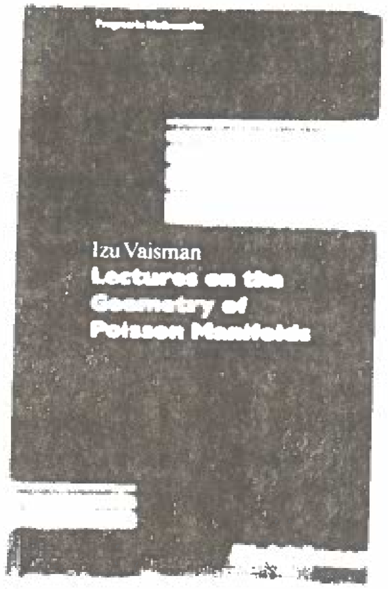

IZU VAISMAN:
LECTURES ON THE GEOMETRY OF POISSON MANIFOLDS,

Birkhäuser, Basel, 1994, VIII+207 pp.,
Progress in Mathematics,
vol. 118, ISBN 3-7643-5016-4

The main purpose of the present book is to give an exposition of the fundamental results on the differential geometry of Poisson manifolds.

A Poisson structure on a C^∞-manifold M is given by a skew-symmetric bivector field $w = (w^{ij})$ that satisfies the Jacobi identity:

$$w^{hi}\partial_h w^{jk} + w^{hj}\partial_h w^{ki} + w^{hk}\partial_h w^{ij} = 0.$$

Thus (M, w) is said to be a Poisson manifold. The Schouten-Nijenhuis bracket is expressed by means of Koszul formula and it provides an invariant meaning of the above identity. As any Poisson manifold carries a symplectic foliation, the author presents a very useful background for general distributions (planes do not have necessarily the same dimension) and the corresponding folliations. Before continuing the general theory of Poisson manifolds, the author devotes a whole chapter to examples of Poisson manifolds. This is a real support for reader to check some theoretical results on concrete Poisson structures. It follows chapters on the topics: Poisson calculus, Poisson cohomology and Poisson morphims. As an important motivation for the study of LP-cohomology of Poisson manifolds the author presents the quantisation of Poisson manifolds. Then there are exposed symplectic realizations of Poisson manifolds and realizations of Poisson manifolds by symplectic groupoids. The last chapter is devoted to Poisson - Lie groups.

This is a fascinating subject, and Vaisman's book is informative and well written. In conclusion, this book is highly recomended both for first-time readers in Poisson manifolds and for those who enjoy a certain familiarity with the subject.

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ON A LORENTZIAN PARACONTACT MANIFOLD

BY

LAKSHMI KANT PANDEY and RAM HIT OJHA

Introduction. T. Suguri and S. Nakayama [2] considered D-conformal deformations on almost contact metric structure. The purpose of this paper is to introduce a D-conformal transformation in a Lorentzian paracontact manifold (an LP-Contact manifold) [1]. The conditions for an LP-Contact manifold to be an LP-Cosymplectic manifold are also obtained.

1.Preliminaries. An n-dimensional differentiable manifold M_n on which there are defined a tensor field F of type (1.1), a vector field t, a 1-form A and a Lorentzian metric g satsifying for arbitrary vector fields $X, Y, Z, \ldots$

(1.1) $\bar{\bar{X}} - X = A(X)t, A(t) = -1, g(\bar X, \bar Y) = g(X,Y) + A(X)A(Y)$, where

(1.2) $$\bar X = FX,\ 'F(X,Y) = g(\bar X, Y), A(X) = g(X,t),$$

then M_n is called a Lorentzian paracontact manifold (an LP - Contact manifold) and the structure (F, t, A, g) is an LP - Contact structure.

