X is complete and F is sub - monotone. Then, for each $y \in X(F,K)$ and each $\varepsilon > 0$ there is a $z \in X(F,K)$ so as*
(a_1^*) $F(y) - F(z) - \varepsilon d(y,z) \in K$,
(b^*) $F(z) - F(w) - \varepsilon d(z,w) \in K^c$, $w \in X, w \neq z$.
Also, let U be a neighbourhood of $o \in E$. If $S = H \cap U^c$ is nonempty then, for each $x \in X(F,K)$ there exists $y \in X$ with
(c) $F(x) - F(y) \in K$ *(hence $y \in X(F,K)$)*
(d) $(F(y) - \varepsilon S - K) \cap F(X) = \emptyset$.
And, finally, for each y with these properties there is an element $z \in X$ fulfilling (a_1^*) plus (b^*) above, as well as

(a_2^*) $d(y,z) \in U$.

A close analysis of the proof (based essentially on the Zorn Maximality Principle) shows it is not strongly related to the locally convex structure of E. So, further extensions of this result are not essentially impossible. It is our main aim in the present exposition to give such an extension of Theorem 2 to the case of E being a topological vector space; this will be done in Section 3. Two main aspects of the argument must be emphasized; these are

i) the normality of the cone K is not indispensable to the substance of the proof (whereas the K -regularity of the subcone H is),

ii) the multiplication by scalars is not effectively used in the reasoning (so, the result in question is valid in the case of E being a topological (abelian) group).

The preliminary material is collected in Sections 1 and 2. It is concerned with appropriate answers to the problems 1) and 2) above. The almost minimality results in Section 2 follow, essentially, the same line as in Nemeth (op.cit.); and the maximality principles we dealt with in Section 1 are being taken from T u r i n i c i [25]. Some variational aspects of the main result will be discussed in Section 4. These involve appropriate interpretations of the conclusion in the main result corresponding to (b^*) in Theorem 2. A serious limitation of the method is related to the strong requirement that int (K) be nonempty; more precisely, significant facts in this direction cannot be obtained beyond the normed context. But, even in this case, a "true" vectorial extension of the well known Ekeland's necessary condition of almost minimum [7] is still lacking. Perhaps the future development of the nonsmooth differential calculus for vector valued functions (under the lines in P a p a g e o r g i o u [18]) will provide us new tools for attacking this problem in its full generality.

1. Preliminaries about maximality principles. Let X be a nonempty set. By a ***quasi-ordering*** over X, denoted $\leq$, we mean any reflexive and transitive relation (over X). Denote also

$$\Delta(X) = \{(x,x); x \in X\} \ \text{ (the diagonal of X).}$$

Under a convention similar to that introduced by N a c h b i n [14, ch.2, Sec.2], we shall term the family $\mathcal{U}$ of subsets in X^2, a ***pseudouniformity*** over X when

$$\Delta(X) \subseteq U, \text{for each U in } \mathcal{U}.$$

The following sort of maximality will be considered in the present exposition. Let $\leq$ and $\mathcal{U}$ be as above. We say $z \in X$ is $(\leq, \mathcal{U})$ - ***maximal***, in case

$$z \leq u \leq v \text{ implies } (\text{u, v}) \in \cap\mathcal{U};$$

the set of all such points will be indicated by $\max(X, \leq, \mathcal{U})$. This may be related, of course, with the standard notion of maximality, introduced as: $z \in X$ is $(\leq)$ – *maximal*, when

$$z \leq w \text{ implies } w \leq z,$$

(the set of all such points being denoted by $\max(X, \leq)$). Namely, assume $\mathcal{U}$ is $(\leq)$ -separated, in the sense

$$\cap \mathcal{U} = (\leq) \cap (\geq).$$

(Here, e.g., $(\leq)$ stands for the *graph* of our quasi–ordering; that is, $\{(x, y) \in X^2; x \leq y\}$.) Then, $(\leq, \mathcal{U})$-maximal and $(\leq)$-maximal are identical. But, in general, the former of these includes the latter. This is manifest in the case of $\leq$ being an ***ordering*** (antisymmetric quasi–ordering) over X, when the right member in the above relation becomes $\Delta(X)$.

Now, it is the moment to specify that the main objective of this section is to give sufficient conditions about $\leq$ and $\mathcal{U}$ so that $\leq$ be a ***(globally) $\mathcal{U}$–normalquasi–ordering,*** in the sense

$$(D_1) \qquad \begin{cases} \text{for each } x \in X \text{ there is a } (\leq, \mathcal{U}) \text{ – maximal element} \\ z \in X \text{ with } x \leq z. \end{cases}$$

For an appropriate answer, a number of notational conventions as well as preliminary facts must be taken into account. We start with the following simple remark. Let $\mathcal{U}$ and $\mathcal{V}$ be two pseudouniformities over X. We say $\mathcal{U}$ is ***coarser*** than $\mathcal{V}$, in case

$$\text{for each } U \in \mathcal{U} \text{ there exists } V \in \mathcal{V} \text{ so that } U \supseteq V.$$

As an immediate consequence, $(\leq, \mathcal{V})$-maximal is stronger than $(\leq, \mathcal{U})$-maximal. The pseudo-uniformities $\mathcal{U}$ and $\mathcal{V}$ will be termed ***equivalent*** when each of them is coarser than the other: in such a circumstance, the maximality properties induced by $\mathcal{U}$ and $\mathcal{V}$ are identical. In particular,

$$\{U\} \text{ is coarser than } \mathcal{U}\text{ , for each } U \in \mathcal{U};$$

hence $(\leq, \mathcal{U})$– maximal is stronger than $(\leq, \{U\})$-maximal, for each $U \in \mathcal{U}$. This motivates the following convention. We call the underlying quasi–ordering $\leq$, locally ***$\mathcal{U}$–normal*** when it is (globally) $\{U\}$–normal for each $U \in \mathcal{U}$; that is

$$(D_2) \qquad \begin{cases} \text{for each } x \in X \text{ there exists a } (\leq, \{U\}) \text{ – maximal element} \\ y \in X \text{ with } x \leq y. \end{cases}$$

By the above discussed facts we have that globally $\mathcal{U}$–normal is a stronger property than locally $\mathcal{U}$–normal, for the considered quasi - ordering. So, the question arises of what can be said about the reverse implication. This is a more accurate formulation of our initial problem. It may be worked out via conditions involving the nets of the ambient space. Before describing them we need, however, a preliminary discussion about their index sets. Precisely, let W stand for the class of all ***ordinals;*** it has a contradictory character, by the well known Burali–Forti paradoxe (see, e.g., S i e r p i n – s k i [22, ch.XIV, Sec.2]). However, when one restricts the considerations to a Grothendieck universe Γ (introduced as in H a s s e and M i – c h l e r [10, ch.I, Sec.2]) this contradictory character is removed for the class W_Γ of all ***admissible*** (mod Γ) ordinals (that is, ordinals generated by the well ordered (non - contradictory) sets in Γ). In the following we drop the subscript Γ for simplicity. So, by an ordinal (in W) we shall understand a Γ - admissible ordinal with respect to a "sufficiently large" Grothendisk universe Γ. This will be refered to as an admissible ordinal (to indicate the fact that a generic universe Γ is considered in its construction). Of course,

$$\alpha = \text{admissible ordinal}, \ \beta \leq \alpha \Rightarrow \beta = \text{admissible ordial}.$$

Hence in the formula

$$W(\alpha) = \{\xi \in W; \zeta < \alpha\}\ , \quad \alpha \in W,$$

the set W in the brackets may be taken as the "absolute" set of all ordinals. We also denote, for each (non–contradictory) set Z of the ambient universe

$$\kappa(Z) = \min\{\alpha \in W;\ \text{card }(Z) = \text{card }(W(\alpha)).$$

That is, $\kappa(Z)$ equals the initial ordinal of card (Z). The existence of this number is a consequence of the (Zermelo) Axiom of Choice; see in this direction A l e x a n d r o v [1,ch.3, Sec.6]. In the following, the notation of ***net*** (ascending net) will be used to designate any mapping (increasing map) $\lambda \vdash x_\lambda$ from some ordinal segment $W(\alpha)$ (where α is an admissible ordinal) to X (respectively $(X, \leq)$). The term ***sequence*** (ascending sequence) will be reserved for the case $\alpha = \omega$ (the first transfinite ordinal). This convention is a particular case of the one introduced by K e l l e y [12,ch.2, Sec.1] under the lines below. Let $(I, \leq)$ be a quasi–ordered set which is supposed to be ***directed*** , in the sense

$$I(i, \leq) \cap I(j, \leq) \neq \emptyset, \text{ for each i,j} \in I.$$

(Here, e.g., $I(i, \leq)$ stands for the set of all $k \in I$ with $i \leq k$). By a ***net*** (ascending net) in X we shall mean any mapping (increasing map) $i \vdash x_i$

from some directed quasi-ordered (index) set I to X (respectively, $(X, \leq)$). The particular aspect of our convention is related to the case of $(I, \leq)$ being a well ordered structure. That is, in the developments below we shall restrict our considerations to those Kelley nets (ascending nets) indexed after well ordered parts (of the ambient universe). The technical effects of this device will be discussed in the next section. For the moment however, we shall be interested in giving an appropriate answer to the above posed problem.

Theorem A. *Suppose that*

$$(C_1)\qquad \begin{cases} \text{each ascending net } (x_\lambda)_{\lambda<\beta} \text{ in } X, \text{ where } \beta \leq \kappa(\mathcal{U}), \\ \text{has an upper bound.} \end{cases}$$

Then, for the ambient quasi-ordering $\leq$, the local $\mathcal{U}$-normality property equals its global counterpart.

Proof. Assume $\leq$ is locally $\mathcal{U}$-normal; that is, (cf. (D_2)) for each $x \in X, U \in \mathcal{U}$, there exists $y = y(x, U) \geq x$ with

$$(1)\qquad y \leq u \leq v \text{ implies } (\mathrm{u}, \mathrm{v}) \in \mathrm{U}.$$

Let $(U_\lambda)_{\lambda<\delta}$ be an enumeration of $\mathcal{U}$, existing by the Zermelo Well Ordering Principle [1,ch.3, Sec.5]; here, of course, $\delta = \kappa(\mathcal{U})$, because card $(W(\delta))$ = card $(\mathcal{U})$. Let x be arbitrary fixed in X. There exists by (1), a $x_0 \geq x$ with the property

$$x_0 \leq u \leq v \text{ implies } (\mathrm{u}, \mathrm{v}) \in \mathrm{U}_0.$$

Of course, it might very well happen that (D_2) be verified with $z = x_0$ (and then we stop the process); otherwise, we go further. Namely, suppose that, for the ordinal $\gamma < \delta$ we obtained an ascending net $(x_\lambda)_{\lambda<\gamma}$ in X so as, for each $\lambda < \gamma$

$$(P(\lambda))\qquad x_\lambda \leq u \leq v \text{ implies } (\mathrm{u}, \mathrm{v}) \in \mathrm{U}_\lambda.$$

If γ is a first kind ordinal, then, by (1) (with $x = x_{\gamma-1}, U = U_\gamma$), we get a point $x_\gamma \geq x_{\gamma-1}$ with the property $(P(\gamma))$. Otherwise (when γ is a second kind (limit) ordinal), use (C_1) to get an upper bound, x', of $(x_\lambda)_{\lambda<\gamma}$ and then (1) (with (x', U_γ) in place of (x,U)) to get a point $x_\gamma \geq x'$ which should satisfy $(P(\gamma))$. As above, if x_γ is $(\leq, \mathcal{U})$-maximal, then we are done; otherwise, we are passing to the $(\gamma + 1)$ step, etc. Summing up, we either end the process after a "number" of iterations less than δ or construct an ascending net $(x_\lambda)_{\lambda<\delta}$ in X satisfying $(P(\lambda))$, for each $\lambda < \delta$. Let z be an upper bound (existing by (C_1)) of this net. It is clear that $z \in \max(X, \leq, \mathcal{U})$ and $x \leq z$. The proof is complete. q.e.d.

