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Perturbed elastic beam problems with nonlinear boundary conditions

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Abstract Based on recent variational methods for smooth functionals defined on reflexive Banach spaces, the existence of three distinct generalized solutions for perturbed elastic beam problems with nonlinear boundary conditions under suitable assumptions on the nonlinear terms is established.

Keywords Multiple solutions · Perturbed differential equation · Fourth-order problem · Elastic beam equation · Variational methods · Critical point theory

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1 Introduction

Consider the following perturbed fourth-order problem

$$\begin{cases} u^{(4)} = \lambda f(t,u) + \mu g(t,u) + p(u), & 0 < t < 1, \\ u(0) = u'(0) = 0, \\ u''(1) = 0, u'''(1) = h(u(1)) \end{cases} \tag{1.1}$$

where λ, μ are two parameters, f, g are two L^2-Carathéodory functions, $p : \mathbb{R} \to \mathbb{R}$ is a Lipschitz continuous function with the Lipschitz constant $L > 0$, i.e.

$$|p(\xi_1) - p(\xi_2)| \leq L|\xi_1 - \xi_2|, \quad \forall\ \xi_1, \xi_2 \in \mathbb{R}$$

such that $p(0) = 0$, and $h \in C(\mathbb{R})$ is real function.

It is well known that these kind of problems are important in describing a large class of elastic deflections. In recent years they have received considerable attention owing

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to their interest to physics. Due to this, many researchers have studied the existence and multiplicity of solutions for fourth-order two-point boundary value problems; we refer the reader to [1–3, 7–9, 13, 14, 17–20, 22–24] and references therein. For example, in [3] the authors using fixed-point theorem and degree theory discussed the existence of one or two positive solutions for nonlinear fourth-order beam equations. Moreover, under the assumption that the nonlinear term is monotone increasing respect to the second variable, they got the uniqueness result and the result of the existence of infinitely many positive solutions for the problem. In [18], by using monotone operator theory and critical point theory, Li et al. established some sufficient conditions for the nonlinear term to guarantee that a class of fourth-order boundary value problems has an unique solution, at least one nonzero solution, or infinitely many solutions. In [14] Han and Xu employing the Morse theory obtained some existence theorems on three solutions and infinitely many solutions for a fourth-order beam equation. In [8] the authors using an infinitely many critical points theorem, without symmetric condition on the nonlinear term established existence results of infinitely many solutions for a fourth-order nonlinear boundary value problem. In [9] the authors employing a local minimum theorem for differentiable functionals, under a non-standard growth condition of the nonlinear term, investigated the existence of at least one non-trivial solution to a boundary value problem for fourth order elastic beam equations. In [1, 7, 8] based on variational methods and critical point theory the existence of multiple solutions for fourth-order elastic beam equations was discussed. Yang et al. in [23] by using variational methods and a three-critical-point theorem, established sufficient conditions under which the problem (1.1), in the case $\mu = 0$ and $p \equiv 0$ possesses two solutions generated from the boundary condition h. Recently, Gao in [13] based on variational methods and critical point theory established some existence results of three solutions for the problem (1.1), in the case $p \equiv 0$. Very recently Song in [22] by using the smooth version of [11, Theorem 2.1], a variant of celebrated Ricceri's variational principle [21, Theorem 2.1], established infinitely many solutions for the problem (1.1), in the case $p \equiv 0$.

In the present paper, motivated by the above works, employing two kinds of three critical points theorems obtained in [4, 10] which we recall in the next section (Theorems 2.1 and 2.2), we establish the existence of at least three generalized solutions for the problem (1.1); see Theorems 3.1 and 3.2. These theorems have been successfully employed to establish the existence of at least three solutions for perturbed boundary value problems in the papers [5, 6, 12, 15, 16].

2 Preliminaries

Our main tools are three critical points theorems that we recall here in a convenient form. The first was obtained in [10] and it is a more precise version of Theorem 3.2 of [4]. The second was established in [4].

Theorem 2.1 ([10], Theorem 2.6) *Let X be a reflexive real Banach space, $\Phi : X \longrightarrow \mathbb{R}$ be a coercive continuously Gâteaux differentiable and sequentially weakly lower semicontinuous functional whose Gâteaux derivative admits a continuous inverse on X^*, $\Psi : X \longrightarrow \mathbb{R}$ be a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact such that $\Phi(0) = \Psi(0) = 0$.*
Assume that there exist $r > 0$ and $\overline{v} \in X$, with $r < \Phi(\overline{v})$ such that

(a_1) $\dfrac{\sup_{\Phi(u)\leq r}\Psi(u)}{r} < \dfrac{\Psi(\overline{v})}{\Phi(\overline{v})}$,

(a_2) *for each* $\lambda \in \Lambda_r := \left]\dfrac{\Phi(\overline{v})}{\Psi(\overline{v})}, \dfrac{r}{\sup_{\Phi(u)\leq r}\Psi(u)}\right[$ *the functional* $\Phi - \lambda\Psi$ *is coercive.*

Then, for each $\lambda \in \Lambda_r$ *the functional* $\Phi - \lambda\Psi$ *has at least three distinct critical points in* X.

