Substitution of (3.9) into (3.3)(a) gives (3.4).

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SOME NEW FIXED POINT THEOREMS FOR WEAKLY INWARD MAPPINGS

BY

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1. Introduction. Recently, two kinds of fixed point index for weakly inward and completely continuous mapping are defined under adequate assumptions (see [3,4]). Since these index theories require for their comprehension quite elaborate preparations and therefore cannot be absorbed quickly. On the other hand, fixed point index for k-set-contractive ($k > 0$) and weakly inward mapping is not available. Inspired by [5], avoid using index theory, we establish some new positive fixed point theorems for weakly inward mappings.

In section 2, we obtain further improvements and generalizations of Deimling [5] and Deimling-Hu [6]. Also, we obtain two fixed point theorems of generalized expansion and compression of conical shells with norm type for completely continuous and weakly inward mappings. In section 3, we investigate the existence of positive fixed point for semidifferentiable and weakly inward mappings. Our results extend and improve the corresponding results of Petryshyn [7] and Amann [1]. In section 4, we introduce the new concept of β-semidifferentiability of a mapping and obtain some nonzero fixed point theorems for β-semidifferentiable and weakly inward mapping. As a consequence, we get some new eigenvector theorems for β-homogeneous weakly inward mappings.

Throughout this paper, X is a real Banach space and $K \subset X$ is a cone. A given mapping F defined on a closed convex set C is said to be weakly inward if $Fx \in \overline{I_C(x)}$ for every $x \in C$, where $I_C(x) = \{x + \lambda(y - x) : \lambda \geq 0, y \in C\}$. In the case C is a cone K, this becomes

$$x \in \partial K,\ x^* \in K^* \quad \text{and} \quad x^*(x) = 0 \Longrightarrow x^*(Fx) \geq 0$$

where ∂K is the boundary of K and $K^* = \{x^* \in X^* : x^*(x) \geq 0 \text{ on } K\}$.

For the definitions and properties of measure of noncompactness and set-contraction the reader may consult [2].

Now, for the sake of convenience, we set

$$K_r = \{x \in K : \|x\| \le r\}, \qquad \partial K_r = \{x \in K : \|x\| = r\},$$

$$K_{\rho,r} = \{x \in K : \rho \le \|x\| \le r\}, \qquad \overset{\circ}{K} = K \setminus \{0\}.$$

2. Fixed point on annulus in cones. To prove our first result, we shall need the following lemma.

Lemma 2.1. *Let $F : K \subset X$ be k-set-contractive with $k \ge 0$; then for any $r > 0$, $\widetilde{F} : K \to X$ defined by*

$$\widetilde{F}x = \begin{cases} \frac{\|x\|}{r} F(\frac{rx}{\|x\|}) & \text{for } \|x\| \ne 0 \\ 0 & \text{for } \|x\| = 0. \end{cases}$$

is also k-set-contractive.

The proof of Lemma 2.1 is easy and therefore omitted.

Theorem 2.1. *Let $F : K_r \to X$ be α-condensing such that for $x \in \partial K$, $\|x\| \le r$, $x^* \in K^*$ and $x^*(x) = 0 \Longrightarrow x^*(Fx) \ge 0$. Suppose that either Condition (1) or Condition (2) holds:*

1-(i) *there exists a compact continuous operator $A : \partial K_r \to X$ such that for $x \in \partial K$, $\|x\| = r$, $x^* \in K^*$ and $x^*(x) = 0 \Longrightarrow x^*(Ax) \ge 0$.*
$Ax \ne \lambda x$ *on* ∂K_r *for all* $\lambda \ge 1$, *and*
$\lambda x \ne Fx + (\lambda - 1)Ax$ *on* ∂K_r *for all* $\lambda > 1$.

1-(ii) *there exists a compact continuous operator $B : \partial K_\rho \to X$ such that for $x \in \partial K$, $\|x\| = \rho$, $x^* \in K^*$ and $x^*(x) = 0 \Longrightarrow x^*(Bx) \ge 0$ (if ∂K_1 is compact, we suppose $B(\partial K_\rho) \subset K$.)*
$x - Fx \ne \lambda Bx$ *on* ∂K_ρ *for all* $\lambda > 0$, $\quad b = \inf_{x \in \partial K_\rho} \|Bx\| > 0$.

2-(i) *there exists a compact continuous operator $A : \partial K_\rho \to X$ such that for $x \in \partial K_\rho$, $\|x\| = \rho$, $x^* \in K^*$ and $x^*(x) = 0 \Longrightarrow x^*(Ax) \ge 0$.*
$Ax \ne \lambda x$ *on* ∂K_ρ *for all* $\lambda \ge 1$, *and*
$\lambda x \ne Fx + (\lambda - 1)Ax$ *on* ∂K_ρ *for all* $\lambda > 1$.

2-(ii) *there exists a compact continuous operator $B : \partial K_r \to X$ such that for $x \in \partial K$, $\|x\| = r$, $x^* \in K^*$ and $x^*(x) = 0 \Longrightarrow x^*(Bx) \ge 0$ (if ∂K_1 is compact, we suppose $B(\partial K_r) \subset K$).*
$x - Fx \ne \lambda Bx$ *on* ∂K_r *for all* $\lambda > 0$, $\quad b = \inf_{x \in \partial K_r} \|Bx\| > 0$.

Then F has a fixed point in $K_{\rho,r}$.

Proof. First case: Condition (1) is satisfied. First of all, we claim that the boundary conditions in 1-(i) and 1-(ii) cannot be violated for all

$k_m F$ as $k_m \to 1-$. In fact, since A is compact continuous and $Ax \ne \lambda x$ on ∂K_r for all $\lambda \ge 1$, we see that $\inf_{x \in \partial K_r} \|x - Ax\| = a > 0$. Now, if there exist $x_m \in \partial K_r$ and $\lambda_m > 1$ such that $\lambda_m x_m = k_m F x_m + (\lambda_m - 1)Ax_m$, then

$$\|\lambda_m(x_m - Ax_m)\| \le \|k_m F x_m - Ax_m\| \le \sup_{x \in \partial K_r} \|Fx\| + \sup_{x \in \partial K_r} \|Ax\|$$

hence $\lambda_m \le (\sup_{x \in \partial K_r} \|Fx\| + \sup_{x \in \partial K_r} \|Ax\|)/a$. So we may assume that $\lambda_m \to \lambda_0^*$ with $\lambda_0^* \ge 1$ and that $Ax_m \to y_0^*$, hence

$$\lambda_m x_m - k_m F x_m = (\lambda_m - 1)Ax_m \longrightarrow (\lambda_0^* - 1)y_0^*,$$

therefore $\alpha(\{\lambda_m x_m : m \ge 1\}) = \alpha(\{k_m F x_m : m \ge 1\}) < \alpha(\{x_m : m \ge 1\})$, (where $\alpha(\cdot)$ is the measure of noncompactness). $\lambda_m > 1$ implies $\alpha(\{x_m : m \ge 1\}) = 0$, i.e. $x_{m_j} \longrightarrow x_0^* \in \partial K_r$ for some subsequence, and

$$\lambda_0^* x_0^* = F x_0^* + (\lambda_0^* - 1)Ax_0^*$$

then, either F has a fixed point on ∂K_r or we have arrived a contradiction to the boundary condition in 1-(i). Similarly, the boundary condition in 1-(ii) cannot be violated for all $k_m F$ as $k_m \longrightarrow 1-$.

