As an immediate consequence of Theorems 4.3 and 4.4 we can obtain the following eigenvector theorem for β–homogeneous weakly inward mapping.

Theorem 4.5. *∂K_1 is not compact, let $F : K \longrightarrow X$ be weakly inward, k–set–contractive ($k \geq 0$) and β–homogeneous with $\beta > 0$, $\beta \neq 1$ (i.e. $F(\lambda x) = \lambda^\beta Fx$ for all $\lambda \geq 0$ and $x \in K$). If $d = \inf_{x \in \partial K_1} \|F(x)\| > 0$, then*

(i) *if $\beta > 1$, then there exists $R > 0$ such that for any $r > R$, there exist $x_r \in \partial K_r$ and $\lambda_r \geq 1$ such that $Fx_r = \lambda_r x_r$ and $\lim_{r \to +\infty} \lambda_r = \infty$.*

(ii) *if $0 < \beta < 1$, then there exists $\rho^* > 0$ such that for any $\rho < \rho^*$, there exist $x_\rho \in \partial K_\rho$ and $\lambda_\rho \geq 1$ such that $Fx_\rho = \lambda_\rho x_\rho$ and $\lim_{\rho \to 0+} \lambda_\rho = \infty$.*

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Received:20.VI.1994

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ANALELE STIITIFICE ALE UNIVERSITATII "AL.I.CUZA" IASI
Tomul XL, s.I.a, Matematică, 1994, f3.

ON THE APPROXIMATION NUMBERS OF THE TENSOR PRODUCT OPERATOR

BY

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Introduction. Let X be a real Banach space and let $T \in \mathcal{L}(X)$ be a linear and bounded operator $T : X \to X$. The n–th approximation number of T is

$$a_n(T) = \inf\{\|T - A\| : A \in \mathcal{L}(X), \text{rank A} < \text{n}\}$$

For the algebraic tensor product $X \otimes X$ we consider a cross–norm τ, [1], [5], $(\tau(x \otimes \bar{x}) = \|x\| \cdot \|\bar{x}\|)$ and we denote by $X\hat{\otimes}_\tau X$ the completion of the topological tensor product $X \otimes_\tau X$ relatively to the norm τ. Let $S, T \in \mathcal{L}(X)$. The cross–norm τ is called tensor–norm if $S\hat{\otimes}_\tau X : X\hat{\otimes}_\tau X \to X\hat{\otimes}_\tau X$ is continuous ($\|S\hat{\otimes}_\tau T\| \leq \|S\| \cdot \|T\|$).

We also consider a real sequence $\alpha = (\alpha_n)$ such that: $\alpha_n \geq \alpha_{n+1} > 0$ and $\alpha_{n^2} \leq \alpha_n \cdot c/n$ for all $n \in N$, where c is a constant.

The purpose of this paper is to prove some inequalities between the approximation numbers of the operator $S\hat{\otimes}_\tau T$ and the approximation numbers of the operators S and T.

Main result. The main result of this paper is the following.

Theorem. *For $S, T \in \mathcal{L}(X), k \in I\!N, p \in (0, \infty)$ and all sequences α as above it holds*

$$\sum_1^k \alpha_n a_n^p(S\hat{\otimes}_\tau T) \leq c(\alpha, p) \cdot \sum_1^k \alpha_n(a_n^p(S) + a_n^p(T)).$$

Proof. First we prove that $a_{n^2}(S\hat{\otimes}_\tau T) \leq M(a_n(S) + a_n(T))$, where $M < 2 \cdot \max(\|S\|, \|T\|)$. To this end we use an argument from [5].

For $\varepsilon > 0$ there exists $A_i \in \mathcal{L}(X)$, rank $A_i < n$ $(i = 1, 2)$, such that $\|S - A_1\| < a_n(S) + \varepsilon$ and $\|T - A_2\| \leq a_n(T) + \varepsilon$.