Theorem 2.2 ([4], Corollary 3.1) *Let* X *be a reflexive real Banach space,* $\Phi : X \longrightarrow \mathbb{R}$ *be a convex, coercive and continuously Gâteaux differentiable functional whose derivative admits a continuous inverse on* X^*, $\Psi : X \longrightarrow \mathbb{R}$ *be a continuously Gâteaux differentiable functional whose derivative is compact, such that*

1. $\inf_X \Phi = \Phi(0) = \Psi(0) = 0$;

2. *for each* $\lambda > 0$ *and for every* $u_1, u_2 \in X$ *which are local minima for the functional* $\Phi - \lambda\Psi$ *and such that* $\Psi(u_1) \geq 0$ *and* $\Psi(u_2) \geq 0$, *one has*

$$\inf_{s\in[0,1]} \Psi(su_1 + (1-s)u_2) \geq 0.$$

Assume that there are two positive constants r_1, r_2 *and* $\overline{v} \in X$, *with* $2r_1 < \Phi(\overline{v}) < \frac{r_2}{2}$, *such that*

(b_1) $\dfrac{\sup_{u\in\Phi^{-1}(]-\infty,r_1[)}\Psi(u)}{r_1} < \dfrac{2}{3}\dfrac{\Psi(\overline{v})}{\Phi(\overline{v})}$;

(b_2) $\dfrac{\sup_{u\in\Phi^{-1}(]-\infty,r_2[)}\Psi(u)}{r_2} < \dfrac{1}{3}\dfrac{\Psi(\overline{v})}{\Phi(\overline{v})}$.

Then, for each

$\lambda \in \left]\dfrac{3}{2}\dfrac{\Phi(\overline{v})}{\Psi(\overline{v})}, \min\left\{\dfrac{r_1}{\sup_{u\in\Phi^{-1}(]-\infty,r_1[)}\Psi(u)}, \dfrac{\frac{r_2}{2}}{\sup_{u\in\Phi^{-1}(]-\infty,r_2[)}\Psi(u)}\right\}\right[$, *the functional* $\Phi - \lambda\Psi$ *has at least three distinct critical points which lie in* $\Phi^{-1}(]-\infty, r_2[)$.

Let

$$X := \Big\{u \in W^{1,2}(0,1) : u(0) = u'(0) = 0\Big\},$$

endowed with the norm

$$\|u\| := \left(\int_0^1 |u''(t)|^2 dt\right)^{1/2}.$$

We observe that the norm $\|\cdot\|$ is equivalent to the usual one.

It is well known that $(X, \|\cdot\|)$ is compactly embedded in the $C([0,1])$, therefore, there exists a constant $\bar{S} > 0$ such that

$$\|u\|_\infty \leq \bar{S}\|u\| \tag{2.1}$$

for every $u \in X$.

We assume throughout, and without further mention, that the following condition holds:

(H) $h(s)$ is nondecreasing on $s \in \mathbb{R}$ and $h(s)s \geq 0$ for any $s \in \mathbb{R}$.

Suppose that the Lipschitz constant $L > 0$ of the function p satisfies $L\bar{S}^2 < 1$.

Proposition 2.3 *Let $T : X \to X^*$ be the operator defined by*

$$T(u)v = \int_0^1 \Big(u''(t)v''(t) - p(u(t))v(t)\Big)dt + h(u(1))v(1)$$

for every $u, v \in X$. Then, T admits a continuous inverse on X^*.

Proof. For any $u \in X \setminus \{0\}$,

$$\begin{aligned}\lim_{\|u\|\to\infty} \frac{\langle T(u), u\rangle}{\|u\|} &= \lim_{\|u\|\to\infty} \frac{\displaystyle\int_0^1 \Big(u''(t)u''(t) - p(u(t))u(t)\Big)dt + h(u(1))u(1)}{\|u\|}\\ &\geq \lim_{\|u\|\to\infty} \frac{(1 - L\bar{S}^2)\|u\|^2}{\|u\|}\\ &= \lim_{\|u\|\to\infty} (1 - L\bar{S}^2)||u|| = \infty.\end{aligned}$$

Thus, the map T is coercive. Now, for any $u, v \in X$, one has

$$\begin{aligned}\langle T(u) - T(v), u - v\rangle = \int_0^1 \Big((u''(t) - v''(t), u''(t) - v''(t)) - (p(u(t)) -\\ p(v(t)))(u(t) - v(t))\Big)dt\\ + (h(u(1)) - h(v(1)))(u(1) - v(1))\\ \geq (1 - L\bar{S}^2)||u - v||^2.\end{aligned}$$

So, $T(u)$ is uniformly monotone. By [25, Theorem 26.A (d)], T^{-1} exists and is continuous on X^*. This completes the proof. □

Corresponding to f, g and p we introduce the functions $F : [0,1] \times \mathbb{R} \to \mathbb{R}$, $G : [0,1] \times \mathbb{R} \to \mathbb{R}$ and $P : \mathbb{R} \to \mathbb{R}$, respectively, as follows

$$F(t,x) = \int_0^x f(t,\xi)d\xi \quad \forall\ (t,x) \in [0,1] \times \mathbb{R},$$

$$G(t,x) = \int_0^x g(t,\xi)d\xi, \quad \forall\ (t,x) \in [0,1] \times \mathbb{R}$$

and

$$P(x) = \int_0^x p(\xi)d\xi, \quad \forall\ x \in \mathbb{R}.$$

Moreover, let $0 < \bar{t} < \hat{t} < 1$ and there exist two functions $d \in C^2([0,\bar{t}]$ and $e \in C^2([\hat{t},1]$ satisfying $d(0) = d'(0) = 0$, $d(\bar{t}) = e(\hat{t}) = 1$, $d'(\bar{t}) = e'(\hat{t}) = 0$ and $K := \int_0^{\bar{t}} |d''(t)|^2dt + \int_{\hat{t}}^1 |e''(t)|^2dt \neq 0$, set $G^\gamma := \displaystyle\int_{[0,1]} \max_{|x|\leq \bar{S}\gamma\sqrt{K}} G(t,x)dt$ and $G_\gamma :$
$= \inf_{[0,1]\times[-\max_{t\in[0,1]}\{e(t)\gamma,\gamma,d(t)\gamma\},\ \max_{t\in[0,1]}\{e(t)\gamma,\gamma,d(t)\gamma\}]} G$, for every $\gamma > 0$. If g is sign-changing, then $G^\gamma \geq 0$ and $G_\gamma \leq 0$.