It is not without importance to say that the property of Theorem A is still refinable, in the following sense. Let $A(\mathcal{U})$ be a family of pseudo-uniformities (over X) whose union is just $\mathcal{U}$. We call the underlying quasi-ordering $\leq$, ***globally** $A(\mathcal{U})$-**normal*** when it is globally $\mathcal{V}$-normal, for each $\mathcal{V}$ in $A(\mathcal{U})$. In particular, if

$$A(\mathcal{U}) = \{\{U\}; U \in \mathcal{U}\} \quad (\text{respectively } , \mathrm{A}(\mathcal{U}) = \mathcal{U})$$

it is clear that globally $A(\mathcal{U})$-normal is identical with locally $\mathcal{U}$-normal (respectively, globally $\mathcal{U}$-normal). But , in general, the introduced property is intermediary between these extreme ones, in the sense

$$\begin{cases} \text{globally } \mathcal{U} \text{ - normal} \Rightarrow \text{ globally A}(\mathcal{U}) \text{ - normal} \Rightarrow \\ \text{locally } \mathcal{U} \text{ - normal.} \end{cases}$$

Now, in view of the above result, the extreme properties of this chain are equivalent. Hence

$$\begin{cases} \text{globally A}(\mathcal{U}) \text{ - normal} \Leftrightarrow \text{ globally } \mathcal{U} \text{ - normal} \\ \text{(for each family A}(\mathcal{U}) \text{ taken as before).} \end{cases}$$

This, for an individual $A(\mathcal{U})$, continues to hold, even if, in (C_1) we replace $\kappa(\mathcal{U})$ by $\kappa(A(\mathcal{U}))$. Indeed, assume $\leq$ is globally $A(\mathcal{U})$-normal and let $(\mathcal{V}_\lambda)_{\lambda<\delta}$ be an enumeration of $A(\mathcal{U})$, where $\delta = \kappa(A(\mathcal{U}))$. Then, for the arbitrary fixed $x \in X$ we may construct (by the same way as in Theorem A) an ascending net $(z_\lambda)_{\lambda<\delta}$ in X with $x \leq z_0$ and

$$z_\lambda \in \max(X, \leq, \mathcal{V}_\lambda), \text{for each } \lambda < \delta.$$

By the accepted variant of (C_1) there exists at least an upper bound, z, for this net which, evidently, has the required properties (in (D_1)); this proves our assertion. The procedure subsumed to this remark is preferable (in arriving at the conclusion of Theorem 4) only if it is known that $\leq$ is globally $A(\mathcal{U})$-normal and

$$\kappa(A(\mathcal{U})) < \kappa(\mathcal{U}) \text{ (hence card } (\mathrm{A}(\mathcal{U})) < \text{ card } (\mathcal{U})).$$

For example, this is not the case when

$$\text{card } (\mathrm{A}(\mathcal{U})) \geq \text{ card } (\mathcal{V}), \text{ for all } \mathcal{V} \in \mathrm{A}(\mathcal{U});$$

because the double inequlity

$$\text{card } (\mathrm{A}(\mathcal{U})) \leq \text{ card } (\mathcal{U}) \leq \text{ card } (\mathrm{A}(\mathcal{U})) \times \mathrm{A}(\mathcal{U})) = \text{card } (\mathrm{A}(\mathcal{U}))$$

(which is known to be fulfilled whenever $A(\mathcal{U})$ is at least denumerable; see, e.g., S i e r p i n s k i [22, ch.XV, Sec.8]) contradicts the requirement (about cardinals) we just formulated.

As a matter of fact, the local $\mathcal{U}$–normality assumption introduced by Theorem A cannot be directly used in practice. So, we must find appropriate substitutes for it. In this direction, let us term the (ascending) net $(x_\lambda)_{\lambda<\alpha}$ in X,$\mathcal{U}$–Cauchy when, for each $U \in \mathcal{U}$ there exists an index $\lambda(U) < \alpha$ such that

$$(x_\mu, x_\nu) \in U, \text{ for all } \mu, \nu < \alpha \text{ with } \lambda(\mathrm{U}) \leq \mu \leq \nu.$$

Of course, for an (ascending) sequence $(x_n)_{n<\omega}$ in X, this property is, in general, stronger than that of being $\mathcal{U}$–asymptotic, in the sense: for each $U \in \mathcal{U}$ there exists $n(U) < \omega$ such that

$$(x_n, x_{n+1}) \in U, \ for\ all\ n \geq n(U).$$

However, when all (ascending) nets/sequence are involved, the reciprocal of this implication also holds, in the form: the global conditions

(C_2) each ascending net $(x_\lambda)_{\lambda<\alpha}$ is $\mathcal{U}$–Cauchy

(C_2') each ascending sequence $(x_n)_{n<\omega}$ is $\mathcal{U}$ - asymptotic

are equivalent each other. In fact, whenever $(x_\lambda)_{\lambda<\alpha}$ is not $\mathcal{U}$–Cauchy, we have that, for some U in $\mathcal{U}$ the following property holds: for each $\lambda < \alpha$ there exists a couple μ, ν in $W(\alpha)$ with

$$\lambda \leq \mu \leq \nu \text{ and } (\mathrm{x}_\mu, \mathrm{x}_\nu) \in \mathrm{U}^c.$$

(Here (cf. a previous notation), U^c stands for the absolute complement of $U \subseteq X^2$). Then, an ascending sequence $(y_n)_{n<\omega}$ may be constructed (with the terms of the initial net) so that

$$(y_{2n}, y_{2n+1}) \in U^c, \text{for all } \mathrm{n} < \omega. \tag{2}$$

But this contradicts (C_2'); hence the assertion follows. This equivalence is no longer valid when the term "net" is taken as in Kelley (op.cit.). Precisely, let $(x_i)_{i\in I}$ be an (ascending) net, indexed after a (generally) directed quasi–ordered set $(I, \leq)$. By convention, it will be called $\mathcal{U}$–Cauchy when, for each $U \in \mathcal{U}$ there exists $i(U) \in I$ such that

$$(x_j, x_k) \in U, \text{for all } \mathrm{j}, \mathrm{k} \in \mathrm{I}, \mathrm{j}, \mathrm{k} \geq \mathrm{i(U)}.$$

Assume now that (C_2) is to be replaced by

(C_2^*) each ascending (Kelley) net $(x_i)_{i\in I}$ is $\mathcal{U}$–Cauchy.

Clearly, $(C_2^*) \Rightarrow (C_2)$; but, the reverse implication is not (in general) valid. To explain this, assume that $(x_i)_{i\in I}$ is not $\mathcal{U}$–Cauchy. Then, for some $U \in \mathcal{U}$, the property below is valid : for each $i \in I$, there exists $j, k \in I$ with

$$j, k \geq i \ \text{ and } \ (\mathrm{x_j}, \mathrm{x_k}) \in \mathrm{U}^c.$$

Now, it might very well happen that j and k be incomparable; that is, neither $j \leq k$ nor $k \leq j$ is holding. But, in this circumstance, we cannot build up the sequence $(y_n)_{n<\omega}$ as in (2) and this proves our claim. The obtained conclusion has, of course, no impact for our developments. But, from a methodological viewpoint, it tells us that the notion under consideration must be handle with care over different classes of nets.

Now, returning to the above development, we note that, under either of the mutually equivalent conditions $(C_2)/(C_2')$, the underlying quasi–order $\leq$ is necessarily locally $\mathcal{U}$–normal. In fact, suppose this is not true. Then, a couple $x \in X, U \in \mathcal{U}$ may be found so that: for each $y \geq x$ there exists $u, v \in X$ with

$$y \leq u \leq v \ \text{ and } \ (\mathrm{u}, \mathrm{v}) \in \mathrm{U}^c.$$

From this, an ascending sequence $(y_n)_{n<\omega}$ may be constructed with the property (2) above. The contradiction at which we arrived proves our assertion. A natural question is to determine of whether or not the converse implication is true. To give it a more adeuqate formulation, let us call the quasi–ordering $\leq$, strongly locally $\mathcal{U}$–normal, when for each (nonempty) part Z of X, the restriction $\leq_Z$ of $\leq$ to Z (introduced as

$$(\leq_Z) = (\leq) \cap Z^2)$$

is locally $\mathcal{U}_Z$–normal, where

$$\mathcal{U}_Z = \{U \cap Z^2; U \in \mathcal{U}\}$$

is the trace of $\mathcal{U}$ over Z. In this perspective, the above argument shows that (C_2) (or (C_2')) is a sufficient premise for

(C_3) $\qquad \leq$ is strongly locally $\mathcal{U}$ – –normal

But, the reciprocal is valid too. In fact, let $(x_n)_{n<\omega}$ be an ascending sequence in X. By the accepted hypothesis, we have (in $Z = \{x_n : n < \omega\}$) that, for each $n < \omega$ and $U \in \mathcal{U}$ there exists $m(n, U) \geq n$ so that

$$m(n, U) \leq p \leq q \ \text{ implies } \ (\mathrm{x_p}, \mathrm{x_q}) \in \mathrm{U}.$$

This, by definition, tells us the sequence in question is $\mathcal{U}$-Cauchy (hence $\mathcal{U}$-asymptotic) and the assertion follows. In other words, (C_3) is but a variant of $(C_2)/(C_2')$ and the above question may be phrased as to what extent it is true that locally $\mathcal{U}$-normal implies (C_3). We conjecture that the answer is negative.

As an immediate consequence of these facts, we have the following practical result.

Theorem B. *Under the same boundedness assumption (C_1), let us admit that either of the mutually equivalent conditions $(C_2)/(C_2')$ is holding. Then, for each $x \in X$ there is a $(\leq, \mathcal{U})$-maximal element $z \in X$ with $x \leq z$. If, in addition,*

$$x \text{ is } (\leq, \{U\}) - \text{maximal}, \ (\text{for some } U \in \mathcal{U})$$

then, necessarily, $(x, z) \in U$.

Let us now return to the framework of Theorem A. It is our main aim in the following to give a counterpark if it which - at least from a methodological viewpoint - may be of some avail for these developments. Let us call the point $z \in X, (\leq, U)$-semi-maximal, when

$$z \leq w \quad \text{implies} \quad (z, w) \in \cap\mathcal{U}.$$

Of course, this notion includes the already introduced one of $(\leq, U)$-maximality. To give an instance in which the converse is also valid, call the pseudo-uniformity $\mathcal{U}, (\leq)$-triangular when, for each $U \in \mathcal{U}$ there exists $V \in \mathcal{U}$ such that

$$x \leq y \leq z, \quad (x, y), (x, z) \in V \Rightarrow (y, z) \in U.$$

Now, an assumption like

(C_4) $\mathcal{U}$ is $(\leq)$-triangular

makes these maximality properties equivalent each other, as it can be directly seen. This conclusion may be also expressed as: under the structural assumption (C_4), global $\mathcal{U}$-normality equals the global $\mathcal{U}$-semi-normality property of the underlying quasi-ordering, introduced as

$$(D_1') \qquad \begin{cases} \text{for each } x \in X \text{ there is a } (\leq, \mathcal{U}) - \text{semi} - \text{maximal element} \\ z \in X \text{ with } z \leq z. \end{cases}$$

However, this equivalence is no longer valid for the local variants of these properties; that is, for an individual member U of $\mathcal{U}$, the (global) $\{U\}$-normality property is stronger, in general, than its corresponding (global) $\{U\}$-semi-normality (even if (C_4) holds). So, we may ask of which additional hypotheses must be imposed to the triplet $(X, \leq, \mathcal{U})$ so that conclusion of Theorem A be valid with local $\mathcal{U}$-normality replaced by a local $\mathcal{U}$-semi-normality of the quasi-ordering $\leq$, expressed as

$$(D_2') \qquad \begin{cases} \text{for each } x \in X, U \in \mathcal{U} \text{ there exists a} \\ (\leq, \mathcal{U}) - \text{semi} - \text{maximal element } y \in X \text{ with } x \leq z. \end{cases}$$

To discuss this, we need some new conventions. Let us call the (ascending) net $(x_\lambda)_{\lambda<\alpha}$ in $X, \mathcal{U}$-semi-Cauchy when, for each $U \in \mathcal{U}$ there exists $\lambda = \lambda(U) < \alpha$ such that

$$(x_\lambda, x_\mu) \in U, \quad \text{for each } \mu < \alpha \quad \text{with} \quad \lambda \leq \mu.$$

This property is, for an individual (ascending) sequence $(x_n)_{n<\omega}$, stronger than that of being $\mathcal{U}$-semi-asymptotic, expressed as: for each $n \in N, U \in \mathcal{U}$,

$$(x_m, x_{m+1}) \in U, \quad \text{for some } m = m(n, U) \geq n.$$

The converse of this assertion is valid in a global sense only: that is, conditions

(C_5) each ascending net is $\mathcal{U}$-semi-Cauchy

(C_5') each ascending sequence is $\mathcal{U}$-semi-asymptotic

are equivalent each other. (The proof of this fact being very similar to the preceding one, we omit the details). Moreover, either of these is a sufficient condition for the local $\mathcal{U}$-semi-normality property (D_2'). Indeed, assume this would not be true. Then, a couple $x \in X, U \in \mathcal{U}$ is to be found with the property: for each $y \geq x$

$$(y, v) \in U^c, \text{ for some } v \geq y.$$

From this, an ascending sequence $(y_n)_{n<\omega}$ may be constructed so that

$$(y_n, y_{n+1}) \in U^c, \text{ for all } n < \omega,$$

a contradiction with respect to (C_5'). This proves our claim. Concerning the converse implication, we note that (cf. a previous argument) either of the conditions $(C_5)/(C_5')$ is equivalent with

$$(C_3^*) \qquad \leq \text{ is strongly locally } \mathcal{U} - \text{semi} - \text{normal}$$

(That is: for each (nonempty) part Z of X, the restriction $\leq_Z$ of $\leq$ to Z is locally $\mathcal{U}_Z$-semi-normal, where $\mathcal{U}_Z$ is the trace of $\mathcal{U}$ over Z). Hence, the implication to be englighten is that between the global $\mathcal{U}$-semi-normality property (defined by $(C_1'))$ and (C_3^*). We again conjecture that the answer is negative.

