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INEQUALITIES RELATED TO HARDY AND COPSON

BY

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1.Introduction. In 1927, C o p s o n [1], proved the following generalization of the H a r d y's inequality given in [3,4].

Theorem A. *Let $p > 1, \lambda_n > 0, a_n \geq 0, \sum_{n=1}^{\infty} \lambda_n a_n^p$ converge, and further let $\Lambda_n = \sum_{i=1}^{n} \lambda_i, A_n = \sum_{i=1}^{n} \lambda_i a_i$. Then*

$$\sum_{n=1}^{\infty} \lambda_n (\frac{A_n}{\Lambda_n})^p \leq (\frac{p}{p-1})^p \sum_{n=1}^{\infty} \lambda_n a_n^p.$$

Nearly fifty years after, in [2], Copson himself give the following integral analogue of the above inequality.

Theorem B. *Let $w(x) > 0, f(x) \geq 0$ for $x \geq 0$. Let $W(x) = \int_0^x w(t)dt$, $F(x) = \int_0^x \omega(t)f(t)dt$. If $p > 1, 0 < b \leq \infty$ and $\int_0^b w(x)f^p(x)ds < \infty$, then*

$$\int_0^b w(x)(\frac{F(x)}{W(x)})^p dx \leq (\frac{p}{p-1})^p \int_0^b w(x)f^p(x)dx.$$

The inequality given in Theorem B is a further generalization of the well known Hardy's integral inequality given in [3,4]. There exists a vast literature on various generalizations, extensions and variants of the above inequalities, see [1-12] and the references given therein. The aim of this paper is to obtain some new generalizations and variants of the inequalities given in Theorem A and B which provide new estimates on inequalities of this type.

2.Statement of results. Our main results are given in the following theorems.

Theorem 1. *Let $H(u)$ be a real - valued nonnegative convex function defined for $u \geq 0$. Let $\lambda_n > 0, a_n \geq 0$ and*

$$\Lambda_n = \sum_{i=1}^{n} \lambda_i \ , \ A_n = \sum_{i=1}^{n} \lambda_i a_i \ , \ Q_n = \sum_{i=1}^{n} \lambda_i H(a_i).$$

(i) If $p \geq 0, q > 1$ be real constants and $\sum_{n=1}^{\infty} \lambda_n [\frac{Q_n}{\Lambda_n}]^p [H(a_n)]^q < \infty$, then

$$(1) \qquad \sum_{i=1}^{\infty} \lambda_n [H(\frac{A_n}{\Lambda_n})]^{p+q} \leq M^q \sum_{n=1}^{\infty} \lambda_n [\frac{Q_n}{\Lambda_n}]^p [H(a_n)]^q,$$

where $M = (p+q)/(p+q-1)$.
(ii) If p, q, r be nonnegative real constants such that $q + r > 1$ and $\sum_{n=1}^{\infty} \lambda_n [\frac{Q_n}{\Lambda_n}]^p [H(a_n)]^{q+r} < \infty$, then

$$(2) \qquad \sum_{i=1}^{\infty} \lambda_n [H(\frac{A_n}{\Lambda_n})]^{p+q} [H(a_n)]^r \leq B^q \sum_{n=1}^{\infty} \lambda_n [\frac{Q_n}{\Lambda_n}]^p [H(a_n)]^{q+r},$$

where $B = (p+q+r)/(p+q+r-1)$.

Remark 1. In the special case when p=0, inequalities (1), (2) reduces respectively to the inequalities

$$(3) \qquad \sum_{i=1}^{\infty} \lambda_n [H(\frac{A_n}{\Lambda_n})]^q \leq (\frac{q}{q-1})^q \sum_{n=1}^{\infty} \lambda_n [H(a_n)]^q,$$

$$(4) \qquad \sum_{i=1}^{\infty} \lambda_n [H(\frac{A_n}{\Lambda_n})]^q [H(a_n)]^r \leq (\frac{q+r}{q+r-1})^q \sum_{n=1}^{\infty} \lambda_n [H(a_n)]^{q+r},$$

The inequality (3) is recently established by the present author in [9]. If we take $H(u) = u$ in (3), then it reduces to the Copson's inequality given in Theorem A, which in itself contains as a special case the series inequality given by H a r d y in [3,4] involving best constant. We believe that the bounds obtained on the right sides of the inequalities in (1) and (2) are new and also of independent interest.