Thus, by the preceding discussion, we may assume without loss of generality that F is k-set-contractive with $k < 1$. Let's prove under this assumption.

By 1-(ii), whether ∂K_1 is compact or not, we find an $e \in \partial K_1$ such that $-\lambda e \notin \overline{B(\partial K_\rho)}$ for all $\lambda \ge 0$. Let $r_0 > 0$ and consider $F_n : K_{r+r_0} \longrightarrow X$ defined by

$$F_n x = \begin{cases} \frac{\|x\|}{r}(1 + \frac{r - \|x\|}{r_0})F(\frac{rx}{\|x\|}) + \frac{\|x\|}{r}\frac{\|x\| - r}{r_0}A(\frac{rx}{\|x\|}) & \text{for } r < \|x\| \le r + r_0 \\ Fx & \text{for } \rho \le \|x\| \le r \\ \frac{\|x\|}{\rho} Fx + \psi_n(\|x\|)B(\frac{\rho x}{\|x\|}) & \text{for } \frac{1}{n} \le \|x\| < \rho \\ \frac{x}{\rho} Fx + n\delta\|x\|B(\frac{\rho x}{\|x\|}) + \varphi_n(\|x\|)e & \text{for } \|x\| < \frac{1}{n}, \end{cases}$$

where $n > (\rho + 1)/\rho$, $\delta > (\rho + M)/b$, $M = \sup_{x \in \partial K_\rho} \|Fx\|$; $\psi_n : [1/n, \rho] \to [0, \delta]$ continuous such that $\psi_n(\rho) = 0$ and $\psi_n(t) = \delta$ for $1/n \le t \le (1 - 1/n)\rho$; $\varphi_n : [0, 1/n] \to [0, \delta_0]$ continuous such that $\varphi_n(1/n) = 0$ and $\varphi_n(t) = \delta_0$ for $0 \le t \le (1 - 1/n)/n$ with $\delta_0 > (1 + M/\rho + \delta m)/\|e\|$, $m = \sup_{x \in \partial K_\rho} \|Bx\|$. By Lemma 2.1, it is easy to see that F_n is k-set-contractive if F is. Moreover, for $x \in \partial K$, $\|x\| \le r + r_0$, $x^* \in K^*$ and $x^*(x) = 0 \Longrightarrow x^*(F_n x) \ge 0$;

$F_n x \neq \lambda x$ on ∂K_{r+r_0} for all $\lambda > 1$. Therefore, by Theorem 1 of [5], there exists $x_n \in K_{r+r_0}$ such that $F_n x_n = x_n$.

(a) if $r < \|x_n\| \leq r + r_0$, and $\|x_n\| = r + r_0$, then we have

$$x_n = \frac{\|x_n\|}{r} A(\frac{r x_n}{\|x_n\|}) \Longrightarrow A(\frac{r x_n}{\|x_n\|}) = \frac{r x_n}{\|x_n\|},$$

a contradiction to the boundary condition in 1-(i); if $r < \|x_n\| < r + r_0$ set $z_n = \dfrac{r x_n}{\|x_n\|}$ and $\lambda_n = (1 + \dfrac{r - \|x_n\|}{r_0})^{-1}$, then $z_n \in \partial K_r$, $\lambda_n > 1$ and

$$F_n x_n \Longrightarrow \lambda_n z_n = F z_n + (\lambda_n - 1) A z_n$$

this is also a contradiction to the boundary condition in 1-(i).

(b) if $\rho(1 - 1/n) \leq \|x_n\| < \rho$, then $\|x_n\| \longrightarrow \rho$ and

$$x_n = F_n x_n = \frac{\|x_n\|}{\rho} F x_n + \psi_n(\|x_n\|) B(\frac{\rho x_n}{\|x_n\|}).$$

Since F is strict-set-contractive and B is compact continuous, then $\alpha(\{x_n : n \geq 1\}) = 0$, i.e. $x_{n_j} \longrightarrow x_0$ for some subsequence and $x_0 \in \partial K_\rho$. On the other hand, we may also assume that $\psi_{n_j}(\|x_{n_j}\|) \longrightarrow \lambda_0$ and $\lambda_0 \in [0, \delta]$, hence $x_0 = F x_0 + \lambda_0 B x_0$ and therefore x_0 is a fixed point of F by 1-(ii).

(c) if $1/n \leq \|x_n\| \leq \rho(1 - 1/n)$, then $\psi_n(\|x_n\|) = \delta$ and

$$x_n = F_n x_n = \frac{\|x_n\|}{\rho} F x_n + \delta B(\frac{\rho x_n}{\|x_n\|})$$

hence $\delta b \leq \delta \|B(\dfrac{\rho x_n}{\|x_n\|})\| \leq \|x_n\| + \dfrac{\|x_n\|}{\rho} \|F x_n\| \leq \rho + M$, a contradiction to the choice of δ.

(d) if $(1 - 1/n)/n \leq \|x_n\| \leq 1/n)$, then $x_n \longrightarrow 0$ and

$$x_n = F_n x_n = \frac{\|x_n\|}{\rho} F x_n + n\delta \|x_n\| B(\frac{\rho x_n}{\|x_n\|}) + \varphi_n(\|x_n\|) e$$

Similarly, let us assume that $\varphi_{n_i}(\|x_{n_i}\|) \longrightarrow \bar{\lambda}_0$ and $B(\dfrac{\rho x_{n_i}}{\|x_{n_i}\|}) \longrightarrow y_0$ for some subsequences, then $\bar{\lambda}_0 \geq 0$ and $y_0 \in \overline{B(\partial K_\rho)}$. Notice that $n\|x_n\| \longrightarrow 1$, we must have $\delta y_0 + \bar{\lambda}_0 e = 0$, i.e. $y_0 = -\dfrac{\bar{\lambda}_0}{\delta} e \in \overline{B(\partial K_\rho)}$, a contradiction to the choice of e.

(e) if $0 \leq \|x_n\| \leq (1 - 1/n)/n$, then $\varphi_n(\|x_n\|) = \delta_0$ and $x_n = F_n x_n = \frac{\|x_n\|}{\rho} F x_n + n\delta \|x_n\| B(\frac{\rho x_n}{\|x_n\|}) + \delta_0 e$, hence

$$\delta_0 \|e\| \leq \|x_n\| + \frac{\|x_n\|}{\rho} \|F x_n\| + n\delta \|x_n\| \|B(\frac{\rho x_n}{\|x_n\|})\| \leq 1 + M/\rho + \delta m,$$

a contradiction to the choice of δ_0.

In view of the above discussion, we claim that F_n, i.e. F has a fixed point in $K_{\rho,r}$.

Second case: Condition (2) is satisfied. We now switch to the situation of first case.