In an LP - Contact manifold we have

(1.3) $$\bar t = 0, A(\bar X) = 0, \text{rank F} = (n-1).$$

In an LP - Contact manifold M_n with the structure (F, t, A, g), it is easily seen that

(1.4)
$$(D_X\, 'F)(\bar Y, Z) + (D_X\, 'F)(Y, \bar Z) = A(Y)(D_X A)(Z) + A(Z)(D_X A)(Y),$$

where D denotes the Riemannian connexion on M_n.

An LP - Contact manifold is an LP - Cosymplectic manifold if

$$D_X F = 0. \tag{1.5}$$

2. Affine connexion. Let D be the Riemannian connexion and B be an affine connexion in M_n. If we put

$$B_X Y = D_X Y + H(X,Y), \tag{2.1}$$

then the torsion tensor of the connexion B is given by

$$S(X,Y) = H(X,Y) - H(Y,X). \tag{2.2}$$

Let us define

$$'S(X,Y,Z) = g(S(X,Y),Z),\ 'H(X,Y,Z) = g(H(X,Y),Z), \tag{2.3}$$

then equation (2.2) means

$$'S(X,Y,Z) = \ 'H(X,Y,Z) - \ 'H(Y,X,Z). \tag{2.4}$$

Theorem 2.1. *Let D be the Riemannian connexion and B be an affine connexion in M_n satisfying*

(2.5) $(a)\quad (B_X\ 'F)(\bar{Y},\bar{Z}) = 0, \qquad (b)\quad 'H(X,\bar{Y},\bar{Z}) = \ 'H(X,\bar{Z},\bar{\bar{Y}}).$

Then
(2.6)
$(a)\ (B_X g)(\bar{Y},\bar{Z}) = 0, \quad (b)\ g(B_X\bar{Y},\bar{Z}) = g(D_X\bar{Y},\bar{Z}) + \frac{1}{2}(D_X\ 'F)(\bar{Y},\bar{\bar{Z}}),$

(c) An LP - Contact manifold is an LP - Cosymplectic manifold if $'H(X,\bar{Y},\bar{\bar{Z}}) = \ 'H(X,\bar{Z},\bar{\bar{Y}}) = 0.$

Proof. In consequence of (2.5) we have

$$X(\ 'F(\bar{Y},\bar{Z})) = \ 'F(B_X\bar{Y},\bar{Z}) + \ 'F(\bar{Y},B_X\bar{Z}) =$$
$$= (D_X\ 'F)(\bar{Y},\bar{Z}) + \ 'F(D_X\bar{Y},\bar{Z}) + \ 'F(\bar{Y},D_X\bar{Z}).$$

By virtue of (2.1), (2.3) and (2.5)(b), this equation assumes the form
(2.7)
$(a)\ \ (D'_X F)(\bar{Y},\bar{Z}) = 2'H(X,\bar{Y},\bar{\bar{Z}}),$ or $(b)\ \ (D'_X F)(\bar{Y},\bar{Z}) = 2'H(X,\bar{Z},\bar{\bar{Y}}),$

From (2.7), (1.4) and taking into account that $\bar{\bar{\bar{X}}} = \bar{X}$, we have

$$'H(X,\bar{Y},\bar{Z}) + \ 'H(X,\bar{Z},\bar{Y}) = 0. \tag{2.8}$$

Now

$$X(g(\bar{Y},\bar{Z})) = (B_X g)(\bar{Y},\bar{Z}) + g(B_X\bar{Y},\bar{Z}) + g(\bar{Y},B_X\bar{Z}) = \tag{2.9}$$
$$= g(D_X\bar{Y},\bar{Z}) + g(\bar{Y},D_X\bar{Z}).$$

Hence, by using (2.1) and (2.8), we obtain (2.6)(a). With the help of (2.1), (2.6)(a), (2.7)(a), (2.8) and (2.9), we obtain (2.6)(b). (2.6)(c) follows from (1.5) and (2.7).