A function $u : [0,1] \to \mathbb{R}$ is a generalized solution to the problem (1.1) if $u \in C^3([0,1])$, $u''' \in AC([0,1])$, $u(0) = u'(0) = u''(1) = 0,\ u'''(1) = h(u(1))$, and $u^{(4)}(t) = \lambda f(t,u(t)) + \mu g(t,u(t)) + p(u(t))$ for almost every $t \in (0,1)$, and it is a weak solution to the problem (1.1) if $u \in X$ and

$$\int_0^1 (u''(t)v''(t) - p(u(t))v(t))dt + h(u(1))v(1) - \lambda \int_0^1 f(t,u(t))v(t)dt -$$

$$\mu \int_0^1 g(t,u(t))v(t)dt = 0$$

for every $v \in X$.

Standard methods (see [7, Proposition 2.2]) show that a weak solution to (1.1) is a generalized one when f, g are L^2-Carathéodory functions.

3 Main results

Fixing a positive constant γ such that

$$\frac{\frac{\gamma^2}{2}(1+L\bar{S}^2)K + \int_0^{e(1)\gamma} h(x)dx}{\int_{\bar{t}}^{\hat{t}} F(t,\gamma)dt} < \frac{\frac{\gamma^2}{2}(1-L\bar{S}^2)K}{\int_0^1 \sup_{|x|\le \bar{S}\gamma\sqrt{K}} F(t,x)dt}$$

and taking

$$\lambda \in \Lambda := \left]\frac{\frac{\gamma^2}{2}(1+L\bar{S}^2)K + \int_0^{e(1)\gamma} h(x)dx}{\int_{\bar{t}}^{\hat{t}} F(t,\gamma)dt}, \frac{\frac{\gamma^2}{2}(1-L\bar{S}^2)K}{\int_0^1 \sup_{|x|\le \bar{S}\gamma\sqrt{K}} F(t,x)dt}\right[,$$

set $\delta_{\lambda,g}$ given by

$$\min\left\{\frac{\frac{\gamma^2}{2}(1-L\bar{S}^2)K - \lambda\int_0^1 \sup_{|x|\le \bar{S}\gamma\sqrt{K}} F(t,x)dt}{G^\gamma}, \right.$$

$$\left.\frac{\frac{\gamma^2}{2}(1+L\bar{S}^2)K + \int_0^{e(1)\gamma} h(x)dx - \lambda\int_{\bar{t}}^{\hat{t}} F(t,\gamma)dt}{G_\gamma}\right\} \tag{3.1}$$

and

$$\overline{\delta}_{\lambda,g} := \min\left\{\delta_{\lambda,g}, \frac{1}{\max\left\{0, \frac{2}{(1-L\bar{S}^2)K}\limsup_{|x|\to\infty}\frac{\sup_{t\in[0,1]} G(t,x)}{x^2}\right\}}\right\}, \tag{3.2}$$

where we read $\gamma/0 = +\infty$, so that, for instance, $\overline{\delta}_{\lambda,g} = +\infty$ when

$$\limsup_{|x|\to\infty} \frac{\sup_{x\in[0,1]} G(t,x)}{x^2} \leq 0,$$

and $G_\gamma = G^\gamma = 0$.

Now,we present our main result.

Theorem 3.1 *Assume that there exists a positive constant γ such that*

(A_1) $F(t,\xi) \geq 0$ *for all* $t \in [0,\bar{t}] \cup [\hat{t},1]$ *and* $\xi \in \mathbb{R}$*;*

(A_2) $$\frac{\int_0^1 \sup_{|x|\leq \bar{S}\gamma\sqrt{K}} F(t,x)dt}{\gamma^2} < \frac{1}{2}(1-L\bar{S}^2)K \frac{\int_{\hat{t}}^{\hat{t}} F(t,\gamma)dt}{\frac{\gamma^2}{2}(1+L\bar{S}^2)K + \int_0^{e(1)\gamma} h(x)dx};$$

(A_3) $\limsup_{|x|\to\infty} \frac{\sup_{t\in[0,1]} F(t,x)}{x^2} \leq 0.$

Then, for each $\lambda \in \Lambda$ and for every L^2-Carathéodory function $g : [0,1]\times\mathbb{R} \to \mathbb{R}$ satisfying the condition

$$\limsup_{|x|\to\infty} \frac{\sup_{t\in[0,1]} G(t,x)}{x^2} < +\infty,$$

there exists $\overline{\delta}_{\lambda,g} > 0$ given by (3.2) *such that, for each $\mu \in [0,\overline{\delta}_{\lambda,g}[$, the problem* (1.1) *admits at least three distinct generalized solutions in X.*