Consider $\widetilde{F}_n : K \longrightarrow X$ given by $\widetilde{F}_n x = \widetilde{\psi}_0(\|x\|) \widetilde{F}_0 P_r x + \widetilde{\varphi}_n(\|x\|) e$, where $P_r : K \longrightarrow K_r$ is the radial projection, i.e. $P_r x = x$ for $\|x\| \leq r$ and $P_r x = rx/\|x\|$ for $\|x\| > r$; $e \in \mathring{K}$ and $\|e\| = 1$; $\widetilde{F}_0 x = \frac{F(\widetilde{\varphi}(\|x\|)x)}{\widetilde{\varphi}(\|x\|)}$; $\widetilde{\varphi} : (0, r] \longrightarrow [\rho/r, +\infty)$, $\widetilde{\varphi}(t) = r/t$ for $0 < t \leq \rho$, $\widetilde{\varphi}(t) = (r + \rho - t)/t$ for $\rho < t \leq r$; $\widetilde{\psi}_0 : [0, +\infty) \longrightarrow [0, 1]$, $\widetilde{\psi}_0(t) = 1$ for $t \geq \rho$, $\widetilde{\psi}_0(t) = (t - \rho_0)/(\rho - \rho_0)$ for $\rho_0 < t < \rho$, $\widetilde{\psi}_0(t) = 0$ for $t \leq \rho_0$; $\widetilde{\varphi}_n : [0, +\infty) \longrightarrow [0, \delta^*]$, δ^* large enough, $\widetilde{\varphi}_n(t) = \delta^*$ for $t \leq \rho - 1/n$, $\widetilde{\varphi}_n(t) = (\rho - t) n\delta^*$ for $\rho - 1/n \leq t \leq \rho$, $\widetilde{\varphi}_n(t) = 0$ for $t \geq \rho$. And $\widetilde{A} : \partial K_r \longrightarrow X$, $\widetilde{A} x = \frac{r}{\rho} A(\frac{\rho x}{r})$; $\widetilde{B} : \partial K_r \longrightarrow X$, $\widetilde{B} x = B(\frac{rx}{\rho})$. Then, by the same arguments as those in the Proof of Theorem 3 in [6], $\widetilde{F}_n$ is k-set-contractive if F is. Also, it is easy to check that $\widetilde{F}_n$, $\widetilde{A}$ and $\widetilde{B}$ satisfy Condition (1), hence $\widetilde{F}_n$ has a fixed point in $K_{\rho,r}$ and therefore F has a fixed point in $K_{\rho,r}$. We omit the details.

Remark 2.1. When ∂K_1 is compact (consequently, K_r is compact), "$B(\partial K_\rho) \subset K$" in 1-(ii) and "$B(\partial K_r) \subset K$" in 2-(ii) cannot be concealed. For example, let $F = -I_{|K_r}$, $A = -I_{|\partial K_\rho}$ and $B = -I_{|\partial K_\rho}$ ($B(\partial K_\rho) \not\subset K$, but B is weakly inward), then Condition (1) is satisfied, but F does not have any fixed point in $K_{\rho,r}$.

Remark 2.2. Theorem 2.1 is a generalization of Theorem 2 in [5] and Theorem 3 in [6] (see also Theorems 1.2 to 1.5 in [8]). On the other hand, when ∂K_1 is compact, we set $B \equiv I : \partial K_\rho \longrightarrow K$, then we get Theorem 1 of [6], therefore Theorem 2.1 is a further extension of Theorem 1 in [6].

Theorem 2.2. *Let $F : K_r \longrightarrow X$ be α-condensing such that for $x \in \partial K$, $\|x\| \leq r$, $x^* \in K^*$ and $x^*(x) = 0 \Longrightarrow x^*(Fx) \geq 0$. Suppose that either Condition (3) or Condition (4) holds:*

3-(i) *same as 1-(i) of Theorem 2.1.*

3-(ii) *there exists a compact continuous operator $B : \partial K_\rho \longrightarrow X$ such that for $x \in \partial K$, $\|x\| = \rho$, $x^* \in K^*$ and $x^*(x) = 0 \Longrightarrow x^*(Bx) \geq 0$ (if ∂K_1 is compact, we suppose $B(\partial K_\rho) \subset K$).*

$b = \inf_{x\in\partial K_\rho} \|Bx\| > 0$ *and* $x - Fx \neq \lambda\frac{Bx}{\|Bx\|}$ *on* ∂K_ρ *for* $\lambda \in [d, D]$, *where* $d = \inf_{x\in\partial K_\rho} \|x - Fx\|$, $D = \sup_{x\in\partial K_\rho} \|x - Fx\|$.

4-(i) *same as* 2-(i) *of Theorem* 2.1.

4-(ii) *there exists a compact continuous operator* $B : \partial K_r \longrightarrow X$ *such that for* $x \in \partial K$, $\|x\| = r$, $x^* \in K^*$ *and* $x^*(x) = 0 \Longrightarrow x^*(Bx) \geq 0$ *(if* ∂K_1 *is compact, we suppose* $B(\partial K_r \subset K)$.
$b' = \inf_{x\in\partial K_r} \|Bx\| > 0$ *and* $x - Fx \neq \lambda\frac{Bx}{\|Bx\|}$ *on* ∂K_r *for* $\lambda \in [d', D']$, *where* $d' = \inf_{x\in\partial K_r} \|x - Fx\|$, $D' = \sup_{x\in\partial K_r} \|x - Fx\|$.

Then F *has a fixed point in* $K_{\rho,r}$.

Proof. We give the proof for the first case. Without loss of generality we can assume that F does not have any fixed point on ∂K_ρ, then $d > 0$. Consider $B_1 : \partial K_\rho \longrightarrow X$, $B_1x = \frac{Bx}{\|Bx\|}$. Then it is sufficient to prove $x - Fx \neq \lambda B_1 x$ on ∂K_ρ for all $\lambda > 0$. In fact, if there exist $x_0 \in \partial K_\rho$ and $\lambda_0 \in (0, d)$ such that $x_0 - Fx_0 = \lambda_0 B_1 x_0$, we have $d \leq \|x_0 - Fx_0\| = \lambda_0\|B_1x_0\| = \lambda_0 < d$ a contradiction. If there exist $x_1 \in \partial K_\rho$ and $\lambda_1 > D$ such that $x_1 - Fx_1 = \lambda_1 B_1 x_1$, then $D \geq \|x_1 - Fx_1\| = \lambda_1 > D$ a contradiction. Hence F has a fixed point in $K_{\rho,r}$ by Theorem 2.1.

In the case F is compact continuous, we can get following theorems of generalized expansion and compression of conical shells with norm type. For the sake of convenience, we set $d' = \inf_{x\in\partial K_r} \|x - Fx\|$, $d = \inf_{x\in\partial K_\rho} \|x - Fx\|$,

$$L' > \sup_{x\in\partial K_r} \|Fx\|, \quad L > \sup_{x\in\partial K_\rho} \|Fx\|, \quad \tau \leq \begin{cases} 1 + \frac{d'}{L'-d'} & \text{for } L' > d', \\ 1 + \frac{d'}{L'} & \text{for } L' \leq d', \end{cases}$$

$\sigma \geq \frac{L}{L+d}$, $W > \max\{d, \rho\}$, $\nu \geq 1 - \frac{d}{W}$, τ' (w.r.t. ∂K_ρ), σ' (w.r.t. ∂K_r) and ν' (w.r.t. ∂K_r) are defined as τ, σ and ν. We have

Theorem 2.3. *Let* $F : K_r \longrightarrow X$ *be compact continuous such that for* $x \in \partial K$, $\|x\| \leq r$, $x^* \in K^*$ *and* $x^*(x) \geq 0 \Longrightarrow x^*(Fx) \geq 0$. *Suppose that either Condition* (5) *or Condition* (6) *holds:*

5-(i) $\|Fx\| \leq \tau\|x\|$ for $x \in \partial K_r$

5-(ii) $\|Fx\| \leq \sigma\|x\|$ for $x \in \partial K_\rho$
(if ∂K_1 *is compact, we suppose* $F(\partial K_\rho) \subset K$*)*

6-(i) $\|Fx\| \leq \tau'\|x\|$ for $x \in \partial K_\rho$

6-(ii) $\|Fx\| \geq \sigma'\|x\|$ for $x \in \partial K_r$
(if ∂K_1 *is compact, we suppose* $F(\partial K_r) \subset K$*)*

Then F *has a fixed point in* $K_{\rho,r}$.