Now, to complete these considerations, the following are in effect. For the (arbitrary fixed) pseudo-uniformity $\mathcal{U}$ over X, let us introduce a $\mathcal{U}$-convergence structure " $\to$ " on the ambient space by saying: $(x_\lambda)_{\lambda<\alpha}$ converges (mod $\mathcal{U}$) to x (and we denote this as $x_\lambda \to x$ (mod $\mathcal{U}$)) when for each $U \in \mathcal{U}$ there exists $\lambda(U) < \alpha$ so that

$$(x_\mu, x) \in U, \quad \text{for all } \mu < \alpha \text{ with } \lambda(\mathrm{U}) \leq \mu.$$

The class of all such points (if any) will be denoted by $\mathcal{U} - lim_\lambda((x_\lambda)_{\lambda<\alpha})$ or, simply, $\lim_\lambda(x_\lambda)$. As a quasi-order variant of this, let us put

$$\begin{cases} \mathrm{x}_\lambda \uparrow \mathrm{x}(\text{mod } \mathcal{U}) \text{ if and only if } \mathrm{x}_\lambda \to \mathrm{x} \text{ (mod } \mathcal{U} \text{) and} \\ x_\lambda \leq x, \lambda < \alpha; \end{cases}$$

the class of these points is being indicated as $\mathcal{U} - o\lim_\lambda((x_\lambda)_{\lambda<\alpha})$ (or, directly, as $o\lim_\lambda(x_\lambda)$ when no confusion can arise). It must be noted that, in general, an information like

$$\lim_\lambda(\mathrm{x}_\lambda) \text{ is nonempty (or, even, } o\lim_\lambda(\mathrm{x}_\lambda) \text{ is nonempty)}$$

cannot imply the $\mathcal{U}$-semi-Cauchy (hence, a fortiori, the $\mathcal{U}$-Cauchy) property of the (ascending) net in question. However, in the presence of a certain counterpart of (C_4), namely

$$(C_4') \qquad \begin{cases} \text{for each } \mathrm{U} \in \mathcal{U} \text{ there exists } \mathrm{V} \in \mathcal{U} \text{ such that} \\ \mathrm{x} \leq \mathrm{y} \leq \mathrm{z} \text{ and } (\mathrm{x},\mathrm{z}),(\mathrm{y},\mathrm{z}) \in \mathrm{V} \text{ imply } (\mathrm{x},\mathrm{y}) \in \mathrm{U}, \end{cases}$$

this is possible, in the sense

$$\begin{cases} \text{each (ascending) convergent (mod } \mathcal{U}) \text{ net in X is} \\ \text{necessarily } \mathcal{U} - \text{Cauchy (hence } \mathcal{U} - \text{semi} - \text{Cauchy).} \end{cases}$$

As a consequence, the convention below is technically correct. We call the underlying pseudo-uniformity $\mathcal{U}$, weakly $(\leq)$ -separated, in case

$$\begin{cases} \lim_\lambda((\mathrm{x}_\lambda)_{\lambda<\delta}) \text{ is } (\leq,\mathcal{U}) \text{ -- flat, for each ascending} \\ \mathcal{U} - \text{semi} - \text{Cauchy net } (\mathrm{x}_\lambda)_{\lambda<\delta} \text{ in X, with } \delta \leq \kappa(\mathcal{U}). \end{cases}$$

(Here, a subset Z of X is termed $(\leq,\mathcal{U})$ -flat, whenever

$$u, v \in Z, u \leq v \Rightarrow (u,v) \in \cap\, \mathcal{U}).$$

For example, a sufficient condition for this property is the $(\leq)$ - triangular assumption (C_4). As a matter of fact, the prefix "weakly" is not very well chosen; that is, the $(\leq)$ -separation property defined at the beginning of this section does not in general assure the one described by (D_3). It, however, does under a condition like

$$(C_6) \qquad \leq \text{ is } \mathcal{U} - \textit{self} - \textit{closed}$$

(in the sense: the limit (mod $\mathcal{U}$) of each ascending net in X is an upper bound of it). The proof is almost evident; so, we do not enter into details. Finally, we call the pseudo-uniformity $\mathcal{U}$, *decreasing* when it may be represented as $(U_\lambda)_{\lambda<\delta}$ (for some admissible ordinal δ depending on $\mathcal{U}$) where $\lambda \vdash U_\lambda$ is decreasing from $(W(\delta),\leq)$ to $(X^2,\subseteq)$. The admissible ordinal in question satisfies

$$\kappa(\mathcal{U}) \leq \delta < \tilde{\kappa}(\mathcal{U}),$$

where, for each nonempty part Z of the ambient universe we put

$$\tilde{\kappa}(Z) = \min\{\alpha \in W; \mathrm{card}(\mathrm{Z}) < \mathrm{card}(\mathrm{W}(\alpha)).$$

With these conventions at hand, let us now return to the matter into discussion. As a first completion of Theorem A, we have

Theorem C. *Suppose (C_1) is accepted with $\tilde{\kappa}(\mathcal{U})$ in place of $\kappa(\mathcal{U})$ and, in addition*

$$\mathcal{U} \textit{ is}(\leq) - \textit{separated and decreasing.}$$

Then, conclusion of Theorem A is retainable if local $\mathcal{U}$-normality is replaced by local $\mathcal{U}$-semi-normality.

Proof. Let $(U_\lambda)_{\lambda<\delta}$ be the representation of $\mathcal{U}$ assured of (C_7). We also take an arbitrary point x (in X). By the accepted hypothesis about $\leq$, there must be a $x_0 \geq x$ with

$$(3) \qquad x_0 \leq w \text{ implies } (\mathrm{x}_0, \mathrm{w}) \in \mathrm{U}_0.$$

This process may (eventually) continue as follows. Assume that, for the ordinal $\gamma < \delta$ we obtained an ascending net $(x_\lambda)_{\lambda<\gamma}$ so as, for each $\lambda < \gamma$.

$$(Q(\lambda)) \qquad x_\lambda \leq w \text{ implies } (\mathrm{x}_\lambda, \mathrm{w}) \in \mathrm{U}_\lambda.$$

If γ is a first kind ordinal then, by using (3) (for $x = x_{\gamma-1}, U = U_\gamma$) we get a point $x_\gamma \geq x_{\gamma-1}$ with the property $(Q(\gamma))$. Suppose γ is a second kind (limit) ordinal. By (C_1) we firstly get an upper bound, x', of $(x_\lambda)_{\lambda<\gamma}$.

This, again combined with (3) (for (x, U) taken as (x', U_γ)) yields a $x_\gamma \geq x'$ for which $(Q(\gamma))$ is fulfilled. Hence, our iterative process is (eventually) continuable beyond $W(\delta)$. Let z be any upper bound (existing by (C_1)) of the obtained (ascending) $\mathcal{U}$–semi–Cauchy net $(x_\lambda)_{\lambda<\delta}$, and assume $u, v \in X$ are such that $z \leq u \leq v$. By $(Q(\lambda); \lambda < \delta)$ plus (C_7) (the second part) we deduce $u, v \in \lim_\lambda(x_\lambda)$; this , in conjunction with the first part of the same gives $(u, v) \in \cap\mathcal{U}$. The proof is complete. q.e.d.

The decreasing property involved in (C_7) is, in a certain sense, characteristic to the pseudometric setting. To be more specific, let us call the function $d : X^2 \to R_+$, a *pseudometric* over X when

$$d(x, x) = 0, \text{ for all } x \in X.$$

Denoting

$$U(d, t) = \{(x, y) \in X^2; d(x, y) < t\},\ t > 0,$$

the family $\mathcal{U}(d) = \{U(d, t); t > 0\}$ is a decreasing pseudouniformity over X. Concerning the associated ordinals entering in the boundeness condition (C_1) to be used in Theorem A (or Theorem C) let

$$a_0, a_1, ..., a_\lambda, ...$$

be the aleph denumeration of all cardinal numbers (a_0 = the first transfinite cardinal) and

$$\omega_0, \omega_1, ..., \omega_\lambda, ...$$

their associated initial ordinals ($\omega_0 = \omega$). (The possibility of this enumeration is assured by the Axiom of Choice; see, in this direction, A l e x a n d r o v [1, ch.3, Sec.6].) Now evidently

$$\text{card } (\mathcal{U}\text{ (d)}) = \text{a}_1 \text{ (hence } \kappa(\mathcal{U}(\text{d})) = \omega_1, \tilde{\kappa}(\mathcal{U}(\text{d})) = \omega_2),$$

if we accept the Continuum Hypothesis (cf. S i e r p i n s k i [22, ch.V, Sec.2]). That is, working with $\mathcal{U}(d)$ in Theorem A (respectively, Theorem C) will necessitate boundedness conditions involving (ascending) nets indexed after ordinals in $W(\omega_1)$ (respectively, $W(\omega_2)$.) But, a direct inspection shows this may be considerably improved in the sense of replacing ω_1 (respectively, ω_2) by $\omega_0 = \omega$. Indeed, letting $(t_n)_{n<\omega}$ be a (strictly) decreasing sequence in $(0, \infty)$ converging to zero, the pseudo - uniformity $\mathcal{V} = \{U(d, t_n); n < \omega\}$ is decreasing and equivalent with $\mathcal{U}(d)$, in the sense of a previous convention. This, in conjunction with the denumerable character of $\mathcal{V}$, proves our assertion, because all notions to be handled may now be related to $\mathcal{V}$. These are independent of the sequence $(t_n)_{n<\omega}$ appearing in this construction (i.e.

may be deemed as generated by d itself) and are applicable to sequences (rather than nets). For a list of these, see T u r i n i c i [23]. It would be interesting to determine whether this construction exhausts the class of decreasing pseudo–uniformities; the answer seems to be negative. Another question is the following: what happens with Theorem C when $\mathcal{U}$ is no longer decreasing ? To give a statement of this type, the remarks made after the statement of Theorem A will be useful. Namely, suppose that a family $A(\mathcal{U})$ of pseudo–uniformities over X exists, whose union is just $\mathcal{U}$ and (C_7') each $\mathcal{V} \in A(\mathcal{U})$ is $(\leq)$-separated and decreasing.
Suppose also that the boundedness condition (C_1') is taken as in Theorem C. Then, for each $\mathcal{V} \in A(\mathcal{U})$ we have that $\leq$ is globally $\mathcal{V}$–normal; this, plus the remark in question shows the above result remains true in such a case. Finally, the local $\mathcal{U}$–seminormality of the underlying quasi-ordering may be replaced in Theorem C (or its variant we just described) by either of the mutually equivalent conditions $(C_5)/(C_5')$, to retain its conclusion.

We are now passing to the basic boundedness condition (C_1). Technically speaking, it may be viewed as a "local" one in all these results, being expressed in terms of a certain ordinal associated to a particular pseudo–uniformity $(\mathcal{U})$. It would be desirable to remove this local character of this condition (hence of these results). This may be done by the use of the "universal" counterpart of it, namely

$$(C_1^*) \quad \begin{cases} \text{each ascending net } (x_\lambda)_{\lambda<\alpha} \text{, where } \alpha \text{ is an} \\ \text{admissible ordinal, has an upper bound.} \end{cases}$$

Formally, (C_1^*) is more restrictive than any variant of (C_1) we encountered until now. Hence, the corresponding "universal" versions of the results above will be particular cases of them. For example, the following "universal" maximality principle is reductible to Theorem B above.

Theorem B^*. *Let the quasi-ordering $\leq$ X be such that (C_1^*) is fulfilled. Then, $\leq$ is globally $\mathcal{U}$–normal, for each pseudo-uniformity $\mathcal{U}$ over X which satisfies either of the mutually equivalent conditions $(C_2)/(C_2')$.*

However, this implication may be reversed, by the fact that Theorem B^* - viewed as an independent result - is but an equivalent form of the Zorn Maximality Principle (in the Bourbaki variant [3]). Indeed, let $(X, \leq)$ be an ordered set in which any well ordered part is bounded above. We denote by $\mathcal{S}(X, \leq)$ the class of all ascending sequences $(x_n)_{n\in N}$ in X and by $N^{\mathcal{S}(X,\leq)}$ the family of all functions $f : \mathcal{S}(X, \leq) \to N$. Let $\mathcal{U}$ be the pseudo–uniformity over X, consisting of all

$$U_f = \cup\{(\Sigma; f(\Sigma)); \Sigma \in \mathcal{S}(X, \leq)\}, f \in N^{\mathcal{S}(X,\leq)}$$

where, by convention,

$$(\Sigma; n) = \{(x_n, x_{n+1}), (x_{n+1}, x_{n+2}, \ldots\}, n \in N,$$

for each $\Sigma = (x_n)_{n\in N}$ in $\mathcal{S}(X, \leq)$. It is easily seen that (C_2') is verified for this $\mathcal{U}$. Moreover, condition (C_1^*) is holding too. Hence, Theorem B^* applies; and, the $(\leq,\mathcal{U})$-maximal element assured by it is necessarily shown to be $(\leq)$-maximal (see T u r i n i c i [23] for details).