Theorem 2. *Let $H(u)$ be a real - valued nonnegative convex function defined for $u \geq 0$,. Let $w(x) > 0, f(x) \geq 0$ be real - valued and integrable functions and*

$$W(x) = \int_0^x w(t)dt, \ F(x) = \int_0^x w(t)f(t)dt, \ Q(x) = \int_0^x w(t)H(f(t))dt.$$

(i) If $p \geq 0, q > 1$ be real constants and $\int_0^\infty w(x)[\frac{Q(x)}{W(x)}]^p [H(f(x))]^q dx < \infty$, then

$$(5) \qquad \int_0^\infty w(x)[H(\frac{F(x)}{W(x)})]^{p+q} dx \leq M^q \int_0^\infty w(x)[\frac{Q(x)}{W(x)}]^p [H(f(x))]^q dx,$$

where $M = (p+q)/(p+q-1)$.
(ii) If p, q, r be nonnegative real constants such that $q + r > 1$ and $\int_0^\infty w(x)[\frac{Q(x)}{W(x)}]^p [H(f(x))]^{q+r} dx < \infty$, then

$$(6) \qquad \begin{aligned} &\int_0^\infty w(x)[H(\frac{F(x)}{W(x)})]^{p+q} [H(f(x))]^r dx \leq \\ &\leq B^q \int_0^\infty w(x)[\frac{Q(x)}{W(x)}]^p [H(f(x))]^{q+r} dx, \end{aligned}$$

where $B = (p+q+r)/(p+q+r-1)$.

Remark 2. If we take $p = 0$, then the inequalities (5), (6) reduces respectively to the following inequalities

$$(7) \qquad \int_0^\infty w(x)[H(\frac{F(x)}{W(x)})]^q dx \leq (\frac{q}{q-1})^q \int_0^\infty w(x)[H(f(x))]^q dx,$$

$$(8) \qquad \begin{aligned} &\int_0^\infty w(x)[H(\frac{F(x)}{W(x)})]^q [H(f(x))]^r dx \leq \\ &\leq (\frac{q+r}{q+r-1})^q \int_0^\infty w(x)[H(f(x))]^{q+r} dx. \end{aligned}$$

Here we note that the inequality (7) is a further generalization of the inequality given by Copson in Theorem B and we believe that the inequality (6) or its special version given in (8) is new to the literature.

3.Proof of Theorem 1. Since H is a convex function, by Jensen's inequality (see,[5,p.133]) we have

$$(9) \qquad H(\frac{A_n}{\Lambda_n}) \leq \frac{Q_n}{\Lambda_n}.$$

We write $\alpha_n = Q_n \Lambda_n^{-1}$ and agree that any number with suffix 0 is 0. Now by using the elementary inequality (see, [4,7]) $x^{m+1} + my^{m+1} \geq$

$(m+1)xy^m$, $x,y \geq 0$ reals and $m > 0$ we observe that

$$\begin{aligned}
&\lambda_n \alpha_n^{p+q} - M\alpha_n^{p+q-1}\lambda_n H(a_n) = \\
&= \lambda_n \alpha_n^{p+q} - M\alpha_n^{p+q-1}[\alpha_n \Lambda_n - \alpha_{n-1}\Lambda_{n-1}] = \\
&= (\lambda_n - M\Lambda_n)\alpha_n^{p+q} + M\Lambda_{n-1}\alpha_{n-1}\alpha_n^{p+q-1} \leq \\
&\leq (\lambda_n - M\Lambda_n)\alpha_n^{p+q} + \frac{\Lambda_{n-1}}{p+q-1}[\alpha_{n-1}^{p+q} + (p+q-1)\alpha_n^{p+q}] = \\
&= \frac{1}{p+q-1}[\Lambda_{n-1}\alpha_{n-1}^{p+q} - \Lambda_n \alpha_n^{p+q}].
\end{aligned} \tag{10}$$

By substituting $n = 1, 2, ..., N$ in (10) and adding the inequalities we see that

$$\sum_{n=1}^{N}[\lambda_n \alpha_n^{p+q} - M\lambda_n H(a_n)\alpha_n^{p+q-1}] \leq -\frac{1}{p+q-1}\Lambda_N \alpha_n^{p+q} \leq 0. \tag{11}$$

From (11) and using Hölder's inequality with indices $q, q/(q-1)$, we observe that

$$\begin{aligned}
&\sum_{n=1}^{N}\lambda_n \alpha_n^{p+q} \leq M\sum_{n=1}^{N}[\lambda_n^{1/q}\alpha_n^{p/q}H(a_n)][\lambda_n^{(q-1)/q}\alpha_n^{p+q-1-p/q} \leq \\
&\leq M[\sum_{n=1}^{N}\lambda_n \alpha_n^{p}[H(a_n)]^q]^{1/q}\cdot[\sum_{n=1}^{N}\lambda_n \alpha_n^{p+q}]^{(q-1)/q}.
\end{aligned} \tag{12}$$

Dividing the inequality (12) by the last factor on the right of (12) and then raising the result to the q th power, we have

$$\sum_{n=1}^{N}\lambda_n \alpha_n^{p+q} \leq M^q \sum_{n=1}^{N}\lambda_n \alpha_n^{p}[H(a_n)]^q. \tag{13}$$

Now from (9) and (13), we have

$$\sum_{n=1}^{N}\lambda_n[H(\frac{A_n}{\Lambda_n})]^{p+q} \leq M^q \sum_{n=1}^{q}\lambda_n[\frac{Q_n}{\Lambda_n}]^p[H(a_n)]^q. \tag{14}$$

By letting N tend to infinity in (14), we obtain the desired inequality in (1).