Theorem 2.2. *Let B satisfy*

(2.10) $(a)\ \ (B_X\ 'F)(Y,Z) = 0, \qquad (b)\ \ 'H(X,Y,\bar{Z}) + \ 'H(Z,Y,\bar{X}) = 0.$

Then $'F$ is closed and an LP - Contact manifold is an LP - Cosymplectic manifold if

$$'S(Y,Z,\bar{X}) = \ 'S(Z,X,\bar{Y}). \tag{2.11}$$

Proof. In consequence of (2.10)(a), we have

$$X(\ 'F(Y,Z)) = \ 'F(B_X Y,Z) + \ 'F(Y,B_X Z) =$$
$$= (D_X\ 'F)(Y,Z) + \ 'F(D_X Y,Z) + \ 'F(Y,D_X Z).$$

From which, we obtain

(2.12)(a) $\qquad (D_X\ 'F)(Y,Z) = \ 'H(X,Y,\bar{Z}) + \ 'H(X,Z,\bar{Y}).$

Similarly, we can write

(b) $\qquad (D_Y\ 'F)(Z,X) = \ 'H(Y,Z,\bar{X}) + \ 'H(Y,X,\bar{Z})$ and

(c) $\qquad (D_Z\ 'F)(X,Y) = \ 'H(Z,X,\bar{Y}) + \ 'H(Z,Y,\bar{X}).$

By virtue of (2.10)(b) and (2.12), we have

$$(D_X\ 'F)(Y,Z) + (D_Y\ 'F)(Z,X) + (D_Z\ 'F)(X,Y) = 0.$$

Consequently $'F$ is closed.

From (2.4), (2.10)(b) and (2.12)(c), we find

(2.13) $$(D_Z\, 'F)(X,Y) = \,'S(Z,X,\bar{Y}) - \,'S(Y,Z,\bar{X}).$$

Inserting (1.5) into (2.13) , we get (2.11).

Theorem 2.3. *Let D be the Riemannian connexion and B be an affine connexion in M_n satisfying*

(2.14)

$$(a)\ (B_X\, 'F)(Y,Z) = 0, \qquad (b)\ 'H(X,Y,\bar{Z}) + \,'H(Z,Y,\bar{X}) = \,'H(Z,X,\bar{Y}).$$

Then

(2.15)

$(a)\, 'S(Z,X,\bar{Y}) + \,'S(Z,Y,\bar{X}) = 2\, 'H(Z,Y,\bar{X})$,

$(b)\, 'F$ *is closed if* $'H(X,Y,\bar{Z}) + \,'H(Y,Z,\bar{X}) + \,'H(Z,X,\bar{Y}) = 0$,

(c) *An LP – Contact manifold is an LP – Cosymplectic manifold if*

$$'H(X,Y,\bar{Z}) = \frac{1}{2}\, 'H(Y,Z,\bar{X}).$$

Proof. As in theorem (2.2), we have

(2.16) $$(D_X\, 'F)(Y,Z) = \,'H(X,Y,\bar{Z}) + \,'H(X,Z,\bar{Y}),$$

hence

(2.17)

$$(a)\ (d\, 'F)(X,Y,Z) = \,'H(X,Y,\bar{Z}) + \,'H(X,Z,\bar{Y}) + \\ + \,'H(Y,Z,\bar{X}) + \,'H(Y,X,\bar{Z}) + \,'H(Z,X,\bar{Y}) + \,'H(Z,Y,\bar{X}).$$

Substituting (2.14)(b) into (2.17)(a), this equation is equivalent to

(2.17)

$$(b)\ (d\, 'F)(X,Y,Z) = \,'H(X,Y,\bar{Z}) + \,'H(Y,Z,\bar{X}) + \,'H(Z,X,\bar{Y}).$$

From above equations, we get

$$2'H(Z,Y,\bar{X}) = \{'H(Z,X,\bar{Y}) - 'H(X,Z,\bar{Y})\} +$$

$$+\{'H(Z,Y\bar{X}) - 'H(Y,Z,\bar{X})\} - \{'H(Z,X,\bar{Y}) + 'H(Y,X,\bar{Z})\} + 'H(Y,Z,\bar{X}).$$

Use of (2.4) and (2.14)(b) in this equation, yields (2.15)(a). By using the fact that if $'F$ is closed then $d'F = 0$, we obtain (2.15)(b). Making the use of (1.5) and (2.14)(b) in (2.16), we get (2.15)(c).