Proof. We give the proof for the first case. Without loss of generality, we can assume that F does not have any fixed point on $\partial K_r \cup \partial K_\rho$, then $d > 0$, $d' > 0$.

Step 1. We claim that $Fx \neq \lambda x$ on ∂K_r for all $\lambda > 1$. Otherwise, we find $x_0 \in \partial K_r$ and $\lambda_0 > 1$ such that $Fx_0 = \lambda_0 x_0$, then $\|\lambda_0 x_0\| = \|Fx_0\| \leq \tau\|x_0\|$, i.e. $\lambda_0 \leq \tau$. If $L' > d'$, we have $\lambda_0 \leq \tau \leq 1 + \frac{d'}{L' - d'}$ and therefore

$$d' \leq \|x_0 - Fx_0\| = \|\frac{Fx_0}{\lambda_0} - Fx_0\| = (1 - \frac{1}{\lambda_0})\|Fx_0\| \leq \frac{d'}{L'}\|Fx_0\| < \frac{d'}{L'}L' = d'$$

a contradiction. If $L' \leq d'$, then $\lambda_0 \leq \tau \leq 1 + \frac{d'}{L'}$ and we have

$$d' \leq \|x_0 - Fx_0\| = \|\frac{Fx_0}{\lambda_0} - Fx_0\| = (1 - \frac{1}{\lambda_0})\|Fx_0\| \leq \frac{d'\|Fx_0\|}{L' + d'} < \frac{d'L'}{L' + d'}$$

this is impossible.

Step 2. Consider $B \equiv F$, then it is sufficient to show that $x - Fx \neq \lambda\frac{Bx}{\|Bx\|}$ on ∂K_ρ for $\lambda \in [d, +\infty)$. If this is not true, that is, there exist $x_1 \in \partial K_\rho$ and $\lambda_1 \in [d, +\infty)$ such that

$$x_1 = Fx_1 + \lambda_1\frac{Bx_1}{\|Bx_1\|} = Fx_1 + \lambda_1\frac{Fx_1}{\|Fx_1\|},$$

by 5-(ii) and the definition of σ, we have

$$\|x_1\| = \frac{\|Fx_1\| + \lambda_1}{\|Fx_1\|}\|Fx_1\| \geq \frac{\|Fx_1\| + \lambda_1}{\|Fx_1\|}\sigma\|x_1\| \geq \frac{\|Fx_1\| + \lambda_1}{\|Fx_1\|}\frac{L}{L + d}\|x_1\|$$

hence $\frac{L}{L+d} \leq \frac{\|Fx_1\|}{\|Fx_1\|+\lambda_1} < \frac{L}{L+\lambda_1}$ and therefore $\lambda_1 < d$, a contradiction to the assumption of $\lambda_1 \in [d, +\infty)$.

Combining step 1 and step 2, we conclude that F has a fixed point in $K_{\rho,r}$ by Theorem 2.2.

Remark 2.3. Obviously, if F satisfies $H : \|Fx\| \leq \|x\|$ for $\|x\| = r$ and $C : \|Fx\| \geq \|x\|$ for $\|x\| = \rho$, then F has a fixed point in $K_{\rho,r}$.

However, we have following example which satisfies Condition (5) of Theorem 2.3, but does not satisfy (H) and (C) just stated.

Example 2.1. Let $X = l^2$ denote the Banach space of all real numbers sequences $x = (x_1, x_2 \ldots)$ with norm $\|x\| = (\sum_{n=1}^{\infty} x_n^2)^{\frac{1}{2}}$. Cone

$K = \{x = (x_1, x_2, \ldots) \in X : x_n \geq 0, n \geq 1\}$, then ∂K_1 is not compact. Let $F : K \longrightarrow X$ be the mapping defined by

$$Fx = (\frac{1}{2} + \frac{x_1}{2}, -x_2, -x_3, 0, 0, \ldots) \quad \text{for} \quad x = (x_1, x_2, x_3 \ldots)$$

then F is compact continuous, and it is easy to see that for $x \in \partial K$, $x^* \in K^*$ and $x^*(x) = 0 \Longrightarrow x^*(Fx) \geq 0$. Now, we consider $K_{\rho,r}$ with $\rho = \dfrac{1}{2} + \dfrac{1}{n}$ $(n \geq 13)$, $r = 2$. Since

$$\|x - Fx\| = (\frac{1}{4} + \|x\|^2 + 3x_2^2 + 3x_3^2 - \frac{1}{4}(2x_1 + 3x_1^2))^{\frac{1}{2}}$$

$$\|Fx\| = ((\frac{1}{2} + \frac{x_1}{2})^2 + x_2^2 + x_3^2)^{\frac{1}{2}} \leq (\frac{1}{4} + \frac{x_1}{2} + \|x\|^2)^{\frac{1}{2}}$$

we have $d' = \inf\limits_{\|x\|=2} \|x - Fx\| = \dfrac{1}{2}$, $d = \inf\limits_{\|x\|=\frac{1}{2}+\frac{1}{n}} \|x - Fx\| = \dfrac{1}{4} - \dfrac{1}{2n}$,

$$\sup_{\|x\|=2} \|Fx\| < 12, \qquad \sup_{\|x\|=\frac{1}{2}+\frac{1}{n}} \|Fx\| \leq \frac{\sqrt{7}}{2} < \frac{n-2}{8},$$

if we choose $L' = 12$, $L = (n-2)/8$, $\tau = 1 + \frac{d'}{L'-d'} = 24/23$, $\sigma = \frac{L}{L+d} = \frac{n}{2+n}$ then we claim that Condition (5) is satisfied, in fact:

if $\|x\| = 2$, then $\|Fx\| = (\frac{1}{4} + \frac{x_1}{2} + \frac{x_1^2}{4} + x_2^2 + x_3^2)^{\frac{1}{2}} \leq (4 + \frac{1}{4} + \frac{x_1}{2} - \frac{3}{4}x_1^2)^{\frac{1}{2}}$ $\sqrt{4 + \frac{1}{3}} < \frac{48}{23} = \tau\|x\|$

if $\|x\| = \frac{1}{2} + \frac{1}{n}$, then $\|Fx\| = (\frac{1}{4} + \frac{x_1}{2} + \frac{x_1^2}{4} + x_2^2 + x_3^2)^{\frac{1}{2}} \geq \frac{1}{2} = \sigma\|x\|$.