Proof. Fix λ, g and μ as in the conclusion. In order to apply Theorem 2.1 to our problem, we introduce the functionals Φ, $\Psi : X \to \mathbb{R}$ for each $u \in X$, as follows

$$\Phi(u) = \frac{1}{2}\|u\|^2 + \int_0^{u(1)} h(x)dx - \int_0^1 P(u(t))dt$$

and

$$\Psi(u) = \int_0^1 [F(t,u(t)) + \frac{\mu}{\lambda} G(t,u(t))]dt.$$

Let us prove that the functionals Φ and Ψ satisfy the required conditions in Theorem 2.1. It is well known that Ψ is a differentiable functional whose differential at the point $u \in X$ is

$$\Psi'(u)(v) = \int_0^1 [f(t,u(t)) + \frac{\mu}{\lambda} g(t,u(t))]v(t)dt$$

for every $v \in X$. Furthermore, $\Psi' : X \to X^*$ is a compact operator. Indeed, it is enough to show that Ψ' is strongly continuous on X. For this end, for fixed $u \in X$, let $u_n \to u$ weakly in X as $n \to \infty$, then u_n converges uniformly to u on $[0,1]$ as $n \to \infty$; see [25]. Since f, g are L^2-Carathéodory functions, f, g are continuous in $\mathbb{R}$ for every $x \in [0,1]$, so

$$f(t,u_n) + \frac{\mu}{\lambda} g(t,u_n) \to f(t,u) + \frac{\mu}{\lambda} g(t,u),$$

as $n \to \infty$. Hence $\Psi'(u_n) \to \Psi'(u)$ as $n \to \infty$. Thus we proved that Ψ' is strongly continuous on X, which implies that Ψ' is a compact operator by Proposition 26.2 of [25]. Moreover, Φ is continuously differentiable whose differential at the point $u \in X$ is

$$\Phi'(u)(v) = \int_0^1 u''(t)v''(t)dt - \int_0^1 p(u(t))v(t)dt + hu(1)v(1)$$

for every $v \in X$, while Proposition 2.3 gives that Φ' admits a continuous inverse on X^*. Furthermore, Φ is sequentially weakly lower semicontinuous. Put $r = \dfrac{\gamma^2}{2}(1 - L\bar{S}^2)K$. Bearing in mind the relation (2.1), one has

$$\begin{aligned}\Phi^{-1}(]-\infty, r]) &= \{u \in X;\ \Phi(u) \le r\} \\ &\subseteq \left\{u \in X;\ \frac{(1 - L\bar{S}^2)||u||^2}{2} \le r\right\} \\ &\subseteq \left\{u \in X : \max_{t\in[0,1]} |u(t)| \le \bar{S}\gamma\sqrt{K}\right\},\end{aligned}$$

and it follows that

$$\begin{aligned}\sup_{u\in\Phi^{-1}(]-\infty,r])} \Psi(u) &= \sup_{u\in\Phi^{-1}(]-\infty,r])} \int_0^1 [F(t,u(t)) + \frac{\mu}{\lambda}G(t,u(t))]dt \\ &\le \int_0^1 \sup_{|x|\le\bar{S}\gamma\sqrt{K}} F(t,x)dt + \frac{\mu}{\lambda}G^\gamma.\end{aligned}$$

Now, fix

$$w(t) := \begin{cases} d(t)\gamma, & t \in [0,\bar{t}), \\ \gamma, & t \in [\bar{t},\hat{t}], \\ e(t)\gamma, & t \in (\hat{t},1]. \end{cases} \tag{3.3}$$

It is easy to see that $w \in X$ and, in particular, one has

$$\begin{aligned}\Phi(w) &= \frac{1}{2}\|w\|^2 + \int_0^{w(1)} h(x)dx - \int_0^1 P(w(t))dt \\ &\le \frac{\gamma^2}{2}(1 + L\bar{S}^2)K + \int_0^{e(1)\gamma} h(x)dx.\end{aligned} \tag{3.4}$$

We observe that by (H),

$$\frac{\gamma^2}{2}(1 - L\bar{S}^2)K < \frac{\gamma^2}{2}(1 + L\bar{S}^2)K + \int_0^{e(1)\gamma} h(x)dx,$$

then

$$0 < r < \Phi(w).$$

On the other hand, by using the condition (A_1), since $0 \leq w(t) \leq \max_{t\in[0,1]}\{e(t)\gamma, \gamma, d(t)\gamma\}$ for each $t \in [0,1]$, we have

$$\begin{aligned}\Psi(w) &\geq \int_{\bar{t}}^{\hat{t}} F(t,\gamma)dt + \frac{\mu}{\lambda}\int_0^1 G(t,w(t))dt \\ &\geq \int_{\bar{t}}^{\hat{t}} F(t,\gamma)dt + \frac{\mu}{\lambda}G_\gamma.\end{aligned}$$

Therefore, we have

$$\begin{aligned}\frac{\sup\limits_{u\in\Phi^{-1}(]-\infty,r])}\Psi(u)}{r} &= \frac{\sup\limits_{u\in\Phi^{-1}(]-\infty,r])}\int_0^1 [F(t,u(t)) + \frac{\mu}{\lambda}G(t,u(t))]dt}{r} \\ &\leq \frac{\int_0^1 \sup\limits_{|x|\leq\bar{S}\gamma\sqrt{K}} F(t,x)dt + \frac{\mu}{\lambda}G^\gamma}{\frac{\gamma^2}{2}(1-L\bar{S}^2)K},\end{aligned} \tag{3.5}$$

and

$$\begin{aligned}\frac{\Psi(w)}{\Phi(w)} &\geq \frac{\int_{\bar{t}}^{\hat{t}} F(t,\gamma)dt + \frac{\mu}{\lambda}\int_0^1 G(t,w(t))dt}{\frac{\gamma^2}{2}(1+L\bar{S}^2)K + \int_0^{e(1)\gamma} h(x)dx} \\ &\geq \frac{\int_{\bar{t}}^{\hat{t}} F(t,\gamma)dt + \frac{\mu}{\lambda}G_\gamma}{\frac{\gamma^2}{2}(1+L\bar{S}^2)K + \int_0^{e(1)\gamma} h(x)dx}.\end{aligned} \tag{3.6}$$

