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ON INTEGRABILITY CONDITIONS OF DIRECTION DEPENDENT f - STRUCTURES SATISFYING $f^{2k+1}+f=0, k \in N^*$

BY

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1.Introduction. In previous paper [5], we studied the notion of a direction dependent $f(s,t)-$ structure, the existence conditions of such structure and the connections which are compatible with them. In the present paper, we are going to find the integrability conditions of a direction dependent $f(s,t)$ - structure in the case $s=2k+1, t=1$, for $k \in N^*$.

2.Preliminaries. Let N be a real differentiable manifold of dimension n. Denote by TN the tangent bundle over N and by π the canonical projection of TN to N. Also, denote by $d\pi : TTN \to TN$ the differential of π and define the vertical subbundle VTN of TTN as the kernel of $d\pi$. A complementary distribution HTN to TN in TTN is called a non linear connection on TN. Of course, the fibres of vector bundles VTN and HTN are of the same dimension n. It is well known, VTN is an integrable distribution on TN.

Let (x^i, y^i) be a canonical coordinate system on TN and $\{\frac{\delta}{\delta x^i}, \frac{\delta}{\delta y^i}\}$ be a local field of frames on TN adapted to the decomposition $TTN = HTN \oplus VTN$, where

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N_i^j(x,y)\frac{\partial}{\partial y^j}$$

and $N_i^j(x,y)$ are n^2 differentiable functions locally defined on TN. The automorphism

$$P : \Gamma(TTN) \to \Gamma(TTN)$$

which is defined by

$$PX = X^i_V(x,y)\frac{\delta}{\delta x^i} + X^i_H(x,y)\frac{\partial}{\partial y^i} \quad \text{for}$$
$$X = X^i_H(x,y)\frac{\delta}{\delta x^i} + X^i_V(x,y)\frac{\partial}{\partial y^i}$$

is the natural almost product structure on TN *i.e.* $P^2 = I$, where $\Gamma(TTN)$ is the $\mathcal{F}(TN)$ - modulo of all differentiable cross sections of TTN. We keep the same notation for any other bundle. If we denote by ν and h the projection morphism of TTN to TN and HTN, respectively, we have

$$P \circ h = \nu \circ P.$$

The automorphism

$$JX = -X^i_V(x,y)\frac{\delta}{\delta x^i} + X^i_H(x,y)\frac{\partial}{\partial y^i}$$

for

$$X = X^i_H(x,y)\frac{\delta}{\delta x^i} + X^i_V(x,y)\frac{\partial}{\partial y^i}$$

is the natural almost complex structure on TN.

Definition 2.1.[5] *We call Finsler f(s,t) - structure of rank r on N, a non null Finsler tensor field f of type (1.1) and a class C^∞ such that $f^s + f^t = 0, s,t \in N, s \geq 2t, t \geq 1$ and rank $f = r$, where r is constant everywhere.*

Definition 2.2.[5] *We call horizontal $f_h(s,t)-$ structure of rank r on N a non null horizontal tensor field f_h on TN of type (1.1) satisfying $f^s_h + f^t_h = 0$, $s,t \in N$, $s \geq 2t$, $t \geq 1$ and rank $f_h = r$, where r is constant everywhere.*

The morphism

$$\ell = -f^{s-t} \ , \ m = f^{s-t} + I_{\Gamma(VTN)}$$

applied to the vertical vector bundle on TN are complementary projection morphisms which satisfy the relations

$$(2.1) \qquad f^t l = l f^t = f^t \ , \ f^t m = m f^t = 0 \ , \ f^{s-t} l = l f^{s-t} = -l.$$

According to the propositions 3.1, 3.2 of [5], if f is a Finsler $f(s,t)$-structure of rank r on N, then an horizontal $f_h(s,t)$-structure of rank r is defined on N and an $F(s,t)-$ structure of rank $2r$ defined on TN by the relations

$$(2.2) \qquad f_P X = P f P X \text{ or } \mathrm{f_j X = -JfJX} \ , \ \forall \ \mathrm{X} \in \Gamma(\mathrm{HTN})$$

$$(2.3) \qquad F_P = f_P h + f\nu \text{ or } \mathrm{F_j = f_j h + f\nu}.$$

Thus, if L and M are the complementary distribution corresponding to the projection morphism l and m respectively, we have the decompositions: [5]

$$TTN = HTN \oplus VTN = PVL \oplus PVM \oplus VL \oplus VM$$

or

$$TTN = (-JVL) \oplus (-JVM) \oplus VL \oplus VM.$$

3. Integrability conditions of a Finsler $f(2k+1,1)$ structure. Let f be a Finsler $f(2k+1,1)$ - structure on N. In this case the identities (2.1) are transformed as follows

$$fl = lf = f \ , \ fm = mf = 0 \ , \ f^{2k} l = l f^{2k} = -l,$$

where now $l = -f^{2k}$ and $m = f^{2k} + I_{\Gamma(VTN)}$.