Now, for an individual $\mathcal{U}$, the self-closedness condition (C_6) reduces the universal boundedness property (C_1^*) to convergence (mod $\mathcal{U}$) properties. On the other hand, by (C_2) (respectively, (C_5)), each ascending net is $\mathcal{U}$-Cauchy (respectively, $\mathcal{U}$- semi-Cauchy). So, the convention below is motivated. We call the ambient space X, $(\leq,\mathcal{U})$-complete (respectively, $(\leq,\mathcal{U})$-super-complete) when each ascending $\mathcal{U}$-Cauchy ($\mathcal{U}$-semi-Cauchy) net is a convergent one. Then, the universal boundness condition in question is necessarily holding (under (C_6)) whenever $(C_2)/(C_2')$ plus
(C_8) X is $(\leq,\mathcal{U})$-complete
(respectively, $(C_5)/(C_5')$ plus
(C_8') X is $(\leq,\mathcal{U})$-super-complete)
is being accepted. Of course, in the context of (C_4), the completeness conditions (C_8) and (C_8') are equivalent. (In general, (C_8) includes (C_8')). As an immediate consequence, we have the following practical result.

Theorem D. *Let the quasi-ordering $\leq$ and the pseudo-uniformity $\mathcal{U}$ over X be such that the self-closedness condition (C_6) is fulfilled. Then*

(a) *under the specific assumptions $(C_2)/(C_2')$ plus (C_8), the quasi-ordering $\leq$ is necessarily globally $\mathcal{U}$-normal.*

(b) *if, in addition, (C_4) is admitted, then $(C_2)/(C_2')$ have the same power as $(C_5)/(C_5')$, and globally $\mathcal{U}$-normal equals globally $\mathcal{U}$-semi-normal.*

To complete this statement, we need one more convention. Let us call $\leq$, *normal* (or, Zorn) *quasi-ordering* when

$$(D_4) \qquad \begin{cases} \text{for each } x \in X \text{ there is a}(\leq) \text{ -- maximal element } z \in X \\ \text{with } x \leq z. \end{cases}$$

By the remarks at the beginning of this section, we therefore have

(c) *if, in addition , $\mathcal{U}$ is $(\leq)$-separated, then globally $\mathcal{U}$-normal equals normal.*

In other words, Theorem D may be also viewed as a Zorn maximality principle. In particular, this is the case when the pseudo-uniformity $\mathcal{U}$ is a *uniformity* over X in the Bourbaki sense [4, ch.II, Sec.1] which, in addition,

be separated ($\cap\mathcal{U} = \Delta(X)$) and $\leq$ is an ordering over X. Then, the result above coincides with Theorem 2 in V a l y i [26] proved by a standard (Zorn) argument; see also I s a c [11].

Perhaps it is not without importance to specify that another (equivalent) language of expressing these results is the abstract pseudometric one; see T u r i n i c i [23] for concrete details. We preferred, however, this pseudo-uniform language in view of some technical reasons imposed by the setting of the main results.

2. Almost minimality in ordered t.v.s. Let E be a linear space over the reals. By a *cone* in E, we shall mean any part K of E, satisfying

$$(D_5) \qquad K + K \subseteq K, tK \subseteq K, \text{for all } t \in R, t \geq 0.$$

(In the Peressini's terminology [19,ch.1, Sec.1], this object is referred to as a *wedge*). Denote by $\leq$ (mod K) the induced quasi-ordering; that is

$$a \leq b \text{ (mod K) if and only if } b - a \in K.$$

Also, indicate by lin(K) the linear space $K \cap (-K)$. When lin(K) $= \{0\}$, then K is termed *acute* (*pointed*); equivalently, this means $\leq$ (mod K) is an ordering over E. A subset C of E will be said to be *bounded* (mod K) from *above* (*below*) in case

$$C \subseteq b - K (\text{resp., } C \subseteq a + K \text{ }), \text{ for some } b \in E \text{ } (a \in E).$$

The intersection of these properties will be referred to as C is *bounded* (mod K); hence, it means

$$C \leq (a + K) \cap (b - K), \text{ for some } a, b \in E.$$

In the following, we also work with a linear topology over E; that is, a topology $\mathcal{T}$ with respect to which the usual operations of the linear structure on E are continuous. Usually, this may be generated be a certain family $\mathcal{B}$ of neighbourhoods of zero, the members of which satisfy
(P_1) each $B \in \mathcal{B}$ is balanced (hence symmetric)
(P_2) $(rB \in \mathcal{B})$, whenever $B \in \mathcal{B}$ and $r > 0$
(P_3) for each $A \in \mathcal{B}$ there exists $B \in \mathcal{B}$ with $B + B \subseteq A$;
see C r i s t e s c u [6, ch.I. Sec.2] for details. Denote for simplicity

$$U(B) = \{(a, b) \in E^2; b - a \in B\} \text{ , } B \in \mathcal{B}.$$

The following properties are immediate (if we take into account the preceding ones)

(P_1') $U(B)^{-1} = U(B)$ for all $B \in \mathcal{B}$
(P_2') $U(rB) = rU(B), r > 0, B \in \mathcal{B}$
(P_3') $U(B) \cdot U(B) \subseteq U(A)$, where A, B are as in (P_3).

The filter $\mathcal{U}(\mathcal{T})$ over E^2 generated by the filterbase $U(\mathcal{B}) = \{U(B); B \in \mathcal{B}\}$ is therefore a uniformity over E in the standard Bourbaki sense (cf. the preceding section), with the supplementary property (P_2'). (The notation adopted is motivated by the fact that the construction above is independent of $\mathcal{B}$).

Now, the symmetry property assured by (P_1') allows us to consider *monotone* (mod K) *nets* (where, by convention monotone means either ascending or descending). Also, the triangular (transitivity) property given by (P_3') will simplify the concepts presented in the previous section. Precisely, for the monotone (mod K) net $(a_\lambda)_{\lambda<\alpha}$ in E, the property of being $\mathcal{U}(\mathcal{T})$-Cauchy may be depicted as: for each $B \in \mathcal{B}$ there exists $\lambda = \lambda(B) < \alpha$ such that

$$a_\mu - a_\lambda \in B, \text{ for all } \mu < \alpha \text{ with } \lambda \leq \mu.$$

Also, for a sequence $(a_n)_{n<\omega}$, the $\mathcal{U}(\mathcal{T})$-asymptotic property becomes: for each $B \in \mathcal{B}$ there exists $n(B) < \omega$ such that

$$a_{m+1} - a_m \in B, \text{ for all m} \geq \text{n(B)}, \text{m} < \omega.$$

Moreover, as already said, the global version of these properties are equivalent over any part of E. Actually, we have a stronger conclusion of this kind. Precisely, suppose H is a *subcone* of K; that is, part of K fulfiling (D_5) above (with H in place of K). We claim that the global conditions

$$(C_9) \quad \begin{cases} \text{each bounded (mod K) monotone (mod H) net in E} \\ \text{is } \mathcal{U}(\mathcal{T}) - \text{-- Cauchy} \end{cases}$$

$$(C_9') \quad \begin{cases} \text{each bounded (mod K) monotone (mod H) sequence in E} \\ \text{is } \mathcal{U}(\mathcal{T}) - \text{-asymptotic} \end{cases}$$

are equivalent each other. For, if the bounded (mod K) monotone (mod H) net $(a_\lambda)_{\lambda<\alpha}$ is not $\mathcal{U}(\mathcal{T})$-Cauchy then, for some $B \in \mathcal{B}$ we must have the property: for each $\lambda < \alpha$ there exists $\mu < \alpha$ with

$$(4) \qquad \lambda < \mu \text{ and } a_\mu - a_\lambda \in B^c.$$

Consequently, a monotone (mod H) sequence $(b_n)_{n<\omega}$ may be found (with the aid of these terms) so that

$$(5) \qquad b_{n+1} - b_n \in B^c, \text{ for all n} < \omega.$$

In addition, this sequence is, by construction, bounded (mod K). But, in this case, (C_9') is contradicted. Hence the assertion.

By convention, (C_9) will be referred to as H is $K-$*regular*. We therefore obtained that an equivalent characterization of K-regularity is (C_9'); it extends the one obtained by N e m e t h [16] in the locally convex case. Another characterization of the concept under discussion is contained in

Proposition 1. *The subcone H (of K) is K-regular, if and only if*

$$(C_{10}) \quad \begin{cases} \textit{whenever } (h_n)_{n<\omega} \textit{ is a sequence in } H \textit{ and } B \textit{ is an} \\ \textit{element of } \mathcal{B} \textit{ with } h_n \in B^c \textit{ for infinitely many } n, \\ \textit{the partial sum sequence } (k_n = \sum_{i\leq n} h_i)_{n<\omega} \\ \textit{cannot be bounded (modK).} \end{cases}$$

Proof. Suppose H is K-regular. Let $(h_n)_{n<\omega}$ and $B \in \mathcal{B}$ be taken as in the above condition. Then, $(k_n)_{n<\omega}$ defined by the same, is necessarily ascending (mod H). Suppose it would be bounded (mod K). Then, by (C_9'), there exists $n(B)$ so that

$$h_{m+1} = k_{m+1} - k_m \in B, \text{ for all m} \geq \text{n(B)}.$$

The contradiction obtained proves the necessity. Conversely, assume (C_{10}) is valid and let $(a_\lambda)_{\lambda<\alpha}$ be a bounded (mod K) monotone (mod H) net in E which is not $\mathcal{U}(\mathcal{T})$-Cauchy. Then, there must be a $B \in \mathcal{B}$ so that, for each $\lambda < \alpha$ there exists $\mu < \alpha$ with the property (4). Now, there is no loss in assuming $(a_\lambda)_{\lambda<\alpha}$ is ascending (mod H). Then, by a previous argument, one may arrive at an ascending (mod H) sequence $(b_n)_{n<\omega}$ (if we single out the terms of the initial net appearing in (4)) so that (5) be valid. Denote

$$h_n = b_{n+1} - b_n, n < \omega.$$

The sequence $(h_n)_{n<\omega}$ is therefore in H and satisfies

$$h_n \in B^c, \text{ for all n} < \omega.$$

But, in this case, by (C_{10}), the partial sum sequence

$$k_n = \sum_{i\leq n} h_i = b_{n+1} - b_0, n < \omega,$$

is not bounded (mod K). On the other hand, this sequence is necessarily bounded (mod K) being a translate of $(b_n)_{n<\omega}$. The contradiction at which we arrived ends the argument. q.e.d.

As a technical consequence of this fact, we have that, in the characterization (C_{10}) of K-regularity one may replace "infinitely many ranks" with "all ranks" without affecting the sphere of the notion. But, for the developments below, this will not suffice. To complete the "portrait" of this concept, we need one more characterization of it, in term of almost minimum points. This requires some preliminary facts. Recall that, when $E = R, K = R_+$, a bounded from below part M of E admits an *infimum*, denoted inf(M), characterized as

(P_4) inf (M) is a lower bound of M

(P_5) for each $t > 0$ there exists $a(t) \in M$ with $a(t) <$ inf (M) + t.