On the other hand,

if we choose $x = (0, 2^{\frac{1}{2}}, 2^{\frac{1}{2}}, 0, 0, \ldots$, then $\|Fx\| = (4 + \frac{1}{4})^{\frac{1}{2}} \nleq \|x\| = 2$;

if we choose $x = (0, 0, 0, \frac{1}{2} + \frac{1}{n}, 0, 0, \ldots$, then $\|Fx\| = \frac{1}{2} \ngeq \|x\| = \frac{1}{2} + \frac{1}{n}$, so F does not satisfy (H) and (C), but F still has a fixed point in $K_{\rho,r}$ by Theorem 2.3.

Theorem 2.4. *Let $F : K_r \longrightarrow X$ be compact continuous such that for $x \in \partial K$, $\|x\| \leq r$, $x^* \in K^*$ and $x^*(x) = 0 \Longrightarrow x^*(Fx) \geq 0$. Suppose that either Condition (7) or Condition (8) holds:*

7-(i) $\|Fx\| \leq \tau\|x\|$ for $x \in \partial K_r$

7-(ii) $\|Fx\| \geq \nu\|x\|$ for $x \in \partial K_\rho$

(*if ∂K_1 is compact, we suppose $F(\partial K_\rho) \subset K$*)

8-(i) $\|Fx\| \leq \tau'\|x\|$ for $x \in \partial K_\rho$

8-(ii) $\|Fx\| \geq \nu'\|x\|$ for $x \in \partial K_r$

(*if ∂K_1 is compact, we suppose $F(\partial K_r) \subset K$*)

Then F has a fixed point in $K_{\rho,r}$.

Proof. We give the proof for the first case. Similarly, we may assume that d and d' are larger than zero. Same as the proof of Theorem 2.3, Condition 7-(i) implies 3-(i) of Theorem 2.2 with $A \equiv 0$. To show that 7-(ii) implies (3-(ii) of Theorem 2.2, we set $B \equiv F$, then it is sufficient to prove that $x \neq Fx + \lambda \dfrac{Bx}{\|Bx\|}$ on ∂K_ρ for all $\lambda \in [d, +\infty)$. In fact, if there exist $x_0 \in \partial K_\rho$ and $\lambda_0 \geq d$ such that $x_0 = Fx_0 + \lambda_0 \dfrac{Bx_0}{\|Bx_0\|} = Fx_0 + \lambda_0 \dfrac{Fx_0}{\|Fx_0\|}$ hence

$$\|x_0\| = \|Fx_0\| + \lambda_0 \geq \nu\|x_0\| + \lambda_0 \geq (1 - \frac{d}{W})\|x_0\| + \lambda_0 \geq (1 - \frac{d}{W})\|x_0\| + d$$

i.e. $\rho = \|x_0\| \geq W$, this is a contradiction to the choice of W, therefore F has a fixed point in $K_{\rho,r}$ by Theorem 2.2.

Remark 2.4. When ∂K_1 is compact, "$F(\partial K_\rho) \subset K$" in 5-(ii) and 7-(ii), "$F(\partial K_r) \subset K$" in 6-(ii) and 8-(ii) cannot be concealed, otherwise, $F = -I_{|K_r}$ is a counter-example.

3. Nonzero fixed point for semidifferentiable weakly inward mappings.

For the sake of convenience, we first recall some definitions (see [7]).

Definition 3.1. *A mapping $F : K \longrightarrow X$ is said to be semidifferentiable at ∞ along K if there exists a mapping F_∞ such that F_∞ is continuous, bounded and positively homogeneous (i.e. $F_\infty(\lambda x) = \lambda F_\infty(x)$ for all $\lambda \geq 0$ and $x \in K$), for all $h \in K$ $F(h) = F_\infty(h) + w_\infty(h)$ with $w_\infty(h) = o(\|h\|)$ as $\|h\| \longrightarrow \infty$.*

Definition 3.2. *A mapping $F : K \longrightarrow X$ is said to be semidifferentiable at 0 along K if there exists a mapping F_0 such that F_0 is continuous, bounded and positively homogeneous (i.e. $F_0(\lambda x) = \lambda F_0(x)$ for all $\lambda \geq 0$ and $x \in K$), for all $h \in K$, $F(h) = F(0) + F_0(h) + w(0, h)$ with $w(0, h) = o(\|h\|)$ as $\|h\| \longrightarrow \infty$.*

Remark 3.1. In Definitions 3.1 and 3.2, we give up the increasingness and linearity of operator.

Theorem 3.1. *Let $F : K \longrightarrow X$ be α-condensing, weakly inward and semidifferentiable at ∞ along K. Assume that:*

9-(i) *$F_\infty(x) \neq \lambda x$ on ∂K_1 for all $\lambda \geq 1$.*

9-(ii) *there exist $\rho > 0$ and a compact continuous operator $B : \partial K_\rho \to X$ such that for $x \in \partial K$, $\|x\| = \rho$, $x^* \in K^*$ and $x^*(x) = 0 \Longrightarrow x^*(Bx) \geq 0$ (if ∂K_1 is compact, we suppose $B(\partial K_\rho) \subset K$).*

$$b = \inf_{x \in \partial K_\rho} \|Bx\| > 0, \text{ and } x - Fx \neq \lambda Bx \text{ on } \partial K_\rho \text{ for all } \lambda > 0.$$

Then F has a fixed point in $\mathring{K}$.

Proof. Since the boundary conditions cannot be violated for all $k_m F$ as $k_m \longrightarrow 1-$, we may assume without loss of generality that F is k-set-contractive with $k \in (0,1)$ and therefore F_∞ is also k-set-contractive (see [7]). To prove this theorem, it is sufficient to show that 9-(ii) implies that there exists $R > \rho$ such that $Fx \neq \lambda x$ for all $\lambda > 1$ and $\|x\| \geq R$. Indeed, if there exist $x_n \in K$ with $\|x_n\| \longrightarrow \infty$ and $\lambda_n > 1$ such that $\lambda_n x_n = Fx_n$, then, in view of the definition of F_∞ we have

$$\lambda_n \frac{x_n}{\|x_n\|} - F_\infty\left(\frac{x_n}{\|x_n\|}\right) = \frac{w_\infty(x_n)}{\|x_n\|} \longrightarrow 0,$$

hence λ_n is bounded. Furthermore, since F_∞ is strict-set-contractive and $\lambda_n > 1$, we must have $\alpha(\{\frac{x_n}{\|x_n\|} : n \geq 1\}) = 0$. So we may suppose $\frac{x_{n_j}}{\|x_{n_j}\|} \longrightarrow \bar{x}_0 \in \partial K_1$ and $\lambda_{n_j} \longrightarrow \bar{\lambda}_0 \geq 1$, hence we have $\bar{\lambda}_0 \bar{x}_0 = F_\infty \bar{x}_0$, a contradiction to 9-(i). Then F has a fixed point in $K_{\rho,R} \subset \mathring{K}$ by Theorem 2.1.

Remark 3.2. In the case F is strict-set-contractive cone mapping (i.e. $F(K) \subset K$) and $B \equiv w \in \mathring{K}$, similar result was obtained by Petryshyn [7]. His arguments cannot be extended to weakly inward mappings.