The Nijenhuis tensor of f is defined by

$$N_f(X,Y) = [fX, fY] - f[fX, Y] - f[X, fY] + f^2[X,Y],$$

for all vertical corss sections of TN. It is easy to see that:

$$(3.1) \qquad Nf(mX, mY) = lN_f(mX, mY) = f^2[mX, mY],$$

$$(3.2) \qquad mN_f(X,Y) = mN_f(lX, lY) = m[fX, fY],$$

$$(3.3) \qquad mN_f(f^{2k-1}X, f^{2k-1}Y) = m[lX, lY],$$

$$(3.4) \qquad lN_f(lX, mY) = N_f(lX, mY), mN_f(lX, mY) = 0,$$

$$(3.5) \qquad lN_f(mX, lY) = N_f(mX, lY), mN_f(mX, lY) = 0,$$

Proposition 3.1. *The following conditions are equivalent to each other:*

$$(3.6) \qquad \begin{aligned} &i) && mN_f(X,Y) = 0, \\ &ii) && mN_f(lX, lY) = 0, \\ &iii) && mN_f(f^{2k-1}X, f^{2k-1}Y) = 0, \quad \forall \ X, Y \in \Gamma(VTN). \end{aligned}$$

Proof. From (3.2) it is easy to see that if $mN_f(X,Y) = 0$, then $mN_f(lX,lY) = 0$ and $mN_f(f^{2k-1}X, f^{2k-1}Y) = 0$. When $mN(lX,lY) = 0$, $\forall\, X, Y \in \Gamma(VTN)$, then

$$mN(lf^{2k-1}X, lf^{2k-1}Y) = 0,$$

because of $lf^{2k-1} = f^{2k-1}l = f^{2k-1}$, we have

$$mN(f^{2k-1}X, f^{2k-1}Y) = 0.$$

When $mN(f^{2k-1}X, f^{2k-1}Y) = 0$, $\forall\, X, Y \in \Gamma(VTN)$ it is clear that $mN(f^{2k-1}fX, f^{2k-1}fY) = 0$ and $mN(lX,lY) = mN_f(X,Y) = 0$.

It is well known that the distributions VL and VM are integrable if and only if $m[lX,lY] = 0$, and $l[mX,mY] = 0$ respectively, for any vector field $X, Y \in \Gamma(VTN)$. From (3.2), (3.3) and (3.6) we have:

Proposition 3.2. *A necessary and sufficient condition for the distribution VL to be integrable is that*

$$mN_f(X,Y) = 0.$$

Proposition 3.3. *A necessary and sufficient condition for the distribution VM to be integrable is that*

$$N_f(mX,mY) = 0 \quad \text{or} \quad lN_f(mX,mY) = 0.$$

Proof. Let VM be integrable, then $l[mX,mY] = 0$ and

$$f^2[mX,mY] = f^2l[mX,mY] = 0.$$

Taking account (3.1) we have

$$N_f(mX,mY) = lN_f(mX,mY) = 0.$$

Conversely, if $N_f(mX,mY) = 0$ we have from (3.1)

$$f^2[mX,mY] = 0$$

and

$$f^{2k-2}f^2[mX,mY] = -l[mX,mY] = 0.$$

Because of $l + m = I, X = lX + mX$, $\forall\, X \in \Gamma(VTM)$ and (3.4), (3.5) we have

$$N_f(X,Y) = lN_f(lX,lY) + mN_f(lX,lY) + N_f(lX,mY) + N_f(mX,lY) + \\ + N_f(mX,mY).$$

Thus we have

Proposition 3.4. *A necessary and sufficient condition for both distributions VL and VM to be integrable is that*

$$N_f(X,Y) = lN_f(lX,lY) + N_f(lX,mY) + N_f(mX,lY),$$

for any vertical vector fields X, Y.