Since $\mu < \delta_{\lambda,g}$, one has

$$\mu < \frac{\frac{\gamma^2}{2}(1-L\bar{S}^2)K - \lambda\int_0^1 \sup\limits_{|x|\leq\bar{S}\gamma\sqrt{K}} F(t,x)dt}{G^\gamma},$$

this means

$$\frac{\int_0^1 \sup\limits_{|x|\leq\bar{S}\gamma\sqrt{K}} F(t,x)dt + \frac{\mu}{\lambda}G^\gamma}{\frac{\gamma^2}{2}(1-L\bar{S}^2)K} < \frac{1}{\lambda}.$$

Furthermore,

$$\mu < \frac{\frac{\gamma^2}{2}(1+L\bar{S}^2)K + \int_0^{e(1)\gamma} h(x)dx - \lambda\int_{\bar{t}}^{\hat{t}} F(t,\gamma)dt}{G_\gamma},$$

this means

$$\frac{\displaystyle\int_{\bar{t}}^{\hat{t}} F(t,\gamma)dt + \frac{\mu}{\lambda}G_\gamma}{\dfrac{\gamma^2}{2}(1+L\bar{S}^2)K + \int_0^{e(1)\gamma} h(x)dx} > \frac{1}{\lambda}.$$

Then,

$$\frac{\displaystyle\int_0^1 \sup_{|x|\le\bar{S}\gamma\sqrt{K}} F(t,x)dt + \frac{\mu}{\lambda}G^\gamma}{\dfrac{\gamma^2}{2}(1-L\bar{S}^2)K} < \frac{1}{\lambda} < \frac{\displaystyle\int_{\bar{t}}^{\hat{t}} F(t,\gamma)dt + \frac{\mu}{\lambda}G_\gamma}{\dfrac{\gamma^2}{2}(1+L\bar{S}^2)K + \int_0^{e(1)\gamma} h(x)dx}. \tag{3.7}$$

Hence, from (3.5)-(3.7), the condition (a_1) of Theorem 2.1 is fulfilled. Finally, since $\mu < \overline{\delta}_{\lambda,g}$, we can fix $l > 0$ such that

$$\limsup_{|x|\to\infty} \frac{\sup_{t\in[0,1]} G(t,x)}{x^2} < l,$$

and $\mu l < \dfrac{1-L\bar{S}^2}{2\bar{S}^2}$. Therefore, there exists a function $\tau \in L^1([0,1])$ such that

$$G(t,x) \le lx^2 + \tau(t), \tag{3.8}$$

for every $t \in [0,1]$ and $x \in \mathbb{R}$. Now, fix $0 < \epsilon < \dfrac{1-L\bar{S}^2}{2\bar{S}^2\lambda} - \dfrac{\mu l}{\lambda}$. From (A_3) there is a function $\sigma \in L^1([0,1])$ such that

$$F(t,x) \le \epsilon x^2 + \sigma(t), \tag{3.9}$$

for every $t \in [0,1]$ and $x \in \mathbb{R}$. Recalling (2.1), from (3.8) and (3.9), for each $u \in X$, we have

$$\begin{aligned}
\Phi(u) - \lambda\Psi(u) &= \frac{1}{2}\|u\|^2 + \int_0^{u(1)} h(x)dx - \int_0^1 P(u(t))dt - \lambda\int_0^1 [F(t,u(t)) + \\
&\qquad\qquad \frac{\mu}{\lambda}G(t,u(t))]dt \\
&\ge \frac{1-L\bar{S}^2}{2}\|u\|^2 - \lambda\epsilon\int_0^1 u^2(t)dt - \lambda\|\sigma\|_1 - \mu l\int_0^1 u^2(t)dt - \mu\|\tau\|_1 \\
&\ge \Big(\frac{1-L\bar{S}^2}{2} - \lambda\bar{S}^2\epsilon - \mu\bar{S}^2 l\Big)\|u\|^2 - \lambda\|\sigma\|_1 - \mu\|\tau\|_1,
\end{aligned}$$

and thus

$$\lim_{\|u\|\to+\infty} (\Phi(u) - \lambda\Psi(u)) = +\infty,$$

which means the functional $\Phi - \lambda\Psi$ is coercive, and the condition (a_2) of Theorem 2.1 is verified. From (3.5)-(3.7) one also has

$$\lambda \in \left]\frac{\Phi(w)}{\Psi(w)}, \frac{r}{\sup_{\Phi(u)\le r}\Psi(u)}\right[.$$

Finally, the conclusion follows from Theorem 2.1 (with $\overline{v} = w$). □

Now, a variant of Theorem 3.1 where no asymptotic conditions on f and g are requested and the functions f and g are supposed to be nonnegative, is pointed out.

For our goal, let us fix positive constants γ_1, γ_2 and γ such that

$$3\frac{\frac{\gamma^2}{2}(1+L\bar{S}^2)K+\int_0^{e(1)\gamma}h(x)dx}{\int_{\bar{t}}^{\hat{t}}F(t,\gamma)dt}$$

$$<(1-L\bar{S}^2)K\min\left\{\frac{\gamma_1^2}{\int_0^1\sup\limits_{|x|\leq\bar{S}\gamma_1\sqrt{K}}F(t,x)dt},\ \frac{\gamma_2^2}{2\int_0^1\sup\limits_{|x|\leq\bar{S}\gamma_2\sqrt{K}}F(t,x)dt}\right\},$$

and taking

$$\lambda\in\Lambda':=\left]\frac{3}{2}\frac{\frac{\gamma^2}{2}(1+L\bar{S}^2)K+\int_0^{e(1)\gamma}h(x)dx}{\int_{\bar{t}}^{\hat{t}}F(t,\gamma)dt},\right.$$

$$\left.\frac{1}{2}(1-L\bar{S}^2)K\min\left\{\frac{\gamma_1^2}{\int_0^1\sup\limits_{|x|\leq\bar{S}\gamma_1\sqrt{K}}F(t,x)dt},\ \frac{\gamma_2^2}{2\int_0^1\sup\limits_{|x|\leq\bar{S}\gamma_2\sqrt{K}}F(t,x)dt}\right\}\right[.$$