This last property may be also written in a form independent of $\inf(M)$; namely

$$(P'_5) \qquad \begin{cases} \text{for each t} > 0 \text{ there exists a} = \text{a(t)} \in \text{M such that} \\ \text{a} - \text{b} < \text{t, whenever b} \in \text{M satisfies b} \leq \text{a.} \end{cases}$$

(By convention, a(t) above will be termed a *t-almost minimum* point for M). Actually, (P'_5) includes (P_4). For, if $(t_n)_{n<\omega}$ were a (strictly) descending sequence in $(0, \infty)$ converging to zero, and $(a_n)_{n<\omega}$ is a descending sequence in M with

$$a_n \text{ is } t_n - \text{almost minimum , for all n,}$$

then $\lim_n a_n$ is necessarily identical to inf(M). Hence, extending (in a vectorial sense) $(P_4) + (P_5)$ above ammounts to extending (P'_5). To this end, return to the initial setting and suppose M is a bounded from below (mod K) subset of E. The vectorial analogu^ of $(P_4) + (P_5)$ is

$$(P_6) \qquad \begin{cases} \text{for each B} \in \text{B there exists a} = \text{a(B)} \in \text{M such that} \\ \text{a} - \text{b} \in \text{B, for all b} \in \text{M , b} \leq \text{a. (mod K)} \end{cases}$$

The points $a(B)$ fulfiling this condition will be termed *B-almost minimum points* of *M*. And, the requirement in question will be referred to as M has the *almost minimum property.*

The following characterization of K regularity in terms of this property is to be noted. (See also N e m e t h [17] for related facts). Denote, for a subcone H of K

$$H(B) = H \cap B^c, \text{ for any } B \in \mathcal{B}.$$

Proposition 2. *The subcone H (of K) is K-regular, if and only if*

$$(C_{11}) \qquad \begin{cases} \textit{whenever } B \in \mathcal{B} \textit{ fulfils } H(B) \neq \emptyset, \textit{ we have that each} \\ \textit{bounded from below (mod K) part of E possesses at} \\ \textit{least one } (H(B) + K)^c - \textit{almost minimum point.} \end{cases}$$

Proof. Assume H is K-regular and let $B \in \mathcal{B}$ be such that H(B) is nonempty. If, for some bounded from below (mod K) part M of E, it is the case that $(H(B) + K)^c$-almost minimum points in M cannot exist, we have that, for each $a \in M$ there may be chosen at least one $b \in M$ with

$$a \geq b \text{ (mod K) , a} - \text{b} \in \text{H(B)} + \text{K.}$$

We therefore, may construct a descending (mod K) sequence $(a_n)_{n<\omega}$ in M with

$$a_n - a_{n+1} \in H(B) + K \ , \ n < \omega;$$

that is, for some sequence $(h_n)_{n<\omega}$ in $H(B) \subseteq H$,

$$a_n - a_{n+1} \geq h_n \text{ (mod K), for all n} < \omega.$$

Now, the sequence $(a_n)_{n<\omega}$ is bounded (mod K) because M is bounded below (mod K); hence, the partial sum sequence

$$k_n = \sum_{i \leq n} h_i \ , \ n < \omega,$$

is bounded above (mod K). On the other hand, $H(B) \subseteq B^c$, by definition; therefore

$$h_n \in B^c, \text{ for all n} < \omega.$$

The obtained properties are, however, contradictory via (C_{10}); this proves the necesity. Conversely , assume (C_{11}) is valid but H is not K-regular. Then, by (C_9), there must be a bounded (mod K) descending (mod H) net $(a_\lambda)_{\lambda<\alpha}$ is E which is not $\mathcal{U}(\mathcal{T})$-Cauchy. This (cf. the argument in Proposition 1) yields a descending (mod H) sequence $(b_n)_{n<\omega}$ in E and a $B \in \mathcal{B}$ with the property (5); that is (by the adopted notation)

$$b_n - b_{n+1} \in H(B) \subseteq H(B) + K \text{ for all n} < \omega.$$

(Hence, in particular, H(B) is not empty.) This shows no point in $M = \{b_n; n < \omega\}$ is $(H(B)+K)^c$-almost minimum. In addition, M is bounded from below (mod K), being a part of $(a_\lambda)_{\lambda<\alpha}$. But, in this case (C_{11}) is contradicted. Hence (C_9) is valid and the sufficiency follows. q.e.d.

With these informations at hand, the question arises of determining those subcones H of K for which the property of Proposition 2 is a sufficient one for the validity of (P_6). This implicitly requires the property subsumed to (C_{11}) be consistent. For example, whenever

(P_7) $H(B) = \emptyset$, for all $B \in \mathcal{B}$, the implication above cannot be retained, because (C_{11}) is without object. We note that (P_7) is equivalent with

$$\text{H is } \mathcal{T} \text{ -- bounded in the ambient space;}$$

see, for instance, C r i s t e s c u [6,ch.I, Sec.2]. Another "disturbing" property is
(P_8) $H(B) = \emptyset$, for some $B \in \mathcal{B}$,
or, equivalently

$$H \subseteq \cap\{tB; t > 0\}, \text{ for some } B \in \mathcal{B}.$$

(The (sub)cone H will be termed $\mathcal{T}$ - *flat* in such a case). The question of whether or not is (P_6) obtainable, via (C_{11}), under the limiting property (P_8) of the subcone H, remains open. In the following, we shall eliminate this possibility; that is, one assumes H is non flat. But, even in such a circumstance, we are not sure that the property (P_6) is holding, because the trace over K of the (nonempty sets) family $\{(H(B)+K)^c; B \in \mathcal{T}\}$ does not coincide, in general, with the trace over K of the family $\mathcal{B}$. It is therefore natural to find conditions under which such a property be valid; that is,

$$K \cap (H(B)+K)^c = K \cap B, \text{ for all } B \in \mathcal{T}. \tag{6}$$

We note that, in the left member of this relation, the following hereditary property is holding

$$a, b \in K, a \leq b \text{ (mod K) }, b \in (H(B)+K)^c \Rightarrow a \in (H(B)+K)^c.$$

So, the right member of (6) must have the same property. This suggests a normality requirement for K. More precisely, denote for each part A of E

$$[A] = (A+K) \cap (A-K) \text{ (the order covering of A),}$$

and call the subset A, *full*, in case it is identical with [A]. Now, the cone K will be said to be *normal*, in case $o \in E$ has a fundamental system of full neighbourhoods. This ammounts to say that all $B \in \mathcal{B}$ are full (see P e r e s s i ni [19,ch.2 Sec.1] for details). Suppose K is normal in what follows. It is also non flat, by the assumption about H plus the inclusion between them. Hence

$$K(B) \text{ is nonempty , for all } B \in \mathcal{B}.$$

In adittion, these are K-invariant, in the sense

$$K(B) + K = K(B), \text{ for all } B \in \mathcal{B},$$

because the members of $\mathcal{B}$ are full subsets. We therefore have

$$H(B) + K \subseteq K(B) + K = K(B), \ B \in \mathcal{B},$$

wherefrom (again for all $B \in \mathcal{B}$)

$$K \cap (H(B)+K)^c \subseteq K \cap (K(B))^c = K \cap (K^c \cup B) = K \cap B;$$

that is, (6) is verified in a sense only. Concerning the opposite inclusion, we note that, in view of

$$H(B) + K \subseteq H(B) \ , \ B \in \mathcal{B},$$

one has

$$K \cap (H(B)+K)^c \subseteq K \cap (H(B))^c = (K \cap H^c) \cup (K \cap B) \ , \ B \in \mathcal{B}$$

Hence, the desired inclusion is holding whenever $K \cap H^c = \emptyset$ or, equivalently, $K = H$; this will be reffered to as K being *regular*. (The question of whether or not is this the unique choice of H to satisfy the discussed inclusion remains open). We therefore obtained the following answer to the above posed question.

Proposition 3. *Let the cone K in E be regular, normal and non flat. Then, each bounded from below (mod K) part of E has the almost minimum property.*

An interesting particular case of these considerations is that described by the initial topology $\mathcal{T}$ being locally convex. Then, the family $\mathcal{B}$ above is to be identified with the one generated by the finite intersections of the sets

$$B_p(t) = \{a \in E; p(a) < t\} \ , \ p \in \mathcal{P}, t > 0, \tag{7}$$

where $\mathcal{P}$ is a certain family of *seminorms* over E. The normality of K may now be expressed as all $p \in \mathcal{P}$ being monotone, i.e.,

$$a, b \in K \ , \ a \leq b \text{ (mod K) } \Rightarrow p(a) \leq p(b);$$

see in this direction P e r e s s i n i [19.ch.2, Sec.1]. Also, the nonflatness of K is equivalent with

$$p(K) \neq \{0\}, \text{ for all } p \in \mathcal{P} \text{ (hence } K \neq \{0\}), \tag{P_9}$$

as it can be readily seen. This requires K be very "large" from a dimensional viewpoint; because, e.g., a sufficient condition for its validity is

$$E = K - K \text{ (referred to as E is K -generated).}$$

It remains now to discuss the regularity of K which, by the accepted convention, may be introduced as

(C_9'') $\left\{\begin{array}{l}\text{each bounded and monotone (mod K) sequence in K}\\ \text{is } \mathcal{U}(\mathcal{T}) \text{ – asymptotic.}\end{array}\right.$

An appropriate answer requires the (accepted in the sequel) assumption that the cone in question be normal. We denote by K^* the *dual cone* associated to K; that is, the set of all $f \in E^*$ with

$$f(a) \geq 0 \text{ , for all a} \in \text{K.}$$

Under a convention proposed by Isac [11], the cone K will be termed *nuclear* when for each $p \in P$ there exists a $f = f_p$ in K^* with

$$p(a) \leq f(a), \quad a \in K.$$

(For example, this is the case when $\mathcal{T}$ coincides with $\mathcal{S} = \mathcal{S}(E, E^*)$ (= the ***weak topology*** over E induced by the natural duality between E and E^*)). It is now clear that, under

(C_{12}) K is (normal and) nuclear

the regularity of K is assured (mod $\mathcal{T}$). As an alternative solution to this question, let the ambient linear topology $\mathcal{T}$ over E be Hausdorff separated. (Note that, by a result in S c h a e f e r [21,ch.V, Sec.3], K is necessarily pointed in such a case). The introduced space E will be said to be *sequentially $\mathcal{S}$–complete* (mod K) when each monotone and bounded (mod K) sequence in E which is $\mathcal{U}(\mathcal{S})$-Cauchy, converges (mod $\mathcal{S}$) to some point of E. (Here, $\mathcal{S} = \mathcal{S}(E, E^*)$ is introduced as above, and $\mathcal{U}(\mathcal{S})$ stands for its associated uniformity).

Proposition 4. *Suppose (C_{12}) is to replaced by*

(C_{12}') *K is (normal and) closed.*

Let also E be sequentially $\mathcal{S}$–complete (mod K). Then, K is necessarily regular (mod $\mathcal{T}$).

Proof. Let $(a_n)_{n<\omega}$ be a descending (mod K) sequence in K which is bounded from below (mod K). For each $f \in K^*$, the (real) sequence $(f(a_n))_{n<\omega}$ is Cauchy (because it is descending in R_+). This property will remain valid over E^* because (see, e.g., S c h a e f e r [21,ch.V,Sec.3]) it is the case that, under (the first half of) $(C_{12}'), E^*$ is K^*–generated. Hence, $(a_n)_{n<\omega}$ is $\mathcal{S}$–Cauchy. As E is sequentially $\mathcal{S}$–complete (mod K), $(a_n)_{n<\omega}$ must converge (mod $\mathcal{S}$) to some $a \in E$. But, for any convex part of E, closed (mod $\mathcal{T}$) is identical with closed (mod $\mathcal{S}$); see C r i s t e s c u [6,ch.III,Sec.3] for details. Therefore

$$a \in K \textit{ and } a_n \geq a \textit{ (mod } K\textit{), for all } n.$$

This, combined with a result in C r i s t e s c u [op.cit.,ch.VI,Sec.1] gives

$$(a_n)_{n<\omega} \textit{ converges (mod } \mathcal{T}\textit{) towards } a.$$

And, consequently, $(a_n)_{n<\omega}$ is $\mathcal{U}(\mathcal{T})$–asymptotic. The conclusion follows now by the characterization (C_9') of regularity. q.e.d.

As a standard example, we let I be a (nonempty) index set and consider the space $E = R^I$ endowed with the product topology; it is clearly locally convex, being introduced by the seminorms

$$p_j(a) = |a_j| \text{ , } a = (a_i)_{i\in I} \in E \text{ , } j \in I.$$

Denote $K = R_+$. That K is normal, follows immediately by the monotonicity of p_j, for all $j \in I$. Moreover, K is non–flat, because (P_9) is fulfilled. (It is also pointed, as it can be directly seen). Finally, by the (sequential) completeness of E, plus the closedness of K, one obtains (via Proposition 4) that K is regular. In other words, Proposition 3 is applicable to these elements. Other examples may be found in N e m e t h [16,17]; see also I s a c [11].

It would be not without interest to see what happens in the normed case. This is not only useful from a technical viewpoint, but also from a methodological one, because the transition from the topological vector setting to the normed one is, in many situations, abrupt (without passing through the locally convex case). The statement below (to be found, e.g., in S c h a e f e r [21,ch.II,Sec.2]) will clarify these facts.

Proposition 5. *Assume that the cone K is pointed, normal, and has nonempty interior. Then, $\mathcal{T}$ is necessarily normable.*

Proof (Sketch). Let a be an interior point of K. The order interval $A = (-a + K) \cap (a - K)$ is then a neighborhood of zero; it is also bounded, as K is normal. Hence, by the Kolmogorov normality result (quoted in the above reference) the linear topology $\mathcal{T}$ over E is generated by a norm. q.e.d.