Let us note one more obvious fact.

Theorem 2.4. *Let B satisfy*

(2.18) $$(a)\ (B_X\, 'F) = 0. \qquad (b)\ 'H(X,Z,\bar{Y}) + \,'H(Y,X,\bar{Z}) = \,'S(Y,X,\bar{Z}).$$

Then a LP–Contact manifold is an LP–Cosymplectic manifold.

Theorem 2.5. *Let D be the Riemannian connexion and B be an affine connexion in M_n satisfying*

(2.19) $$(a)\ (d\, 'F)(X,Y,Z) = 0, \qquad (b)\ 'H(X,Y,\bar{Z}) + \,'H(Z,Y,\bar{X}) = 0.$$

Then

(2.20)
$$(a)\ (B_X\, 'F)(Y,Z) + (B_Y\, 'F)(Z,X) + (B_Z\, 'F)(X,Y) = 0.$$
$$(b)\ 2\, 'H(Z,X,\bar{Y}) = \,'S(X,Y,\bar{Z}) - \,'S(Y,Z,\bar{X}) + \,'S(Z,X,\bar{Y}).$$

Proof. We have

$$X(\,'F(Y,Z)) = (B_X\, 'F)(Y,Z) + \,'F(B_XY,Z) + \,'F(Y,B_XZ) = \\ = (D_X\, 'F)(Y,Z) + \,'F(D_XY,Z) + \,'F(Y,D_XZ).$$

Which implies

$$(D_X\, 'F)(Y,Z) = (B_X\, 'F)(Y,Z) + \,'H(X,Y,\bar{Z}) + \,'H(X,Z,\bar{Y}).$$

Writing two other equations by cyclic permutations of X,Y,Z and adding these, we get

(2.21)

$$(d\, 'F)(X,Y,Z) = (B_X\, 'F)(Y,Z) + (B_Y\, 'F)(Z,X) + (B_Z\, 'F)(X,Y) + \\ + \,'H(X,Y,\bar{Z}) + \,'H(X,Z,\bar{Y}) + \,'H(Y,Z,\bar{X}) + \\ + \,'H(Y,X,\bar{Z}) + \,'H(Z,X,\bar{Y}) + \,'H(Z,Y,\bar{X}).$$

Using (2.19) in above equation, we obtain (2.20)(a). Finally to establish (2.20)(b), the equations (2.19)(a), (2.20)(a) and (2.21) imply

$$2'H(Z,X,\bar{Y}) = \{'H(Z,X,\bar{Y}) - 'H(X,Z,\bar{Y})\} + \{'H(X,Y,\bar{Z}) - 'H(Y,X,\bar{Z})\} -$$

$$-\{'H(Y,Z,\bar{X}) - 'H(Z,Y\bar{X})\} - 2\{'H(X,Y,\bar{Z}) + 'H(Z,Y,\bar{X})\}.$$

Combining this equation with (2.4) and (2.19)(b), we obtain (2.20)(b).

3. *D*–Conformal transformation. Let the corresponding Jacobian map J of the transformation b transforms the structure (F, t, A, g) to the structure (F, V, ν, h) such that

$$(3.1)\quad \begin{aligned} &(a)\ \ J\bar{X} = \overline{JX}, \quad (b)\ \ h(JX, JY)ob = e^{\sigma} g(\bar{X}, \bar{Y}) - e^{2\sigma} A(X)A(Y), \\ &(c)\ \ V = e^{-\sigma} Jt, \quad (d)\ \ \nu(JX)ob = e^{\sigma} A(X), \end{aligned}$$

where σ is a differentiable function on M_n. Then the transformation is said to be D -conformal transformation. If σ is constant, the transformation is known as D - homothetic.