Theorem 3.2 *Let $f:[0,1]\times\mathbb{R}\to\mathbb{R}$ be a nonnegative L^2-Carathéodory function. Assume that there exist three positive constants γ_1, γ_2 and γ with*

$$\gamma_1<\sqrt{\frac{\frac{\gamma^2}{2}(1-L\bar{S}^2)K+\int_0^{e(1)\gamma}h(x)dx}{(1-L\bar{S}^2)K}}$$

and

$$2\sqrt{\frac{\frac{\gamma^2}{2}(1+L\bar{S}^2)K+\int_0^{e(1)\gamma}h(x)dx}{(1-L\bar{S}^2)K}}<\gamma_2$$

such that the assumption (A_1) in Theorem 3.1 holds. Furthermore, suppose that

$$(B_1) \quad \max\left\{\frac{\int_0^1 \sup_{|x|\le \bar{S}\gamma_1\sqrt{K}} F(t,x)dt}{\frac{\gamma_1^2}{2}(1-L\bar{S}^2)K}, \frac{2\int_0^1 \sup_{|x|\le \bar{S}\gamma_2\sqrt{K}} F(t,x)dt}{\frac{\gamma_2^2}{2}(1-L\bar{S}^2)K}\right\} < \frac{2}{3}\frac{\int_{\bar{t}}^{\hat{t}} F(t,\gamma)dt}{\frac{\gamma^2}{2}(1-L\bar{S}^2)K+\int_0^{e(1)\gamma} h(x)dx}.$$

Then, for each $\lambda \in \Lambda'$ and for every nonnegative L^2-Carathéodory function $g : [0,1] \times \mathbb{R} \to \mathbb{R}$, there exists $\delta^*_{\lambda,g} > 0$ given by

$$\min\left\{\frac{\gamma_1^2(1-L\bar{S}^2)K - 2\lambda\int_0^1 \sup_{|x|\le \bar{S}\gamma_1\sqrt{K}} F(t,x)dt}{2G\gamma_1}, \frac{\gamma_2^2(1-L\bar{S}^2)K - 4\lambda\int_0^1 \sup_{|x|\le \bar{S}\gamma_2\sqrt{K}} F(t,x)dt}{4G\gamma_2}\right\}$$

such that, for each $\mu \in [0, \delta^*_{\lambda,g}[$, the problem (1.1) admits at least three distinct generalized solutions u_i for $i = 1,2,3$, such that

$$0 \le u_i(t) < \bar{S}\gamma_2\sqrt{K}, \quad \forall\ t \in [0,1], (i = 1,2,3).$$

Proof. Fix λ, g and μ as in the conclusion and take Φ and Ψ as in the proof of Theorem 3.1. We observe that the regularity assumptions of Theorem 2.2 on Φ and Ψ are fulfilled. Then, our aim is to verify (b_1) and (b_2). To this end, put w as given in (3.3), as well as

$$r_1 := \frac{\gamma_1^2}{2}(1-L\bar{S}^2)K$$

and

$$r_2 := \frac{\gamma_2^2}{2}(1-L\bar{S}^2)K.$$

By using the condition

$$\gamma_1 < \sqrt{\frac{\frac{\gamma^2}{2}(1-L\bar{S}^2)K + \int_0^{e(1)\gamma} h(x)dx}{(1-L\bar{S}^2)K}} \quad \text{and}$$

$$2\sqrt{\frac{\frac{\gamma^2}{2}(1+L\bar{S}^2)K + \int_0^{e(1)\gamma} h(x)dx}{(1-L\bar{S}^2)K}} < \gamma_2,$$

we get $2r_1 < \Phi(w) < \dfrac{r_2}{2}$. Since $\mu < \delta^*_{\lambda,g}$ and $G_\gamma = 0$, one has

$$\begin{aligned}
\frac{\sup\limits_{u\in\Phi^{-1}(]-\infty,r_1])}\Psi(u)}{r_1} &= \frac{\sup\limits_{u\in\Phi^{-1}(]-\infty,r_1])}\int_0^1 [F(t,u(t))+\frac{\mu}{\lambda}G(t,u(t))]dt}{r_1}\\
&\leq \frac{\int_0^1 \sup\limits_{|x|\leq \bar{S}\gamma_1\sqrt{K}} F(t,x)dt + \frac{\mu}{\lambda}G^{\gamma_1}}{\frac{\gamma_1^2}{2}(1-L\bar{S}^2)K}\\
&< \frac{1}{\lambda} < \frac{2}{3}\frac{\int_{\bar{t}}^{\hat{t}} F(t,\gamma)dt + \frac{\mu}{\lambda}G_\gamma}{\frac{\gamma^2}{2}(1-L\bar{S}^2)K + \int_0^{e(1)\gamma} h(x)dx}\\
&\leq \frac{2}{3}\frac{\Psi(w)}{\Phi(w)},
\end{aligned}$$

and

$$\begin{aligned}
\frac{2\sup\limits_{u\in\Phi^{-1}(]-\infty,r_2])}\Psi(u)}{r_2} &= \frac{2\sup\limits_{u\in\Phi^{-1}(]-\infty,r_2])}\int_0^1 [F(t,u(t))+\frac{\mu}{\lambda}G(t,u(t))]dt}{r_2}\\
&\leq \frac{2\int_0^1 \sup\limits_{|x|\leq \bar{S}\gamma_2\sqrt{K}} F(t,x)dt + 2\frac{\mu}{\lambda}G^{\gamma_2}}{\frac{\gamma_2^2}{2}(1-L\bar{S}^2)K}\\
&< \frac{1}{\lambda} < \frac{2}{3}\frac{\int_{\bar{t}}^{\hat{t}} F(t,\gamma)dt + \frac{\mu}{\lambda}G_\gamma}{\frac{\gamma^2}{2}(1+L\bar{S}^2)K + \int_0^{e(1)\gamma} h(x)dx}\\
&\leq \frac{2}{3}\frac{\Psi(w)}{\Phi(w)}.
\end{aligned}$$