Theorem 3.2. *Let $F : K \to X$ be α-condensing, weakly inward and semidifferentiable at $x = 0$ along $K, F(0) = 0$. Assume that:*

10-(i) *$F_0 x \neq \lambda x$ on ∂K_1 for all $\lambda \geq 1$.*

10-(ii) *same as 9-(ii) of Theorem 3.1.*

Then F has a fixed point in $\mathring{K}$.

Proof. Similar to the proof of Theorem 3.1.

Remark 3.3. Theorem 3.2 is a generalization of the corresponding results due to Petryshyn [7] and Amann [1].

In the case F is compact continuous, we have following

Theorem 3.3. *∂K_1 is not compact, let $F : K \to X$ be compact continuous, weakly inward and semidifferentiable at ∞ along K. Assume that:*

11-(i) *there exist $r > 0$ and a compact continuous operator $A : \partial K_r \to X$ such that for $x \in \partial K$, $\|x\| = r$, $x^* \in K^*$ and $x^*(x) = 0 \Longrightarrow x^*(Ax) \geq 0$. $Ax \notin \lambda x$ on ∂K_r for all $\lambda \geq 1$, and $\lambda x \neq Fx + (\lambda - 1)Ax$ on ∂K_r for all $\lambda > 1$.*

11-(ii) *$F_\infty x \neq \lambda x$ on ∂K_1 for all $\lambda \in (0,1]$ and $\inf_{x \in \partial K_1} \|F_\infty(x)\| > 0$.*

Then F has a fixed point in $\mathring{K}$.

Proof. Let $B \equiv F_\infty$, then B is compact continuous (see [7]), and by applying arguments similar to those in the proof of Theorem 4.3 of [3], it is easy to see that B is weakly inward. Now, we claim that there exists $R > r$, such that

$$x - Fx \neq \lambda Bx \quad \text{on} \quad \partial K_R \quad \text{for all} \quad \lambda > 0.$$

Otherwise, we have x_n with $\|x_n\| \to \infty$ and $\lambda_n > 0$ such that $x_n - Fx_n = \lambda_n Bx_n = \lambda_n F_\infty(x_n)$, by the definition of F_∞ we get

$$\frac{x_n}{\|x_n\|} = (\lambda_n + 1) F_\infty\left(\frac{x_n}{\|x_n\|}\right) + \frac{w_\infty(x_n)}{\|x_n\|},$$

since $\inf_{x \in \partial K_1} \|F_\infty(x)\| > 0$ and F_∞ is compact continuous, we have $(x_{n_j})/\|x_{n_j}\| \longrightarrow x_0 \in \partial K_1$ and $1/(\lambda_{n_j} + 1) \longrightarrow \lambda_0 \in (0,1]$ for some subsequence and therefore $F_\infty(x_0) = \lambda_0 x_0$, a contradiction. Hence F has a fixed point in $\mathring{K}$ by Theorem 2.1.

Similarly we have

Theorem 3.4. *∂K_1 is not compact, let $F : K \to X$ be compact continuous, weakly inward and semidifferentiable at $x = 0$ along K, $F(0) = 0$. Assume that:*smallskip

12-(i) *same as 11-(i) of Theorem 3.3.*

12-(ii) *$F_0(x) \neq \lambda x$ on ∂K_1 for all $\lambda \in (0,1]$ and $\inf_{x \in \partial K_1} \|F_0(x)\| > 0$.*

Then F has a fixed point in $\mathring{K}$.

Proof. Similar to the proof of Theorem 3.4.

Remark 3.4. It is easy to see that Theorem 3.3 and 3.4 is wrong if ∂K_1 is compact.

4. β-semidifferentiable mappings. In this section, we introduce the new concept of β-semidifferentiability of a mapping and show how it can be used to obtain nonzero fixed point theorems for weakly inward mappings.

Definition 4.1. *A mapping $F : K \longrightarrow X$ is said to be β-semidifferentiable at ∞ along K if there exists a mapping $F^\infty : K \longrightarrow X$ such that F^∞ is continuous, bounded and β-positively homogeneous with $\beta > 0$ (i.e. $F^\infty(\lambda x) = \lambda^\beta F^\infty(x)$ for all $x \in K$ and $\lambda \geq 0$), and for all $h \in K$*

$$F(h) = F^\infty(h) + w^\infty(h) \text{ with } w^\infty(h) = o(\|h\|^\beta) \text{ as } \|h\| \longrightarrow \infty.$$

We call F^∞ a β-semiderivative of F at ∞ along K.

Definition 4.2. *A mapping $F : K \longrightarrow X$ is said to be β-semidifferentiable at $x = 0$ along K if there exists a mapping $F^0 : K \longrightarrow X$ such that F^0 is continuous, bounded and β-positively homogeneous with $\beta > 0$ (i.e. $F^0(\lambda x) = \lambda^\beta F^0(x)$ for all $x \in K$ and $\lambda \geq 0$), and for all $h \in K$*

$F(h) = F(0)+F^0(h)+w^0(0,h)$ with $w^0(0,h) = o(\|h\|^\beta)$ as $\|h\| \longrightarrow 0$.

We call F^0 a β-semiderivative of F at $x = 0$ along K.

Remark 4.1. If $F : K \longrightarrow X$ is continuous bounded and there exists $\beta > 0$ such that $F(\lambda x) = \lambda^\beta F(x)$ for all $x \in K$ and $\lambda \geq 0$, then $F^\infty = F^0 = F$.

Example 4.1. Let $X = C[0,1]$ be Banach space of all continuous real function $x(t)$ on R^1 with norm $\|x\| = \|x\|_C = \sup\limits_{0 \leq t \leq 1} |x(t)|$, $K = \{x \in C[0,1] : x(t) \geq 0\}$ a cone, $f(x,t)$ is continuous on $[0,1] \times K$ (need not nonnegative). Consider $F : K \longrightarrow X$ defined by

$$Fx(t) = \int_0^t f(s, x(s))ds, \quad t \in [0,1].$$

Then:

(i) if there exists $\beta > 0$ such that $\lim\limits_{x \to +\infty} \dfrac{f(t,x)}{x^\beta} = b(t)$ uniformly with respect to $t \in [0,1]$ then

$F^\infty(x(t)) = \int_0^t b(s)(x(s))^\beta ds.$

(ii) if $f(t,0) \equiv 0$ for $t \in [0,1]$ and there exists $\beta > 0$, such that $\lim\limits_{x \to +0} \dfrac{f(t,x)}{x^\beta} = c(t)$ uniformly with respect to $t \in [0,1]$ then

$F^0(x(t)) = \int_0^t c(s)(x(s))^\beta ds.$

The proof of example 4.1 is trivial and therefore omitted.

To establish nonzero fixed point theorems, we shall need the following lemmas.

Lemma 4.1. *Let $F : K \longrightarrow X$ be weakly inward and β-semidifferentiable at ∞ along K, then $F^\infty : K \longrightarrow X$ is also weakly inward. Moreover, if F is k-set-contractive ($k \geq 0$) and $\beta > 1$, then F^∞ is compact continuous.*

Proof. For any $x \in K$, by the definition of F^∞ and notice that $\bar{I}_K(x)$ is a wedge (see Lemma 4.2 [3]), we have

$$F^\infty(x) = \lim_{\lambda \longrightarrow +\infty} \frac{F(\lambda x)}{\lambda^\beta} \in \bar{I}_K(x),$$

this proves the first of our assertions. To prove the second part, let D be any bounded subset of K.