By rewriting the left side of (2) and using the Hölder's inequality with indices $(q+r)/r$ and $(q+r)/q$, the inequalities (9) and (1), we observe that

$$\begin{aligned}
&\sum_{n=1}^{N}\lambda_n[H(\frac{A_n}{\Lambda_n})]^{p+q}[H(a_n)]^r = \\
&= \sum_{n=1}^{N}[\lambda_n^{r/(q+r)}[H(\frac{A_n}{\Lambda_n})]^{pr/(q+r)}[H(a_n)]^r]\cdot[\lambda_n^{q/(q+r)}[H(\frac{A_n}{\Lambda_n})]^{p+q-\frac{pr}{q+r}}] \leq \\
&\leq [\sum_{n=1}^{\infty}\lambda_n[H(\frac{A_n}{\Lambda_n})]^p[H(a_n)]^{q+r}]^{r/(q+r)}\cdot[\sum_{n=1}^{\infty}\lambda_n[H(\frac{A_n}{\Lambda_n})]^{p+q+r}]^{q/(q+r)} \leq \\
&\leq [\sum_{n=1}^{\infty}\lambda_n[\frac{Q_n}{\Lambda_n}]^p[H(a_n)]^{q+r}]^{r/(q+r)}\cdot[B^{q+r}\sum_{n=1}^{\infty}\lambda_n[\frac{Q_n}{\Lambda_n}]^p[H(a_n)]^{q+r}]^{q/(q+r)} \\
&= B^q\sum_{n=1}^{\infty}\lambda_n[\frac{Q_n}{\Lambda_n}]^p[H(a_n)]^{q+r}.
\end{aligned}$$

This is the required inequality in (2) and the proof is complete.

4. Proof of Theorem 2. Since H is convex, by Jensen's inequality (see[5,p.133]) we have

$$H(\frac{F(x)}{W(x)}) \leq \frac{Q(x)}{W(x)}. \tag{15}$$

Let $o < a < b < \infty$ and define

$$Q_a(x) = \int_a^x w(t)H(f(t))dt \ , \ Q_0(x) = Q(x).$$

Integrating by parts we have

$$\begin{aligned}
\int_a^b w(x)[\frac{Q_a(x)}{W(x)}]^{p+q}dx &= [\frac{W(x)^{-(p+q)+1}}{-(p+q)+1}Q_a(x)^{p+q}]_a^b + \\
&+M\int_a^b w(x)H(f(x))[\frac{Q_a(x)}{W(x)}]^{p+q-1}dx.
\end{aligned} \tag{16}$$

From (16) and using the Hölder's inequlity with indices $q, q/(q-1)$ we observe that

(17)

$$\int_a^b w(x)[\frac{Q_a(x)}{W(x)}]^{p+q}dx \le M\int_a^b w(x)H(f(x))[\frac{Q_a(x)}{W(x)}]^{p+q-1}dx =$$

$$= M\int_a^b [w(x)^{1/q}[\frac{Q_a(x)}{W(x)}]^{p/q}H(f(x))][w(x)^{(q-1)/q}[\frac{Q_a(x)}{W(x)}]^{p+q-1-\frac{p}{q}}]dx \le$$

$$\le M[\int_a^b w(x)[\frac{Q_a(x)}{W(x)}]^p[H(f(x))]^q dx]^{1/q}[\int_a^b w(x)[\frac{Q_a(x)}{W(x)}]^{p+q}dx]^{(q-1)/q}.$$

Dividing the inequality (17) by the last factor on the right of (17) and then raising the result to the q th power, we have

(18)
$$\int_a^b w(x)[\frac{Q_a(x)}{W(x)}]^{p+q}dx \le M^q\int_a^b w(x)[\frac{Q_a(x)}{W(x)}]^p[H(f(x))]^q dx.$$

Now from (18) and using the arguments as in the proof of Theorem 1 given in [8,pp.270-271], we obtain

(19)
$$\int_0^\infty w(x)[\frac{Q(x)}{W(x)}]^{p+q}dx \le M^q\int_0^\infty w(x)[\frac{Q(x)}{W(x)}]^p[H(f(x))]^q dx.$$

The desired inequality in (5) follows from (15) and (19).

The proof of the inequality (6) follows by the arguments as in the proof of inequality (2) in Theorem 1 given above with suitable modifications. We omit the details.

Remark 3. We note that the inequalities established in Theorems 1 and 2 can also be extended to the sequences and functions involving many independent variables by using the methods employed by the present author in [10,11]. However, the precise formulations and detailed discussion of such inequalities is left to another place.

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