Definition 3.1. *We say that $f(2k+1,1)$ - structure is partially integrable if the distribution VL is integrable and $N_f(lX,lY) = 0$.*

Proposition 3.5.*Suppose VL and VM are both integrable and that an adapted coordinate system has been chosen. The components of f are independent of the coordinates which are constant along the integral manifold of VL if and only if*

$$N_f(X,Y) = lN(lX,lY), \text{ or } \mathrm{N_f(lX,mY) = 0}$$

The proof is analogous to the proofs (1.6), (1.7) p.390 [6].

Definition 3.2. *We say that the Finsler $f(2k+1,1)$ structure is integrable if*

a) The $f(2k+1,1)$ -structure is partially integrable,

b) The distribution VM is integrable and

c) The components of f are independent of the coordinates which are constant along the integral manifold of VL.

Combining the proposition 3.2, 3.3, 3.4 and 3.5 we obtain the following theorem:

Theorem 3.1. *A necessary and sufficient condition for the Finsler $f(2k+1,)$ - structure to be integrable is that $N_f(X,Y) = 0$ for any vertical vector fields on TN.*

Let $CTTN$ be the complexified tangent bundle to TN. A CR-structure on TN is a complex subbundle H of $CTTN$, such that $H \cap \bar{H} = \{0\}$, where $\bar{H}$ is the complex conjugate of H and H is involutive [6].

Let f be a partially integrable Finsler $f(4k-1)$-structure on N.

From the relation (2.1) we have

$$f^{4k-2}l = -l,$$

so the tensor f^{2k-1} defines an almost complex structure on VL, which is called induced almost complex structure of f. We suppose that this almost complex structure is integrable i.e. $N_{f^{2k-1}}(X,Y) = 0$. for any $X, Y \in \Gamma(VL)$. Then we can define the complex subbundle H of $CVTN$ by

$$H = \{X - \sqrt{-1}f^{2k-1}X,\ X \in \Gamma(VL)\}.$$

Then Re $H = VL$ and $H \cap \bar{H} = \{0\}$. If $U = X - \sqrt{-1}f^{2k-1}X$ and $V = Y - \sqrt{-1}f^{2k-1}Y$ are two cross sections of H, by direct computation we obtain:

(3.7)
$$[U,V] = [X,Y] - [f^{2k-1}X, f^{2k-1}Y] - \sqrt{-1}([X, f^{2k-1}Y] + [f^{2k-1}X, Y]).$$

From $N_{f^{2k-1}}(X,Y) = 0$ we obtain:

$$[f^{2k-1}X, f^{2k-1}Y] + f^{4k-2}[X,Y] = f^{2k-1}([X, f^{2k-1}Y] + [f^{2k-1}X, y]).$$
$$[f^{2k-1}X, f^{2k-1}Y] - l[X,Y] = f^{2k-1}([X, f^{2k-1}Y] + [f^{2k-1}X, Y]).$$

Because of $f^{2k-1}l = lf^{2k-1} = f^{2k-1}$ we have

$$f^{2k-1}([f^{2k-1}X, f^{2k-1}Y] - [X,Y]) = -l([X, f^{2k-1}Y] + [f^{2k-1}X, Y]).$$

But f is partially integrable, so VL is an integrable distribution and

$$[X, f^{2k-1}Y], [f^{2k-1}X, Y] \in \Gamma(VL).$$

Thus we have

$$f^{2k-1}([f^{2k-1}X, f^{2k-1}Y] - [X,Y]) = -([X, f^{2k-1}Y] - [f^{2k-1}X, Y]).$$

Then from (3.7) we take:

$$[U,V] = [X,Y] - [f^{2k-1}X, f^{2k-1}Y] - \sqrt{-1}f^{2k-1}([X,Y] - [f^{2k-1}X, f^{2k-1}Y]).$$

that is $[U,V] \in H$, H is involutive and H defines a CR - structure on TN.

Conversely: Let H be a CR-structure on TN such that Re$H \subset VTN$ and K be the complementary distribution of Re H to TN, that is $TN = ReH \oplus K$. We define a morphism of vector bundles

$$f : \Gamma(VTN) \to \Gamma(VTN)$$

given by

$$f(Z) = 0, \quad \forall\, Z \in \Gamma(K)$$

and

(3.8)
$$f(X) = \frac{1}{2}\sqrt{-1}(U - \bar{U}),$$

where

$$U = X + \sqrt{-1}Y \in \Gamma(H).$$

From (3.8) we have

(3.9)
$$f(X) = -Y\ ,\ f^2(X) = -X\ ,\ f^3(X) = Y\ ,\ f^4(X) = X\ ,\ f^5(X) = -Y,$$
$$+ \ldots\ldots + f^{4k-2}(X) = -X$$

and

(3.10)
$$f^{4k-1} + f = 0\ ,\ k = 1, 2, 3, \ldots$$

So we see that the morphism, which is defined by (3.8) is a Finsler $f(4k-1, 1)-$ structure on N such that the distributions VL and VM, which are defined by f coincide to Re H and K respectively.