Step 1. If $\inf\limits_{x \in D} \|x\| = d > 0$. Then for any $\varepsilon > 0$, by the definition of F^∞, there exists $R > 0$ such that $\|w^\infty(x)\| \leq \varepsilon\|x\|^\beta$ for all $x \in K$ with $\|x\| \geq R$. If we choose $\lambda > \max\{1, R/d\}$, then $\|\lambda x\| \geq \lambda d \geq R$ for all $x \in D$ and therefore $\|w^\infty(\lambda x)\| \leq \varepsilon\|\lambda x\|^\beta \leq \varepsilon\lambda^\beta M^\beta$, where $M = \sup\limits_{x \in D} \|x\|$, hence

$$\alpha(w^\infty(\lambda D)) \leq \operatorname{diam}(w^\infty(\lambda D)) \leq 2 \sup_{x \in D} \|w^\infty(\lambda x)\| \leq 2\varepsilon\lambda^\beta M^\beta,$$

using the definition of F^∞ again, we have $\lambda^\beta \alpha(F^\infty(D)) = \alpha(F^\infty(\lambda D)) \leq \alpha(F(\lambda D)) + \alpha(w^\infty(\lambda D)) \leq \lambda k \alpha(D) + 2\varepsilon\lambda^\beta M^\beta$, therefore $\alpha(F^\infty(D)) \leq \lambda^{1-\beta} k\alpha(D) + 2\varepsilon M^\beta$. Notice that $\beta > 1$, setting $\lambda \to +\infty$, $\varepsilon \to 0$, we get $\alpha(F^\infty(D)) = 0$.

Step 2. If $\inf\limits_{x \in D} \|x\| = 0$. For any $\varepsilon > 0$, we set $D_1\{x : x \in D, \|x\| \leq \varepsilon\}$, $D_1' = D_1 \setminus \{0\}$, $D_2\{x : x \in D, \|x\| > \varepsilon\}$, then $\alpha(F^\infty(D_1)) = \alpha(F^\infty(D_1')) \leq$ $\leq \operatorname{diam}(F^\infty(D_1')) \leq 2 \sup\limits_{x \in D_1'} \|F^\infty(x)\| = 2 \sup\limits_{x \in D_1'} \|F^\infty(\frac{x\|x\|}{\|x\|})\| \leq$ $\leq 2\varepsilon^\beta \sup\limits_{z \in \partial K_1} \|F^\infty(z)\| = 2\varepsilon^\beta |F^\infty|$, where $|F^\infty| = \sup\limits_{x \in \partial K_1} \|F^\infty(x)\|$, therefore $\alpha(F^\infty(D)) = \alpha(F^\infty(D_1 \cup D_2)) \leq \max\{\alpha(F^\infty(D_1)), \alpha(F^\infty(D_2))\} \leq \max\{2\varepsilon^\beta|F^\infty|, 0\} = 2\varepsilon^\beta|F^\infty|$ since $\varepsilon > 0$ is arbitrary, this implies that $\alpha(F^\infty(D)) = 0$, this completes the proof.

Lemma 4.2. *Let $F : K \longrightarrow X$ be weakly inward and β-semidifferentiable at $x = 0$ along K, $F(0) = 0$, then $F^0 : K \longrightarrow X$ is also weakly inward. Moreover, if F is k-set-contractive ($k \geq 0$) and $\beta < 1$, then F^0 is compact continuous.*

Proof. For any $x \in K$, since $\bar{I}_K(x)$ is a wedge, we have

$$F^0 = \lim_{\lambda \longrightarrow 0+} \frac{F(\lambda x)}{\lambda^\beta} \in \bar{I}_K(x),$$

i.e. F^0 is weakly inward.

To prove the second conclusion of the lemma, we take any bounded subset D of K and any $\varepsilon > 0$, then, by the definition of F^0, it is easy to show that $\alpha(w^0(0,\lambda D)) \leq 2\varepsilon\lambda^\beta M^\beta$ for sufficiently small $\lambda > 0$, where $M = \sup\limits_{x \in D} \|x\|$. Therefore $\lambda^\beta \alpha(F^0(D)) = \alpha(F^0(\lambda D)) \leq \alpha(F(\lambda D)) + \alpha(w^0(0,\lambda D)) \leq \lambda k\alpha(D) + 2\varepsilon\lambda^\beta M^\beta$ i.e. $\alpha(F^0(D)) \leq \lambda^{1-\beta} k\alpha(D) + 2\varepsilon M^\beta$, since $\beta < 1$, $\lambda \longrightarrow 0$ and ε is arbitrary, we conclude that $\alpha(F^0(D)) = 0$ and this completes the proof.

Theorem 4.1. *∂K_1 is not compact, let $F : K \longrightarrow X$ be weakly inward, α-condensing and β-semidifferentiable at ∞ along K, $\beta > 1$. Assume that:*

13-(i) *there exist $\rho > 0$ and a compact continuous operator $A : \partial K_\rho \to X$ such that for $x \in \partial K$, $\|x\| = \rho$, $x^* \in K^*$ and $x^*(x) = 0 \Longrightarrow x^*(Ax) \geq 0$. $Ax \neq \lambda x$ on ∂K_ρ for all $\lambda \geq 1$, and $\lambda x \neq Fx + (\lambda - 1)Ax$ on ∂K_ρ for all $\lambda > 1$.*

13-(ii) $d^\infty = \inf_{x \in \partial K_1} \|F^\infty(x)\| > 0$.

Then F has a fixed point in $\overset{\circ}{K}$.

Proof. We set $B \equiv F^\infty$, then, by Lemma 4.1, B is compact continuous and weakly inward with respect to K, i.e. $x \in \partial K$, $x^* \in K^*$ and $x^*(x) = 0 \Longrightarrow x^*(Bx) \geq 0$. We claim that there exists $r > \rho$ such that $x - Fx \neq \lambda Bx$ for $\|x\| \geq r$ and all $\lambda > 0$. If not, then we find $x_n \in K$ with $\|x_n\| \to \infty$ and $\lambda_n > 0$ such that $x_n - Fx_n = \lambda_n Bx_n = \lambda_n F^\infty(x_n)$. By the definition of F^∞, we have

$$x_n = (1 + \lambda_n)F^\infty(x_n) + w^\infty(x_n),$$

hence

$$\frac{x_n}{\|x_n\|^\beta} = (1 + \lambda_n)F^\infty\left(\frac{x_n}{\|x_n\|^\beta}\right) + \frac{w^\infty(x_n)}{\|x_n\|^\beta}.$$

Notice that $\beta > 1$ and $\|x_n\| \to \infty$, we must have $F^\infty(\frac{x_n}{\|x_n\|^\beta}) \longrightarrow 0$, this contradicts $b^\infty > 0$ in 13-(ii). On the other hand, 13-(ii) implies that $\inf_{x \in \partial K_r} \|F^\infty(x)\| = r^\beta b^\infty > 0$. Thus, Condition (2) of Theorem 2.1 is satisfied, hence F has a fixed point in $K_{\rho,r} \subset \overset{\circ}{K}$.