Therefore, (b_1) and (b_2) of Theorem 2.2 are satisfied. Finally, we prove that $\Phi - \lambda\Psi$ satisfies the assumption *2.* of Theorem 2.2. For this, let u_1 and u_2 be two local minima for $\Phi-\lambda\Psi$. Then u_1 and u_2 are critical points for $\Phi-\lambda\Psi$, and so, they are weak solutions for the problem (1.1). We want to prove that they are nonnegative. Let u_0 be a weak solution of problem (1.1). Arguing by a contradiction, assume that the set $A = \big\{t \in]0,1] : u_0(t) < 0\big\}$ is non-empty and of positive measure. Put $\bar{v}(t) = \min\{0, u_0(t)\}$ for all $t \in [0,1]$. Clearly, $\bar{v} \in X$ and one has

$$\int_0^1 u_0''(t)\bar{v}''(t)dt + h(u_0(1))\bar{v}(1) - \int_0^1 p(u_0(t))\bar{v}(t)dt$$

$$-\lambda \int_0^1 f(t,u_0(t))\bar{v}(t)dt - \mu \int_0^1 g(t,u_0(t))\bar{v}(t)dt = 0.$$

Thus, from our sign assumptions on the data we have

$$0 \leq (1 - L\bar{S}^2) \int_A |u_0''(t)|^2 dt \leq \int_A u_0''(t)u_0''(t)dt + h(u_0(1))u_0(1)$$

$$- \int_A p(u_0(t))u_0(t)dt \leq 0.$$

Hence, since $1 - L\bar{S}^2 > 0$, $u_0 = 0$ in A and this is absurd. Then, we observe $u_1(t) \geq 0$ and $u_2(t) \geq 0$ for every $t \in [0,1]$. Thus, it follows that $su_1 + (1-s)u_2 \geq 0$ for all $s \in [0,1]$, and that

$$(\lambda f + \mu g)(x, su_1 + (1-s)u_2) \geq 0,$$

and so, $\Psi(su_1 + (1-s)u_2) \geq 0$, for every $s \in [0,1]$. From Theorem 2.2, for every

$$\lambda \in \left] \frac{3}{2}\frac{\Phi(w)}{\Psi(w)}, \ \min\left\{ \frac{r_1}{\sup\limits_{u \in \Phi^{-1}(]-\infty, r_1[)} \Psi(u)}, \ \frac{r_2/2}{\sup\limits_{u \in \Phi^{-1}(]-\infty, r_2[)} \Psi(u)} \right\} \right[,$$

the functional $\Phi - \lambda\Psi$ has at least three distinct critical points which are the generalized solutions of the problem (1.1) and we have the result. □

Remark 3.1 If either $f(t,0) \neq 0$ for some $t \in (0,1)$ or $g(t,0) \neq 0$ for some $t \in (0,1)$, or both are true, the solutions ensured by Theorem 3.1 are nontrivial and the solutions ensured by Theorem 3.2 are positive.

We now present the following examples to illustrate Theorems 3.1 and 3.2, respectively.

Example 3.1 Consider the problem

$$\begin{cases} u^{(4)} = \lambda f(u) + 2\mu u + sin(u), & 0 < t < 1, \\ u(0) = u'(0) = 0, \\ u''(1) = 0, u'''(1) = h(u(1)) \end{cases} \tag{3.10}$$

where

$$f(x) = \begin{cases} 20x^{19}, \text{ if } x < 1, \\ \frac{20}{x^2}, \quad \text{if } 1 \leq x. \end{cases}$$

By the expression of f we have

$$F(x) = \begin{cases} x^{20}, \quad \text{if } x < 1, \\ 21 - \frac{20}{x}, \text{ if } 1 \leq x. \end{cases}$$

Choose $\bar{t} = \frac{8}{10}$, $\hat{t} = \frac{9}{10}$, $\gamma = 1$, $d(t) = \frac{75}{16}t^2 - \frac{250}{64}t^3$, $e(t) = \frac{20}{9}t - \frac{10^2}{9^2}t^2$ for all $t \in [0,1]$ and $h(s) = 2s$ for every $s \in \mathbb{R}$. By simple calculations, we obtain

$$K = \frac{12 \times 5^3 \times 9^4 + 4 \times 40^3}{4^3 \times 9^4}.$$

We easily see that all conditions in Theorem 3.1 are satisfied. Therefore, since $G_1 = 0$ it follows that for each

$$\lambda \in \left]5(1+\bar{S}^2)K + \frac{2^6 \times 10^3}{9^4}, \frac{(1-\bar{S}^2)K}{2\bar{S}^{20}K^{10}}\right[$$

and for every

$$\mu \in \left[0, \min\left\{\frac{\frac{1}{2}(1-\bar{S}^2)K - \lambda\bar{S}^{20}K^{10}}{\bar{S}^2K}, \frac{(1-\bar{S}^2)K}{2}\right\}\right[,$$

the problem (3.10) admits at least three distinct classical solutions in X.