Then we can write

$$a_{n^2}(S \hat{\otimes}_\tau T) \leq \|S \hat{\otimes}_\tau T - A_1 \hat{\otimes}_\tau A_2\| = \|(S - A_1) \hat{\otimes}_\tau T + A_1 \hat{\otimes}_\tau (T - A_2)\| \leq$$

$$\leq \|(S - A_1\| \cdot \|T\| + \|A_1\| \cdot \|T - A_2\| \leq (a_n(S) + \varepsilon)\|T\| +$$

$$+ \|A_1 - S + S\|(a_n(T) + \varepsilon) \leq (a_n(S) + \varepsilon)\|T\| + (a_n(S) + \varepsilon + \|S\|)(a_n(T) + \varepsilon)$$

Since ε is arbitrary and $\|S\| = a_1(S) \geq \ldots \geq a_n(S) \geq \ldots$ it results the relation. Now we observe that for all $k \in \mathbb{N}$ there exists $j \in \mathbb{N}$ such that $j^2 \leq k < (j+1)^2$ and hence we obtain

$$\sum_{n=1}^{k} \alpha_n a_n^p (S \hat{\otimes}_\tau T) \leq \sum_{n=1}^{j} (2n + 1) \alpha_{n^2} a_{n^2}^p (S \otimes T) \leq$$

$$\leq 3 \sum_{1}^{j} n \cdot \alpha_n \cdot c/n a_{n^2}^p (S \hat{\otimes}_\tau T) \leq 3 \sum_{1}^{j} c \cdot \alpha_n [M(a_n(S) + a_n(T))]^p \leq$$

$$\leq c(\alpha, p) \sum_{1}^{j} \alpha_n (a_n^p(S) + a_n^p(T)) \leq c(\alpha, p) \sum_{1}^{k} \alpha_n (a_n^p(S) + a_n^p(T)).$$

Remarks. A similar result we obtain if $\alpha = (\alpha_n)$ is replaced by an other sequence such that $\max\{\alpha_{n^2}, \alpha_{n^2+1}, \ldots, \alpha_{(n+1)^2-1}\} \leq \alpha_n \cdot c/n$, where c is a constant which depends on α.

Applications. (1) If $\alpha_n = \frac{1}{n}$ it results, in a simple way, that the approximation ideal $\mathcal{L}_{\infty,p}^{(a)}(X) := \{T : \sum_{1}^{\infty} \frac{a_n^p(T)}{n} \leq \infty\}$ is tensor product stable. (If $S, T \in \mathcal{L}_{\infty,p}(X)$ then $S \hat{\otimes}_\tau T \in \mathcal{L}_{\infty,p}(X \hat{\otimes}_\tau X)$). This problem is solved in [5], [1], [2]. We remark that $\sum_{1}^{\infty} \frac{a_n^p(T)}{n} \leq \infty$ iff $\sum_{1}^{\infty} 2^{n-1} \frac{a_{2^{n-1}}^p(T)}{2^{n-1}} = \sum_{1}^{\infty} a_{2^{n-1}}^p(T) < \infty$ and hence $\mathcal{L}_{\infty,p}(X) \equiv \mathcal{L}_p^*(X) =$ $= \{T : \sum_{1}^{\infty} a_{2^{n-1}}^p(T) < \infty\}$ [5].

(2) If $\alpha_n = \dfrac{(1 + \log n)^{\gamma p}}{n}$ we obtain the Lorentz - Zygmund ideals $\mathcal{L}_{\infty,p,\gamma}^{(a)}(X) := \{T : \sum((1 + \log n)^\gamma a_n(T))^p n^{-1} < \infty\}$ which are also tensor product stable for all $0 < p < \infty$, $\frac{1}{p} < \gamma < \infty$.

(3) If $\alpha_n = (1 + \log n)^{\gamma p}(1 + \log(1 + \log n))^{1-p} \cdot n^{-1}, p \in (1, \infty)$ we obtain an other operator ideal

$$\tilde{\mathcal{L}}_{p,\gamma}(X) := \{T : \sum (1 + \log n)^{\gamma p} (1 + \log(1 + \log n))^{1-p} \cdot n^{-1} \cdot a_n^p(T) < \infty\},$$

which is tensor product stable. In this way the last theorem from [1] is improved.

(4) If Φ is a symmetric norming function [3] we consider the operator ideals $\mathcal{L}_{\Phi_\alpha}^{(a)}(X) := \{T : \Phi(\{\alpha_n a_n(T)\}) < \infty$. Here $\alpha_1 = 1 \geq \alpha_2 \geq \ldots > 0$, and $\alpha_{n^2} \leq \alpha_n \cdot c/n$. From the properties of the function Φ and from the above theorem it results that these ideals are also tensor product stable.

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Received: 11.VII.1994

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