So, let $(E, \|\cdot\|)$ be a normed space. The fundamental neighborhoods system $\mathcal{B}$ is that defined by (7), where the family $\mathcal{P}$ of seminorms is reduced to $\{\|\cdot\|\}$. In such a case, the Cauchy property of a net $(a_\lambda)_{\lambda<\alpha}$ in E means: for each $t > 0$, there exists $\lambda(t) < \alpha$, such that

$$\|a_\nu - a_\mu\| < t \ , \ \text{whenever } \lambda(t) \leq \mu \leq \nu < \alpha.$$

Also, the asymptotic property of the sequence $(a_n)_{n<\omega}$ in E may be depicted as : $\lim_n \|a_{n+1} - a_n\| = 0$. Let H be a (nonzero) subcone of K. The K–regularity of this object may be expressed by the normed versions (constructed under the above lines) of (C_9) and (C_9'). It may be also described, via Proposition 1, in terms of (C_{10}), as follows

(C_{10}') $\left\{\begin{array}{l}\text{whenever the sequence } (h_n)_{n<\omega} \text{ in H and the number} \\ t > 0 \text{ satisfy } \|h_n\| \geq t \text{ for infinitely many ranks n,} \\ \text{the partial sum sequence } (k_n = \sum_{i\leq n} h_i)_{n<\omega} \\ \text{cannot be bounded (mod K),}\end{array}\right.$

or, via Proposition 2, in terms of (C_{11}); namely

(C_{11}') $\left\{\begin{array}{l}\text{for each } t > 0 \text{ we have that any bounded from below} \\ \text{(mod K) part of E possesses at least one} \\ (H(t)+K)^c - \text{almost minimum point.}\end{array}\right.$

Here, $H(t)$ stands for $H(B)$, where $B = B_{\|\cdot\|}(t)$ is that introduced by (7). Note that

$$H(t) \neq \emptyset, t > 0 \ \text{(because H is a nonzero cone);}$$

or, in other words, H is non flat. These characterizations remain valid when $H = E$; that is, for the case of K being regular. Concerning the normality property, the following converse to Proposition 4 is to be noted. Recall that, be the considerations involving the locally convex case, K is a normal cone if and only if an equivalent norm over E may be determined with the property of being monotone. In terms of $\|\cdot\|$ (the initial norm over E) this may be also expressed as (when K is, in addition, pointed)

$$\nu(K) = \sup\{\|a\|/\|b\|; 0 < a \leq b\} < \infty;$$

see in this direction C r i s t e s c u [6, ch.VI, Sec.1]. The normed structure $(E, \|\cdot\|)$ will be said to be *complete* (mod K) when each monotone (mod K) Cauchy sequence in E converges.

Proposition 6. *Assume K is a closed regular (pointed) cone and E is complete (mod K). Then, K is necessarily normal.*

Proof. Suppose by contradiction K is not normal. Then, for each $t > 0$, there exists a couple $a_t, b_t \in K$ with

$$0 < a_t \leq b_t \ , \ \|a_t\| \geq t\|b_t\|.$$

And, from this, a couple of sequence $(a_n)_{n<\omega}, (b_n)_{n<\omega}$ in K may be determined in such a way that

$$a_n \leq b_n \ , \ \|a_n\| = 1 \ , \ \|b_n\| < 2^{-n} \ , \ n < \omega.$$

Denote,

$$c_n = a_0 + \ldots + a_n \ , \ d_n = b_0 + \ldots + b_n \ , \ n < \omega.$$

As $(d_n)_{n<\omega}$ is ascending (mod K) and Cauchy, it must converge (by the completeness hypothesis we admitted) to some $d \in K$. Moreover (as K is closed)

$$d_n \leq d \text{ (mod K)} \ , \ \text{for all } n < \omega.$$

This, along with

$$c_n \leq d_n \text{ (mod K)} \ , \ n < \omega,$$

gives $(c_n)_{n<\omega}$ is bounded above (mod K) by d, in contradiction to (C_{10}'). Hence, K is normal. q.e.d.

As a consequence we have that, in (the normed version of) Proposition 3, the normality of K may be substituted by the closedness of the same, plus completeness (mod K) of the ambient space. (The non–flatness property is, as above said, trivially valid whenever K is different from $\{0\}$). We do not know whether the statement above (originally due to K r a s n o s e l s k i i [13, ch.I, Sec.5]) is valid beyond the normed setting.

3. The main result. Let X be a nonempty set. Given the (real) linear space E, the (nonzero) cone K in E and the subcone H of K, by a *H–pseudometric* over X we shall mean any mapping $d : X^2 \to H$ with

(8) $$d(x, x) = 0, \text{ for all } x \in X.$$

In this case, the couple (X, d) will be termed a *H – pseudometric space.* Now, technically speaking, the objective to be reached is that of applying the results in Section 1 to such structures. Hence, we must introduce the class of quasi–orderings and pseudouniformities over X involved in these statements. Let $F : X \to E$ be a mapping. The relation $\leq$ over X defined as

$$x \leq y \text{ if and only if } d(x,y) \leq F(x) - F(y) \text{ (mod K)}$$

is clearly reflexive, by (8). To have transitivity as well, as must impose a condition like

(9) $d(x,z) \le d(x,y) + d(y,z)$ (mod K), whenever $x \le_{(F)} y \le_{(F)} z$

where, by convention,

$$x \le_{(F)} y \text{ if and only if } F(x) \ge F(y) \text{ (mod K)}.$$

(The initial H-pseudometric d will be termed (K,F)-*transitive* in such a case). This solves the first question. For the second, we suppose that a linear topology, $\mathcal{T}$ has been introduced over E. Denote by $\mathcal{A}$ the family of all zero neighborhoods in the sense of $\mathcal{T}$ (see, e.g., S c h a e f e r [21,ch.I,Sec.1]). The pseudo-uniformity under discussion is then defined as the class $U(d,\mathcal{T})$ of all

$$U(d,A) = \{(x,y) \in X^2; d(x,y) \in A\}\ ,\ A \in \mathcal{A}.$$

As a variant of this construction, let $\mathcal{B}$ stand for a neighbourhoods basis of $o \in E$ with the properties $(P_1)-(P_3)$ of the preceding section. Then, the pseudo-uniformity (over X)

$$U(d,\mathcal{B}) = \{U(d,B); B \in \mathcal{B}\}$$

is equivalent with $U(d,\mathcal{T})$ above (in the sense of Section 1); therefore, it may be used in all notions related to $U(d,\mathcal{T})$. For example, the fact that $U(d,\mathcal{B})$ is (or is not) $(\le)$ -triangular may be deemed as a property for $U(d,\mathcal{T})$. Concerning this fact, it is interesting to note that, whenever d fulfils

(9') $d(y,z) \le d(x,y) + d(x,z)$ whenever $x \le_{(F)} y \le_{(F)} z$,

and the cone K is normal, this necessarily happens. Indeed, for each $B \in \mathcal{B}$ there exists, by (P_3), a $C \in \mathcal{T}$ with $C + C \subseteq B$. This shows that, if one accepts the premises

$$x \le y \le z \text{ and } d(x,y), d(x,z) \in C$$

one has $d(x,y) + d(x,z) \in B$. Now, the normality of K ammounts to saying that all members of $\mathcal{B}$ are full subsets. Hence, combining with (9'), we necessarily derive $d(y,z) \in B$, and the assertion follows. As a matter of fact, the usual device is to substitute (9) and (9') by their common stronger (amorph) counterpart

(10) $d(x,z) \le d(x,y) + d(y,z)$ (mod K), for all $x,y,z \in X$.

(The corresponding object, d, will be termed a *H-semimetric* in such a case. For a topological investigation of these structures we refer to C o l l a t z [5,ch.I,Sec.4]). But, for the results in Section 1, the $(\le)$ -triangular property is not essential. So, we shall ignore (9') in the sequel, to maintain a reasonable degree of generality for the present exposition.

Now the basis assumptions encountered in Section 1 are those codified as $(C_1)/(C_1^*)$ or $(C_2)/(C_2^*)$. To give appropriate interpretations of them in our (vector) pseudometric setting, we shall follow the line of $(C_6)+(C_8)$ in that section. Precisely, by the considerations above, the $\mathcal{U}(d,\mathcal{T})$-Cauchy property of a net $(x_\lambda)_{\lambda<\alpha}$ in X may be described as: for each $B \in \mathcal{B}$ there must be a $\lambda(B) < \alpha$ so that

$$d(x_\mu, x_\nu) \in B, \text{ for all } \mu,\nu < \alpha \text{ with } \lambda(\mathrm{B}) \le \mu \le \nu.$$

Also, the convergence (modulo d) of a net $(x_\lambda)_{\lambda<\alpha}$ in X towards a point $x \in X$ is, by definition, equivalent with the convergence (modulo $\mathcal{T}$) of the net $(d(x_\lambda,x))_{\lambda<\alpha}$ in E towards $o \in E$; that is, for each $B \in \mathcal{B}$ there exists $\lambda(B) < \alpha$ so that

$$d(x_\mu, x) \in B, \text{ for all } \mu < \alpha \text{ with } \lambda(\mathrm{B}) \le \mu.$$

After the model of Section 1, we shall denote this property as $x_\lambda \to x$; if, in addition, x is an upper bound (mod $\le_{(F)}$) of the net in question, the resulting property will be depicted as $x_\lambda \uparrow x$. The quasi-ordering $(\le_{(F)})$ over X will be said to be *d-self-closed* when, for each ascending (mod $\le_{(F)}$) net $(x_\lambda)_{\lambda<\alpha}$ in X and each $x \in X$,

$$x_\lambda \to x \text{ if and only if } x_\lambda \uparrow x.$$

(That, is whenever the left part of this relation is holding, the right part is holding too). By convention, this will be referred to as F being $d-self-lsc$; and, by the definition of $\le_{(F)}$, it means

$$(D_6)\qquad \begin{cases} \text{whenever } (F(x_\lambda))_{\lambda<\alpha} \text{ is descending (mod K) and } x_\lambda \to x, \\ \text{we have that } F(x_\lambda) \ge F(x) \text{ (mod K)}\ ,\ \lambda < \alpha. \end{cases}$$

This notion is of the same nature as the one introduced by N e m e t h [16] is a more restriction (modulo d) context. It may be viewed as a weakened form of the usual $d-lsc$ property for the function F, expressed as

(D_6') $x_\lambda \to x$ and $F(x_\lambda) \le a(\text{mod } K)$, $\lambda < \alpha$, imply $F(x) \le a$ (mod K).

The d–convergence property we introduced is, in general, not reductible to the $\mathcal{U}(d,\mathcal{T})$ -Cauchy property. (With some extra conditions, however, this is possible, if we restrict our considerations to ascending (modulo $\leq_{(F)}$) nets. For example, whenever K is a normal cone, F is d–self–lsc and the H–pseudometric D satisfies

$$(9'') \qquad d(x,y) \leq d(x,z) + d(y,z)\ ,\ \text{if } x \leq_{(F)} y \leq_{(F)} z.$$

we have that each ascending (mod $\leq_{(F)}$) convergent (modulo d) net in X is necessarily $\mathcal{U}(d,\mathcal{T})$-Cauchy. (The proof being almost evident we omit the details). But, in the sequel, (9″) will be not accepted, because it is not essential to the argument). However, we shall term the ambient space $X,(d,F) - -complete$, when each ascending (mod $\leq_{(F)}$) $\mathcal{U}(d,\mathcal{T})$-Cauchy net in X converges (modulo d). This notion is trivially comprising the standard one, attached to the H–semimetric setting (expressed by (10)). Its technical interest consists in the intervention of the function F at the level of this structural property of the ambient space (which, as a rule, is of a pseudometric nature only).

Under these preliminaries, a first answer to the above posed question is that concerning the couple of conditions $(C_2)/(C_2')$. It may be given in the form below.

Lemma 1. *Suppose the subcone H (of K) is K-regular. Then, each ascending (mod $\leq$) net $(x_\lambda)_{\lambda<\alpha}$ in X for which $(F(x_\lambda))_{\lambda<\alpha}$ is bounded from below (mod K) is necessarily $\mathcal{U}(d,\mathcal{T})$-Cauchy.*

Proof. The ascending (mod $\leq$) property of the net in question means

$$(11) \qquad d(x_\lambda, x_\mu) \leq F(x_\lambda) - F(x_\mu)(mod\ K), \lambda \leq \mu < \alpha,$$

Of course, $(F(x_\lambda))_{\lambda<\alpha}$ is, by this property, a descending (mod K) net in E. We shall show that rejecting the conclusion above yields a contradiction. For, suppose the net $(x_\lambda)_{\lambda<\alpha}$ would be not $\mathcal{U}(d,\mathcal{T})$-Cauchy. Then, for some $B \in \mathcal{B}$, it is the case that, for each $\lambda < \alpha$ there exists $\mu, \nu < \alpha$ with

$$\lambda \leq \mu \leq \nu \ \text{ and } \ d(x_\mu, x_\nu) \in B^c.$$

As a consequence, an ascending (mod $\leq$) sequence $(y_n)_{n<\omega}$ in X may be obtained (via these terms of $(x_\lambda)_{\lambda<\alpha}$) so that, for the sequence $(h_n = d(y_n, y_{n+1}))_{n<\omega}$ in H, the following property be verified

$$h_{2n} \in B^c, \quad for\ all \quad n < \omega.$$

On the other hand, we have, by (11), that

$$h_n \leq F(y_n) - F(y_{n+1})(mod\ K)\ ,\ n < \omega.$$

This, combined with the boundednees (mod K) of the sequence $(F(y_n))_{n<\omega}$, tells us the partial sum sequence $(k_n = \sum_{i\leq n} h_i)_{n<\omega}$ is bounded (mod K). The obtained conclusions, are, however, contradictory in the light of Proposition 1. Hence , $(x_\lambda)_{\lambda<\alpha}$ is necessarly $\mathcal{U}(d.\mathcal{T})$-Cauchy, as claimed. q.e.d.

Now, the second answer to be formulated concerns the boundness condition (C_1^*). The assumption in Lemma 1 are being maintained. Also, we admit that K is an *ord-closed* cone; that is

the limit of any convergent net in K is an element of K.