Remark 4.2. The following example shows that Theorem 4.1 is wrong if ∂K_1 is compact. Consider $F : K \to X$ defined by $Fx = -\|x\|^{\beta-1}x$ ($\beta > 1$). Then F is compact continuous if ∂K_1 is compact, and weakly inward with respect to K. Moreover, $F = F^\infty$ by Remark 4.1. Now, if we set $A = F$, then the all conditions of Theorem 4.1 are satisfied. But F does not have any fixed point in $\overset{\circ}{K}$.

By applying Theorem 2.1 and Lemma 2.4, we easily obtain the following nonzero fixed point theorem, its proof is similar to those of Theorem 4.1, so we state it without the proof.

Theorem 4.2. *∂K_1 is not compact, let $F : K \longrightarrow X$ be weakly inward, α-condensing and β-semidifferentiable at $x = 0$ along K, $\beta \in (0, 1)$, $F(0) = 0$. Assume that:*

14-(i) *same as 13-(i) of Theorem 4.1.*

14-(ii) $d^0 = \inf_{x \in \partial K_1} \|F^0(x)\| > 0$.

Then F has a fixed point in $\overset{\circ}{K}$.

As further consequences of the preceding theorems and lemmas we have the following new eigenvector/eigenvalue results.

Theorem 4.3. *∂K_1 is not compact, let $F : K \to X$ be weakly inward, α-condensing and β-semidifferentiable at ∞ along K with $\beta > 1$. If*

$$d^\infty = \inf_{x \in \partial K_1} \|F^\infty(x)\| > 0,$$

then there exists $R > 0$ such that for any $r > R$, there exist $x_r \in \partial K_r$ and $\lambda_r \geq 1$ such that $F^\infty(x_r) = \lambda_r x_r$.

Proof. Putting $A \equiv B \equiv F^\infty : K \longrightarrow X$, it follows from Lemma 4.1 that A and B are compact continuous and weakly inward with respect to K. By applying arguments similar to those in the Proof of Theorem 4.1, we find $R_0 > 0$ such that

(*) $\qquad \lambda x \neq Fx + (\lambda - 1)Ax \quad$ for $\|x\| \geq R_0$ and $\lambda > 1$

(**) $\qquad x - Fx \neq \lambda Bx \quad$ for $\|x\| \geq R_0$ and $\lambda > 0$.

Now, if the conclusion of Theorem 4.3 is not true, that is, there exist $r_n > R_0$ ($r_n \to \infty$) such that $F^\infty(x) = Ax \neq \lambda x$ on ∂K_{r_n} for all $\lambda \geq 1$. Combining (*), (**) and Theorem 2.1, we have $x_n \in K$ with $\|x_n\| \longrightarrow \infty$ such that $Fx_n = x_n$, hence

$$\frac{x_n}{\|x_n\|^\beta} = \frac{Fx_n}{\|x_n\|^\beta} = F^\infty\left(\frac{x_n}{\|x_n\|}\right) + \frac{w^\infty(x_n)}{\|x_n\|^\beta},$$

therefore, $F^\infty(\frac{x_n}{\|x_n\|}) \to 0$ as $\|x_n\| \to \infty$. This contradicts the fact $d^\infty > 0$.

Theorem 4.4. *∂K_1 is not compact, let $F : K \longrightarrow X$ be weakly inward, α-condensing and β-semidifferentiable at $x = 0$ along K, $\beta \in (0, 1)$, $F(0) = 0$. If*

$$d^0 = \inf_{x \in \partial K_1} \|F^0(x)\| > 0,$$

then there exists ρ^* such that for any $\rho < \rho^*$, there exist $x_\rho \in \partial K_\rho$ and $\lambda_\rho \geq 1$ such that $F^0(x_\rho) = \lambda_\rho x_\rho$.

Proof. Similar to the proof of Theorem 4.3 by means of Lemma 4.2 and Theorem 2.1.

As an immediate consequence of Theorems 4.3 and 4.4 we can obtain the following eigenvector theorem for β-homogeneous weakly inward mapping.

Theorem 4.5. *∂K_1 is not compact, let $F : K \longrightarrow X$ be weakly inward, k-set-contractive ($k \geq 0$) and β-homogeneous with $\beta > 0$, $\beta \neq 1$ (i.e. $F(\lambda x) = \lambda^\beta Fx$ for all $\lambda \geq 0$ and $x \in K$). If $d = \inf_{x \in \partial K_1} \|F(x)\| > 0$, then*

(i) *if $\beta > 1$, then there exists $R > 0$ such that for any $r > R$, there exist $x_r \in \partial K_r$ and $\lambda_r \geq 1$ such that $Fx_r = \lambda_r x_r$ and $\lim_{r \to +\infty} \lambda_r = \infty$.*

(ii) *if $0 < \beta < 1$, then there exists $\rho^* > 0$ such that for any $\rho < \rho^*$, there exist $x_\rho \in \partial K_\rho$ and $\lambda_\rho \geq 1$ such that $Fx_\rho = \lambda_\rho x_\rho$ and $\lim_{\rho \to 0+} \lambda_\rho = \infty$.*

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ON THE APPROXIMATION NUMBERS OF THE TENSOR PRODUCT OPERATOR

BY

NICOLAE TIŢA

Introduction. Let X be a real Banach space and let $T \in \mathcal{L}(X)$ be a linear and bounded operator $T : X \to X$. The n-th approximation number of T is

$$a_n(T) = \inf\{\|T - A\| : A \in \mathcal{L}(X), \text{rank } A < n\}$$

For the algebraic tensor product $X \otimes X$ we consider a cross–norm τ, [1], [5], ($\tau(x \otimes \bar{x}) = \|x\| \cdot \|\bar{x}\|$) and we denote by $X\hat{\otimes}_\tau X$ the completion of the topological tensor product $X \otimes_\tau X$ relatively to the norm τ. Let $S, T \in \mathcal{L}(X)$. The cross–norm τ is called tensor–norm if $S\hat{\otimes}_\tau X : X\hat{\otimes}_\tau X \to X\hat{\otimes}_\tau X$ is continuous ($\|S\hat{\otimes}_\tau T\| \leq \|S\| \cdot \|T\|$).

We also consider a real sequence $\alpha = (\alpha_n)$ such that: $\alpha_n \geq \alpha_{n+1} > 0$ and $\alpha_{n^2} \leq \alpha_n \cdot c/n$ for all $n \in N$, where c is a constant.

The purpose of this paper is to prove some inequalities between the approximation numbers of the operator $S\hat{\otimes}_\tau T$ and the approximation numbers of the operators S and T.

Main result. The main result of this paper is the following.

Theorem. *For $S, T \in \mathcal{L}(X), k \in \mathbb{N}, p \in (0, \infty)$ and all sequences α as above it holds*

$$\sum_1^k \alpha_n a_n^p(S\hat{\otimes}_\tau T) \leq c(\alpha, p) \cdot \sum_1^k \alpha_n (a_n^p(S) + a_n^p(T)).$$

Proof. First we prove that $a_{n^2}(S\hat{\otimes}_\tau T) \leq M(a_n(S) + a_n(T))$, where $M < 2 \cdot \max(\|S\|, \|T\|)$. To this end we use an argument from [5].