Theorem 3.1. *The structure (F, V, ν, h) is Lorentzian paracontact.*

Proof. By virtue of (1.1) , (1.3) and (3.1)(b),(d), we have

$$h(J\bar{X}, J\bar{Y})ob = e^{\sigma} g(\bar{X}, \bar{Y}) = h(JX, JY)ob + e^{2\sigma} A(X)A(Y) =$$

$$= h(JX, JY)ob + \{\nu(JX)ob\}\{\nu(JY)ob\}.$$

or

$$(3.2)(a)\qquad h(J\bar{X}, J\bar{Y}) = h(JX, JY) + \nu(JX)\nu(JY).$$

Making the use of (1.1), (3.1)(a), (3.1)(c) and (3.1)(d), we obtain

$$(3.2)(b)\qquad \overline{\overline{JX}} = J\bar{\bar{X}} = JX + A(X)Jt = JX + \nu(JX)obV.$$

Also,

$$(3.2)(c)\qquad \bar{V} = e^{-\sigma}\overline{JT} = 0.$$

(3.2) prove the statement.

Theorem 3.2. *Let E and D be the Riemannian connexions with respect to h and g such that*

(3.3)

$$(a)\ \ E_{JX}JY = JD_XY + JH(X, Y), \qquad (b)\ \ {}'H(X, Y, Z) = g(H(X, Y), Z).$$

Then

$$(3.4)\quad \begin{aligned} 2E_{JX}JY = {} & 2JD_XY - J[2e^{\sigma}\{(X\sigma)A(Y)t + +(Y\sigma)A(X)t - \\ & - ({}^{-1}G\nabla\sigma)A(X)A(Y)\} + (e^{\sigma} - 1)\{(D_XA)(Y) + (D_YA)(X) - \\ & - 2A(H(X, Y))\}t + (e^{\sigma} - 1)\{A(X)(D_Yt) + A(Y)(D_Xt) - \\ & - A(X)({}^{-1}G\nabla A)(Y) - (A(Y)({}^{-1}G\nabla A)(X)\}]. \end{aligned}$$

Proof. In consequence of (3.1)(b), we have

$$JX(h(JY, JZ))ob = X(e^{\sigma} g(\bar{Y}, \bar{Z}) - e^{2\sigma} A(Y)A(Z)).$$

Consequently

$$(3.5)\quad \begin{aligned} & h(E_{JX}JY, JZ)ob + h(JY, E_{JX}JZ)ob = (X\sigma)e^{\sigma} g(\bar{Y}, \bar{Z}) + \\ & + e^{\sigma} g(D_X\bar{Y}, \bar{Z}) + e^{\sigma} g(\bar{Y}, D_X\bar{Z}) - 2(X\sigma)e^{2\sigma} A(Y)A(Z) - \\ & - e^{2\sigma}(D_XA)(Y)A(Z) - e^{2\sigma}(D_XA)(Z)A(Y) - \\ & - e^{2\sigma} A(D_XY)A(Z) - e^{2\sigma} A(Y)A(D_XZ). \end{aligned}$$

Also,

$$(3.6)\quad \begin{aligned} & h(E_{JX}JY, JZ)ob + h(JY, E_{JX}JZ)ob = e^{\sigma} g(\overline{D_XY}, \bar{Z}) - \\ & - e^{2\sigma} A(D_XY)A(Z) + e^{\sigma} g(\overline{H(X, Y)}, \bar{Z}) - e^{2\sigma} A(H(X, Y))A(Z) + \\ & + e^{\sigma} g(\bar{Y}, \overline{H(X, Z)}) - e^{2\sigma} A(Y)A(H(X, Z)) + \\ & + e^{\sigma} g(\bar{Y}, \overline{D_XZ}) - e^{2\sigma} A(Y)A(D_XZ). \end{aligned}$$