Concerning this fact, remember that, by the conventions in Section 1, the term "net" is reserved for those mappings defined over some ordinal segment $W(\alpha)$ (where α is an admissible ordinal). Hence, the above convention includes the standard closedness property of the cone K obtained (cf. Kelley [12,ch.2, Sec.2]) by replacing the word "net" by "Kelley net" in the discussed convention. This inclusion may be very well effective. But, for the developments below, this has a marginal importance only. The announced answer may now be given as follows.

Lemma 2. *Assume F is d-self-lsc and X is (d,F) - complete. Then, each ascending (mod $\leq$) net $(x_\lambda)_{\lambda<\alpha}$ in X for which $(F(x_\lambda))_{\lambda<\alpha}$ is bounded below (mod K) has necessarily an upper bound (mod $\leq$).*

Proof. By the above lemma, $(x_\lambda)_{\lambda<\alpha}$ is $\mathcal{U}(d,\mathcal{T})$-Cauchy; and, on the other hand, it is ascending (mod $\leq_{(F)}$). These, plus the (d,F)-completeness assumption about X, yield

$x_\lambda \to x$ (that is, $d(x_\lambda, x) \to 0$), for some $x \in X$.

Moreover, by the d–self–lsc property about F,

$x_\lambda \leq_{(F)} x$ (hence $F(x_\lambda) \geq F(x)$ (mod K)), for all $\lambda < \alpha$.

We have now, by the (K,F)–transitivity of d that, for the arbitrary fixed $\lambda < \alpha$,

$$d(x_\lambda, x) \leq d(x_\lambda, x_\mu) + d(x_\mu, x) \leq F(x_\lambda) - F(x_\mu) + d(x_\mu, x) \leq$$
$$\leq F(x_\lambda) - F(x) + d(x_\mu, x)\ (\text{mod K})\ , \lambda \leq \mu < \alpha.$$

Hence, passing to limit over μ we derive (by the ord-closedness of K)

$$d(x_\lambda, x) \leq F(x_\lambda) - F(x)\ (\text{mod K}),\ \text{for each } \lambda < \alpha;$$

that is, x appears as an upper bound (mod $\leq$) of the net in question. The proof is complete. q.e.d.

We are now prepared for stating the main result. Let X be a nonempty set. One takes a H–pseudometric d over X and a mapping $F : X \to E$. The usual hypotheses involving these elements are contained in

$$(H_1) \qquad \begin{cases} \text{d is (K,F) - transitive, F is d - self - lsc and} \\ \text{X is (d,F) - complete.} \end{cases}$$

Here, the triplet (E, K, H) is that introduced at the beginning of this section. Precisely, E is a (real) topological vector space and K is a cone in E, admitting H as a subcone. The following hypotheses are in use (about this couple)
(H_2) K is an ord-closed cone and H is K - regular.
We also denote by $\mathcal{B}$ a certain neighborhoods basis of $o \in E$, satisfying $(P_1) - (P_3)$. Remember that, by H(B) we denoted the intersection $H \cap B^c$, for all $B \in \mathcal{B}$. The hereditary property of the family $((H(B) + K)^c; B \in \mathcal{B})$ has been precised in Section 2. We also note the homogeneous property (of the same)

$$(H(tB) + K)^c = t(H(B) + K)^c \ , \ t > 0.$$

Finally, by the specific form of the auxiliary statements above, we are forced to introduce an appropriate admissibility notion. Precisely, we say $x \in X$ is (F, K)*-admissible,* when

$$(F(x) - K) \cap F(X) \text{ is bounded below (mod K).}$$

The set of all such points will be denoted $X(F, K)$. Now, the last hypothesis to be accepted is
(H_3) $X(F, K)$ is not empty.

The main result of the present paper is

Theorem 3. *Let the hypotheses above be admitted. And, for the arbitrary fixed $r > 0$, let $\leq$ be the quasi - ordering over X introduced at the beginning of this section, with $(1/r)F$ in place of F; that is*

$x \leq y$ if and only if $d(x, y) \leq (1/r)(F(x) - F(y))$ (mod K).

Then, the following conclusions are valid:

(A_1) *For each $y \in X(F, K)$, there exists a $z \in X(F, K)$ with $y \leq z$ and*

(13) $$z \leq u \leq v \text{ implies } d(u, v) \in \cap\mathcal{B}.$$

(A_2) *For each $x \in X(F, K)$ and each $B \in \mathcal{B}$ with $H(B) \neq \emptyset$, there exists $y \in X(F, K)$ with $x \leq_{(F)} y$ and*

(13) $$F(y) - F(w) \in r(H(B) + K)^c, \quad \textit{whenever} \quad y \leq_{(F)} w.$$

Moreover, if $z \in X(F, K)$ is the point given by (A_1), we have, in addition, the localizing relation

(14) $$d(y, z) \in B.$$

Proof. (A_1) Denote, for simplicity

$$X(y, \leq) = \{u \in X; y \leq u\};$$

here, $y \in X(F, K)$ is arbitrary fixed, By the evident inclusion

$$F(X(y, \leq)) \subseteq (F(y) - K) \cap F(X),$$

we deduce that the left part in this relation is bounded from below (mod K). Moreover, the property in question will remain valid when F is replaced by $(1/r)F$. These, combined with Lemma 1, yield

(a) $$\begin{cases} \text{each ascending (mod } \leq) \text{ net } (x_\lambda)_{\lambda<\alpha} \text{ in } X(y, \leq) \\ \text{is necessarily } \mathcal{U}(d, \mathcal{T}) - \text{Cauchy.} \end{cases}$$

Now, the property of being d-self-lsc may be transfered from F to $(1/r)F$, as results by the definition of this concept. At the same time , X is (d,F)-complete if and only if it is $(d, (1/r)F)$-complete; in fact, $\leq_{(F)}$ and $\leq_{(G)}$ are identical whenever $F = rG$ (or viceversa). Adding to these the boundness property above, it will follow by Lemma 2 that

(b) $$\begin{cases} \text{each ascending (mod } \leq) \text{ net } (x_\lambda)_{\lambda<\alpha} \text{ in } X(y, \leq) \\ \text{is bounded from above (mod } \leq). \end{cases}$$

Hence, conditions of Theorem B are fulfield in $X(y, \leq)$ (endowed with the induced quasi-ordering and induced pseudo - uniformity). The conclusion in that statement (with y in place of x) is just (12) above.

(A_2) Let $B \in \mathcal{B}$ be as in the statement. By a previous remark $H(tB)$ is nonempty too; this, combined with Proposition 2 in the preceding section shows that, for each $x \in X(F, K)$, the subset $(F(x) - K) \cap F(X)$ has at least one $(H(rB)+K)^c$-almost minimum point, say $F(y)$; that is, (13) holds if we take into account the homogeneous property we already precised. Finally, by the relation

$$d(y, z) \leq (1/r)(F(y) - F(z)) \text{ (mod K)}$$

(where $z \in X(F, K)$ is given as in the statement) plus (13) (with z in place of w) one derives (by the hereditary property of the subset $(H(B) + K)^c$; cf. Section 2)

$$d(y, z) \in (H(B) + K)^c \subseteq H^c \cup B.$$

But, the codomain of d is H. Hence, (14) follows. The proof is thereby finished.

q.e.d.

Concerning this statement, some remarks are in order. Firstly, the normality of the cone K is not entering among the working hypotheses; hence it is not indispensable to the substance of the argument. However, under such a supplementary assumption, some inprovements of the conclusions above may be obtained. Precisely, let us admit that, in (H_2), ord–closedness is replaced by closedness plus normality, and

(H_4) d fulfils the requirement (9').

An immediate consequence of this fact is (cf.a previous remark)

$$\mathcal{U}(d,\mathcal{T})\text{is } (\leq) - \text{triangular.}$$

Here, $\leq$ is the quasi–ordering over X defined as in the statement above. As another sequel, we have

$$\text{lin (K)} = \cap\mathcal{B}. \tag{15}$$

Indeed, the normality of K ammounts to saying each $B \in \mathcal{B}$ is a full subset of E. On the other hand, the closedness of K may be also expressed as (cf. C r i s t e s c u [6, ch.I, Sec.2])

$$K = \cap\{K + B; B \in \mathcal{B}\}(= \cap\{K - B; B \in \mathcal{B}\}).$$

Therefore

$$\text{lin (K)} = \text{K} \cap (-\text{K}) = \cap\{(\text{K}+\text{B})\cap(-\text{K}+\text{B}); \text{B} \in \mathcal{B}\} =$$
$$= \cap\{[B]; B \in \mathcal{B}\} = \cap\mathcal{B}$$

and the assertion follows. Putting together these informations we have that, in the statement above, conclusion (A_1) may be written in the form

(A_1') *For each $y \in X(F,K)$ there exists a $z \in X(F,K)$ with $y \leq z$ and*

$$z \leq w \ \textit{implies} \ d(z,w) \in lin\ (K). \tag{12'}$$

This, in the particular case of K being pointed, says our main result is a direct extension of Nemeth's (codified as Theorem 2). Of course, an alternative assumption to this one is the the ambient linear topology $\mathcal{T}$ over E being Hausdorff separated (if we take (15) into account). A second remark about these developments is related to the eventuality that $(H_1) - (H_3)$ be fulfilled with $H = K$. More specifically, let us adjoin to the initial working hypotheses

(H_5) K is regular and non flat.

The, by the considerations in the preceding section, it is clear that conclusion (A_2) in our main result will become

(A_2') *For each $x \in X(F,K)$ and each $B \in \mathcal{B}$ there exists $y \in X(F,K)$ with $x \leq_{(F)} y$ and*

$$F(y) - F(w) \in rB, \text{ whenever } y \leq_{(F)} w. \tag{13'}$$

Note that the localizing relation (14) is trivially obtainable in this case. The important aspect to be emphasized is that, under (H_5), almost minimum points (y) for the function F may be determined, with any prescribed "accuracy" (B). And, the corresponding version of our main result is a true vector extension of the variational Ekeland's principle subsumed to Theorem 1. (It, however, cannot include the smooth version of this principle obtained by B o r w e i n and P r e i s s [2]). Finally, we must remark that, in case r=1, the scalar-multiplication properties of the ambient space are not used in the argument. In other words, he conclusions of the main result will continue to hold in the case of E being a topological abelian (additive) group and K, H semigroups in E (that is, parts of E satisfying (D_5), with R substituted by Z). The question of further enlarging these results, beyond the group setting, seems to be difficult. Some elementary facts have been discussed in T u r i n i c i [23].

4. A variational interpretation. The information contained in the main result is, technically speaking, variational in nature. It is our main aim in the following to clarify this aspect. These developments may be also deemed as a characterization - in terms of Gâteaux derivatives - of the almost minimum points of the non–constrained minimization problem (attached to the function $F : X \to E$)

$$\min\ \{F(x); x \in X\}. \tag{MP}$$

To begin with, let X be a (real) linear space. We also give a (real) topological vector space $(E,\mathcal{T})$, a cone K in E and a subcone H of K. By a H – *–seminorm* over X we shall mean any mapping $e : X \to E$ with the properties

$$e(tx) = te(x), \text{ for all } t \geq 0, x \in X \tag{8'}$$

$$e(x + y) \leq e(x) + e(y)\ (mod\ K)\ ,\ \textit{for}\ x, y \in X. \tag{10'}$$

The introduced object induces a H–semimetric over X by the formula

$$d(x,y) = e(y - x)\ ,\ x, y \in X.$$

In this perspective, it is natural to consider all notions related to d, as e-notions. For example, let $F : X \to E$ be a mapping. The quasi-ordering $\leq$ over X induced by d and F under the model of Section 3 is depicted as

$$x \leq y \text{ if and only if } e(y-x) \leq F(x) - F(y) \pmod{K}.$$

(That $\leq$ is indeed quasi-ordering is a consequence of the fact that, by (10'), d is (K,F)-transitive.) Also, the pseudo-uniformity over X associated to d is now introduced as the class $U(e,\mathcal{T})$ of all

$$U(e,A) = \{(x,y) \in X^2; e(y-x) \in A\}, A \in \mathcal{A}.$$

(Here, $\mathcal{A}$ is the family of all zero neighbourhoods (mod $\mathcal{T}$).) It is readily shown to be equivalent with

$$U(e,\mathcal{B}) = \{U(e,B); B \in \mathcal{B}\},$$

where $\mathcal{B}$ is a neighborhoods basis of $o \in E$ endowed with the properties $(P_1)-(P_3)$. Moreover, under a normality assumption about K, a stronger triangular property is to be verified for this pseudo-uniformity ; namely (cf. also (P_3'))

$$U(e,B)\cdot U(e,B) \subseteq U(e,A), \text{ where A, B are as in } (P_3).$$

This may be retained in the sequel, because the working hypothesis about the couple (K,H) is

(H_2^*) K is a pointed normal closed cone and H is K–regular.