Consequently every cross section of H can be written in the form

$$U = X - \sqrt{-1}fX$$

or

$$U = X - \sqrt{-1}f^{2k-1}X, \ \forall\, k = 2\lambda + 1, \ \forall\, \lambda \in N$$

or

$$U = X + \sqrt{-1}f^{2k-1}X, \ \forall\, k = 2\lambda.$$

If V is another cross section of H it can be written in the form

$$V = Y - \sqrt{-1}fY \text{ or } \mathrm{V} = \mathrm{Y} - \sqrt{-1}\mathrm{f}^{2k-1}\mathrm{Y}, \ \forall\, \mathrm{k} = 2\lambda + 1, \ \forall\, \lambda \in \mathrm{N}$$

or

$$V = Y + \sqrt{-1}f^{2k-1}Y, \ \forall\, k = 2\lambda \text{ and } \lambda \in \mathrm{N}.$$

Then we have

$$[U,V] = [X,Y] - \sqrt{-1}\{[X, fY] + [fX, Y]\} - [fX, fY]$$

or

$$[U,V] = [X,Y] - \sqrt{-1}\{[X, f^{2k-1}Y] + [f^{2k-1}X, Y]\} - [f^{2k-1}X, f^{2k-1}Y]$$

$\forall\, k = 2\lambda + 1$ or

$$[U,V] = [X,Y] + \sqrt{-1}\{[X, f^{2k-1}Y] + [f^{2k-1}X, Y]\} - [f^{2k-1}X, f^{2k-1}Y]$$

$\forall\, k = 2\lambda$.

Taking account of H is involutive we have

$$f([X,Y] - [fX,fY]) = [X,fY] + [fX,Y], \tag{3.11}$$

$$f^{2k-1}([X,Y] - [f^{2k-1}X, f^{2k-1}Y]) = [X, f^{2k-1}Y] + [f^{2k-1}X, Y], \tag{3.12}$$

for every $k = 1,2,3,..$

From (3.11) implit

$$f^2([X,Y] - [fX,fY]) = f[X,fY] + f[fX,Y]$$

and by virtue of (3.9) we find

$$f^2([X,Y] - [fX,fY]) = [fX,fY] - [X,Y] = f[X,fY] + f[fX,Y],$$
$$[fX,fY] + f^2[X,Y] = f[X,fY] + f[fX,Y].$$

Thus,

$$N_f(X,Y) = 0. \tag{3.13}$$

Also, from (3.12) for any $X, Y \in \Gamma(VL)$ we have

$$-l([X,Y] - [f^{2k-1}X, f^{2k-1}Y]) = f^{2k-1}([X, f^{2k-1}Y] + [f^{2k-1}X, Y])$$

or

$$[f^{2k-1}X, f^{2k-1}Y] - [X,Y] = f^{2k-1}([X, f^{2k-1}Y] + [f^{2k-1}X, Y])$$

and

$$[f^{2k-1}X, f^{2k-1}Y] + f^{4k-2}[X,Y] = f^{2k-1}([X, f^{2k-1}Y] + [f^{2k-1}X, Y]).$$

Hence

$$N_{f^{2k-1}}(X,Y) = 0, \quad \forall\, X, Y \in \Gamma(VL).$$

Gathering the results, we have

Theorem 3.2. *A necessary and sufficient condition for an n-dimensional manifold N to admit a partially integrable Finsler $f(4k-1,1)$-structure with integrable induced almost complex structure is that TN has a CR-structure such that*

$$ReH \subset VTN.$$

Corollary 3.1. *If $ReH \equiv VTN$ then we have a direction dependent integrable almost complex structure on N.*

Similarly we can prove.

Theorem 3.3. *A neccessary and sufficient condition for an n-dimensional manifold N to admit a partially integrable horizontal $f_h(4k-1,1)$-structure with integral induced almost complex structure is that TN has a CR-structure Π such that $Re\,\Pi \subset HTN$.*

Theorem 3.4. *A neccessary and sufficient condition for the tangent space TN of an n-dimensional manifold N to admit a partially integrable horizontal $F(4k-1,1)$-structure with integral induced almost complex structure is that TN has a CR-structure.*

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