The equations (1.4), (3.5) and (3.6) imply

$$(3.7)\quad \begin{aligned} & (X\sigma)g(\bar{Y}, \bar{Z}) - 2(X\sigma)e^{\sigma} A(Y)A(Z) - \\ & (e^{\sigma} - 1)[(D_XA)(Y)A(Z) + (D_XA)(Z)A(Y)] = \\ & = {}'H(X, Y, Z) + {}'H(X, Z, Y) - (e^{\sigma} - 1)[A(H(X, Y))A(Z) + \\ & + A(H(X, Z))A(Y)]. \end{aligned}$$

Writting two other equations by cyclic permutation of X, Y, Z and subtracting the third equation from the sum of the first two equations and using the symmetry of $'H$ in the first two slots, we get

$$(3.8)\quad \begin{aligned} 2\,{}'H(X, Y, Z) = {} & -2e^{\sigma}[(X\sigma)A(Y)A(Z) + (Y\sigma)A(Z)A(X) - \\ & - (Z\sigma)A(X)A(Y)] - (e^{\sigma} - 1)[A(Z)\{(D_XA)(Y) + \\ & + (D_YA)(X) - 2A(H(X, Y))\} + A(X)\{(D_YA)(Z) - \\ & - (D_ZA)(Y)\} + A(Y)\{(D_XA)(Z) - (D_ZA)(X)\}], \end{aligned}$$

which gives

$$(3.9)\quad \begin{aligned} & 2H(X, Y) = \\ & = -2e^{\sigma}[(X\sigma)A(Y)t + (Y\sigma)A(X)t - ({}^{-1}G\nabla\sigma)A(X)A(Y)] - \\ & - (e^{\sigma} - 1)[\{(D_XA)(Y) + (D_YA)(X) - 2A(H(X, Y))\}t + A(X)D_Yt + \\ & + A(Y)D_Xt - A(X)({}^{-1}G\nabla A)(Y) - A(Y)({}^{-1}G\nabla A)(X)]. \end{aligned}$$

Substitution of (3.9) into (3.3)(a) gives (3.4).

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SOME NEW FIXED POINT THEOREMS FOR WEAKLY INWARD MAPPINGS

BY

WENMING ZOU

1. Introduction. Recently, two kinds of fixed point index for weakly inward and completely continuous mapping are defined under adequate assumptions (see [3,4]). Since these index theories require for their comprehension quite elaborate preparations and therefore cannot be absorbed quickly. On the other hand, fixed point index for k-set-contractive ($k > 0$) and weakly inward mapping is not available. Inspired by [5], avoid using index theory, we establish some new positive fixed point theorems for weakly inward mappings.

In section 2, we obtain further improvements and generalizations of Deimling [5] and Deimling–Hu [6]. Also, we obtain two fixed point theorems of generalized expansion and compression of conical shells with norm type for completely continuous and weakly inward mappings. In section 3, we investigate the existence of positive fixed point for semidifferentiable and weakly inward mappings. Our results extend and improve the corresponding results of Petryshyn [7] and Amann [1]. In section 4, we introduce the new concept of β-semidifferentiability of a mapping and obtain some nonzero fixed point theorems for β-semidifferentiable and weakly inward mapping. As a consequence, we get some new eigenvector theorems for β-homogeneous weakly inward mappings.

Throughout this paper, X is a real Banach space and $K \subset X$ is a cone. A given mapping F defined on a closed convex set C is said to be weakly inward if $Fx \in \overline{I_C(x)}$ for every $x \in C$, where $I_C(x) = \{x + \lambda(y - x) : \lambda \geq 0, y \in C\}$. In the case C is a cone K, this becomes

$$x \in \partial K,\ x^* \in K^* \quad \text{and} \quad x^*(x) = 0 \Longrightarrow x^*(Fx) \geq 0$$

where ∂K is the boundary of K and $K^* = \{x^* \in X^* : x^*(x) \geq 0 \text{ on } K\}$.