(Note that, in view of (15), $\mathcal{T}$ is Hausdorff separated). As a consequence, the $\mathcal{U}$ $(e,\mathcal{T})$-Cauchy property of a net $(x_\lambda)_{\lambda<\alpha}$ on X may be expressed as: for each $B \in \mathcal{B}$, there must be a $\lambda = \lambda(B) < \alpha$ with

$$e(x_\mu - x_\lambda) \in B, \text{ for all } \mu < \alpha \text{ with } \lambda \leq \mu.$$

The convergence property of the above net towards a point in X has an obvious interpretation in our context. Also, for the quasiordering $\leq$ over X introduced as in Section 3, e-self-closed will mean d-self-closed, where d is the associated H-semimetric; equivalently, this will be referred to as F being e-self-lsc and described by (D_6). It is worth notting that, in view of (H_2^*) plus (10'), the e-convergence property of a net is reductible to the $\mathcal{U}(e,\mathcal{T})$-Cauchy property of the same. The ambient space X will be said to be *e-complete* when the converse implication is also true: that is, any

$\mathcal{U}(e,\mathcal{T})$-Cauchy net converges (mod e). The working hypotheses about the triplet (X,e,F) may now be phrased as

(H_1^*) F is e-self-lsc and X is e-complete.

Finally, the notion of (F,K)-admissible point has the same meaning as in Section 3, and (H_3) is again accepted in our context.

Under these preliminaries, we have, as an immediate consequence of Theorem 3, the following practical result.

Theorem 4. *Let the hypotheses $(H_1^*),(H_2^*)$ and (H_3) be admited; also, let $r > 0$ be arbitrary fixed. Then, the conclusion below are valid:*

(A_1^*) *For each $y \in X(F,K)$ there exists $z \in X(F,K)$ with the properties*

(a) $-(F(z) - F(y) + re(z-y)) \in K$ *(hence $F(y) \geq F(z)$ (mod K))*
(b) $-(F(w) - F(z) + re(w-z)) \in K^c$, *whenever $e(w-z) \neq 0$.*

(A_2^*) *For each $x \in X(F,K)$ and $B \in \mathcal{B}$ with $H(B) \neq \emptyset$, there exists $y \in X(F,K)$ with $F(x) \geq F(y)$ (mod K) and*

(c) $F(y) - F(w) \in r(H(B)+K)^c$, *whenever $F(y) \geq F(w)$ (mod K).*

Moreover, if $z \in X(F,K)$ is the point given by (A_1), *we have, in addition, the localizing relation*

(d) $e(z-y) \in B$.

It may be also emphasized that (cf. the developments in the preceding section) relation (c) in (A_2^*) is valid with B in place of $(H(B)+K)^c$ whenever H = K. Moreover, condition $H(B) \neq \emptyset$ is fulfilled in (A_2^*) provided H is non flat.

As a matter of fact, a more general version of this result may be stated. We are, however, not interested in this device, but in deriving an appropriate variational consequence of (A_1^*). To do this, we need some notational conventions. It will be assumed further that X is endowed with a certain linear topology $\mathcal{S}$. We denote by $\Delta F(x)(v)$ (the ***upper Dini derivate*** of the mapping F at the point $x \in X$, in the direction $v \in X$) tye set of all

$$\lim_n((1/t_n)(F(x+t_n v) - F(x))_{n<\omega}),$$

with $(t_n)_{n<\omega}$ in (o,∞) converging to zero. Also, call the considered mapping, F, *upper Dini derivable* when

$$\Delta F(x)(v) \text{ is not empty , for each } x \in X, v \in X.$$

In particular, $\Delta F(x)(v)$ may consist of a single point

$$F'(x)(v) = \lim_{t\to 0+}(1/t)(F(x+tv) - F(x)),$$

called the *Gâteaux derivative* of the mapping F at the point $x \in X$ in the direction $v \in X$. This happens, e.g., in the convex case, described as

$$F(sx + (1-s)y) \leq sF(x) + (1-s)F(y) \ (mod\ K),\ 0 \leq s \leq 1, x, y \in X.$$

Precisely, the following statement is true (cf. also V a l a d i e r [25]). Let us call the (image) space E, sequentially $\mathcal{T}$ - complete (mod K) when each monotone and bounded (mod K) $\mathcal{U}(\mathcal{T})$-Cauchy sequence in E converges (mod T). (Here, $\mathcal{U}(\mathcal{T})$ is the natural uniformity over E, attached to $\mathcal{T}$; cf. Section 2).

Lemma 3. *Suppose K is a closed regular normal cone and E is sequentially $\mathcal{T}$-complete (mod K). Then , F is Gâteaux derivable (at any point and in any direction) whenever it is convex.*

Proof. Denote, for a fixed couple $x, v \in X$

$$f(t) = F(x + tv)\ ,\ t \in R.$$

By the convexity assumption we admitted, f is convex too. As a consequence, the associated mapping

$$g(t) = (1/t)(f(t) - f(c))\ ,\ t \neq 0,$$

is increasing (mod K) from $R\backslash\{0\}$ to E; hence its restriction to $(0, \infty)$ is increasing and bounded from below (mod K). Let $(t_n)_{n<\omega}$ be a (strictly) descending sequence is (o, ∞) converging to zero. As $(g(t_n))_{n<\omega}$ is descending and bounded below (mod K), it is necessarily $\mathcal{U}(\mathcal{T})$-Cauchy, because K is regular. Therefore, $g(t_n) \to b$, for some $b \in E$ (by the sequential completeness of E). Moreover, as K is closed,

$$g(t_n) \geq b\ (mod\ K), \text{for all } n < \omega$$

In other words,

$$o \leq g(t) - b \leq g(t_n) - b \ (\text{mod K})\ , \text{whenever } o < t \leq t_n, n < \omega.$$

This, in conjunction with the normality of K, gives

$$\lim_{t \to o+} (g(t) - b) = 0 \ (\text{that is, } g(t) \to b \text{ as } t \to o+).$$

But, the existence of a limit for g as $t \to o+$ is equivalent with the existence of $F'(x)(v)$. The proof is completed. q.e.d.

Now, it is the moment to specify that the working hypothesis we accept in the sequel is

(H_6) F is upper Dini derivable.

The result above may therefore be viewed as a motivation of the fact that such a hypotheses is not without object. We, however, shall not use any sort of convexity for the mapping F, in the variational type result to be stated below. But. as a last ingredient, we need

Lemma 4. *For the nonzero cone K in E, we have*

$$Z - K \subseteq Z,\ \text{whenever } Z = K^c \text{ or } Z = cl\ (K^c).$$

Proof. The first part is immediate: if $a \in K^c$ and $b \in K$, then $a - b \in K$ gives $a = a - b + b \in K$, an impossible fact. For the second part, let $a \in cl\ (K^c), b \in K$, be arbitrary fixed. If a - b is not belonging to $cl(K^c)$ then

$$(a - b + B) \cap (K^c) = \emptyset, \text{for some } B \in \mathcal{B}.$$

Hence $a - b + B \subseteq K$ (for that $B \in \mathcal{B}$). But then,

$$a + B = a - b + B + b \subseteq K + b \subseteq K,$$

which tells us $a \in int(K)$. This , however, is in contradiction to $a \in cl(K^c)$. So, necessarily, $a - b \in cl(K^c)$, s claimed. q.e.d.

Under these preliminaries, the following variational consequence of Theorem 4 is to be noted (cf. N e m e t h [15]). Let K_0 stand for the set of all nonzero points in K.

Theorem 5. *Let the hypotheses (H_1^*), (H_2^*), (H_3) and (H_6) be admitted. Let also $r > 0$ be arbitrary fixed, and $z \in X(F, K)$ be the point claimed in (A_1^*) with $r/2$ in place of r. Then, the following conclusions are true*

(A_3) $-(\Delta F(z)(v) + (r/2)e(v)) \subseteq cl(K^c)$, *for each direction* $v \in X$ *with* $e(v) \in K_0$

(A_4) $-(\Delta F(z)(v) + re(v)) \subseteq cl(K^c)$, *for each direction* $v \in X$ *with* $e(v) \in int\ (K)$.

Proof. Let $b \in \Delta F(z)(v)$ be arbitrary fixed. As an easy consequence of the relations in (A_1^*) plus the homogeneous property (8') of the ambient H-seminorm,

(16) $$-(1/t_n)(F(z + t_n v) - F(z)) - (r/2)e(v) \in K^c\ ,\ \text{for all } n < \omega.$$

(Here, $(t_n)_{n<\omega}$ is the zero-convergent sequence in (o,∞) promised by the definition of b). It suffices now to take the limit (as $n \to \infty$) in this relation, to get (A$_3$). Now, assume by contradiction that

$$-b - re(v) \in K \quad \text{(hence } b + re(v) \in -K).$$

By the above lemma one gets, via (16)

$$-(1/t_n)(F(z+t_n v) - F(z)) + b + (r/2)e(v) \in K^c \quad , \quad n < \omega.$$

Since the left member of this relation tends to $(r/2)e(v) \in int\ (K)$, one arrives at a contradiction provided n is large enough. This ends the argument.

q.e.d.

The latter conclusion in this result requires (implicity)

$$int(K)\ is\ nonempty;$$

this, by (H$_2^*$) plus Proposition 5, is not valid unless $\mathcal{T}$ is normable. Concerning the former conclusion, a useful interpretation of it may be obtained in case

$x \mapsto e(x)$ is a sufficient mapping ($e(x) = 0 \Rightarrow x - 0$) and
$v \mapsto \Delta F(z)(v)$ is univalent.

To this, let $\mathcal{F}(X,E)$ indicate the class of all mappings from X to E. For $G \in \mathcal{F}(X,E)$, denote by $\Lambda_e(G)$ the set of all $s \geq 0$ with

$$-(G(x) + se(x)) \in cl(K^c)\ ,\ \text{for all } x \in X.$$

We denote that, by Lemma 4,

$$s \in \Lambda_e(G) \text{ and } t \geq s \text{ imply } t \in \Lambda_e(G).$$

So, the convention below is (technically) justifiable. We put

$$\|G\|_e = \infty, \text{when } \Lambda_e(G) \text{ is empty}$$
$$= inf\ \Lambda_e(G), \text{ in the opposite case.}$$

The extended real valued function $G \mapsto \|G\|_e$ is readily shown to be positively homogeneous; unfortunately, it is not sub-additive unless K^c is a cone in E or, equivalently (cf. C r i s t e s c u [6, ch.V, Sec.1]), unless the image space E is one-dimensional (hence isomorphic with R). In other words, $G \mapsto \|G\|_e$ does not in general act as an extended seminorm over

$\mathcal{F}(X,E)$. Nevertheless, it becomes an interesting mathematical object once it is noted that conclusion (A_3) may be written in the form

(A$_3'$) $\|\Delta F(z)(v)\|_e \leq (r/2)$.

For, in the particular case we just evoked, it is nothing but Ekeland's necessary condition of almost extremality [7].

Summing up, the variational interpretations of the main result are not very productive beyond the normed/real setting. This motivates, in part, the limited interest in these vector extensions of Theorem 1 obtained along the preceding section. The difficulty in handling conclusions like (A_3) or (A_4) in the general case (modulo ($E,\mathcal{T}$)) must be added to the fact that, a genuine subdifferential theory for vector valued functions is not obtainable beyond the normed lattice framework; we refer to P a p a g e ò r g i o u [18] for details. Nevertheless, it is our firm conviction that the situation will change in the future.

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INEQUALITIES RELATED TO HARDY AND COPSON

BY

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1.Introduction. In 1927, C o p s o n [1], proved the following generalization of the H a r d y's inequality given in [3,4].

Theorem A. *Let $p > 1, \lambda_n > 0, a_n \geq 0, \sum_{n=1}^{\infty} \lambda_n a_n^p$ converge, and further let $\Lambda_n = \sum_{i=1}^{n} \lambda_i, A_n = \sum_{i=1}^{n} \lambda_i a_i$. Then*

$$\sum_{n=1}^{\infty} \lambda_n (\frac{A_n}{\Lambda_n})^p \leq (\frac{p}{p-1})^p \sum_{n=1}^{\infty} \lambda_n a_n^p.$$

Nearly fifty years after, in [2], Copson himself give the following integral analogue of the above inequality.

Theorem B. *Let $w(x) > 0, f(x) \geq 0$ for $x \geq 0$. Let $W(x) = \int_0^x w(t)dt$, $F(x) = \int_0^x w(t)f(t)dt$. If $p > 1, 0 < b \leq \infty$ and $\int_0^b w(x)f^p(x)ds < \infty$, then*

$$\int_0^b w(x)(\frac{F(x)}{W(x)})^p dx \leq (\frac{p}{p-1})^p \int_0^b w(x)f^p(x)dx.$$

The inequality given in Theorem B is a further generalization of the well known Hardy's integral inequality given in [3,4]. There exists a vast literature on various generalizations, extensions and variants of the above inequalities, see [1-12] and the references given therein. The aim of this paper is to obtain some new generalizations and variants of the inequalities given in Theorem A and B which provide new estimates on inequalities of this type.

2.Statement of results. Our main results are given in the following theorems.