Example 3.2 Consider the problem

$$\begin{cases} u^{(4)} = \lambda f(u) + \mu e^u + \arctan(u), & 0 < t < 1, \\ u(0) = u'(0) = 0, \\ u''(1) = 0, u'''(1) = h(u(1)) \end{cases} \tag{3.11}$$

where

$$f(x) = \begin{cases} \frac{25}{16}x^2, & \text{if } x \leq \frac{4}{5}, \\ 1, & \text{if } \frac{4}{5} < x \leq \frac{9}{10}, \\ \frac{10^2}{81}x^2, & \text{if } \frac{9}{10} < x \leq 1, \\ \frac{10^2}{81}\frac{1}{x^2}, & \text{if } 1 < x. \end{cases}$$

Direct calculation gives

$$F(x) = \begin{cases} \frac{25}{48}x^3, & \text{if } x \leq \frac{4}{5}, \\ \frac{9}{10}x - \frac{8}{15}, & \text{if } \frac{4}{5} < x \leq \frac{9}{10}, \\ \frac{10^2}{3\times 9^2}x^3 - \frac{7}{300}, & \text{if } \frac{9}{10} < x \leq 1, \\ -\frac{10^2}{81x} + \frac{39433}{3\times 90^2}, & \text{if } 1 < x. \end{cases}$$

Choosing $\bar{t} = \frac{8}{10},\ \hat{t} = \frac{9}{10},\ d(t) = \frac{3\times 5^2}{4^2}t^2 - \frac{2\times 5^3}{4^3}t^3,\ e(t) = \frac{20}{9}t - \frac{10^2}{9^2}t^2$ for all $t \in [0,1]$, simple calculations give

$$K = \frac{12 \times 5^3 \times 9^4 + 4 \times 40^3}{4^3 \times 9^4}.$$

We clearly observe that all assumptions of Theorem 3.2 are satisfied by choosing $\gamma = 1,\ \gamma_1 = 10^{-2},\ \gamma_2 = 10^2$ and $h(s) = 2s$ for all $s \in \mathbb{R}$. Therefore, it follows that for each

$$\lambda \in \left]\frac{3}{2}\frac{\frac{1}{2}(1+\bar{S}^2)K + \frac{2^6\times 10^2}{9^4}}{\frac{10}{3\times 9^2} - \frac{7}{3000}}, \frac{1}{2}(1-\bar{S}^2)K\frac{48\times 10^2}{25\bar{S}^3K^{\frac{3}{2}}}\right[.$$

and for every

$$\mu \in \left[0, \min\left\{\frac{10^{-4}(1-\bar{S}^2)K - \lambda\dfrac{25\bar{S}^3K^{\frac{3}{2}}}{48\times 10^6}}{2e^{10^{-2}}}, \frac{10^4(1-\bar{S}^2)K - 4\lambda\left(-\dfrac{1}{81\bar{S}\sqrt{K}} + \dfrac{39433}{3\times 90^2}\right)}{4e^{10^2}}\right\}\right[$$

the problem (3.11) admits at least three distinct classical solutions u_i for $i = 1, 2, 3$, such that

$$0 \leq u_i(t) < 10^2\bar{S}\sqrt{K}, \quad \forall\ t \in [0,1],\ (i = 1,2,3).$$

A special case of Theorem 3.1 is the following theorem.

Theorem 3.3 *Let $f : \mathbb{R} \to \mathbb{R}$ be a nonnegative continuous function and $p : \mathbb{R} \to \mathbb{R}$ be a Lipschitz continuous function. Put $F(x) := \int_0^x f(\xi)d\xi$ for each $x \in \mathbb{R}$. Assume that*

$$\liminf_{x\to 0} \frac{F(x)}{x^2} = \limsup_{|x|\to\infty} \frac{F(x)}{x^2} = 0.$$

Then, there is $\lambda^* > 0$ such that for each $\lambda > \lambda^*$ and for every continuous function $g : \mathbb{R} \to \mathbb{R}$ satisfying the asymptotical condition

$$\limsup_{|x|\to\infty} \frac{\int_0^x g(s)ds}{x^2} < +\infty,$$

there exists $\delta > 0$ such that, for each $\mu \in [0, \delta[$, the problem

$$\begin{cases} u^{(4)} = \lambda f(u) + \mu g(u) + p(u), & 0 < t < 1, \\ u(0) = u'(0) = 0, \\ u''(1) = 0, u'''(1) = h(u(1)) \end{cases} \tag{3.12}$$

admits at least three generalized solutions.

Proof. Fix $\lambda > \lambda^* := \dfrac{\frac{\gamma^2}{2}(1 + L\bar{S}^2)K + \int_0^{e(1)\gamma} h(x)dx}{(\hat{t} - \bar{t})F(\gamma)}$ for some $\gamma > 0$. From the condition

$$\liminf_{\xi\to 0} \frac{F(\xi)}{\xi^2} = 0,$$

there is a sequence $\{\gamma_n\} \subset]0, +\infty[$ such that $\lim_{n\to\infty} \gamma_n = 0$ and

$$\lim_{n\to\infty} \frac{\sup_{|\xi|\le \bar{S}\gamma_n\sqrt{K}} F(\xi)}{\gamma_n^2} = 0.$$

Indeed, one has

$$\lim_{n\to\infty} \frac{\sup_{|\xi|\le \bar{S}\gamma_n\sqrt{K}} F(\xi)}{\gamma_n^2} = \lim_{n\to\infty} \frac{F(\xi_{\gamma_n})}{\xi_{\gamma_n}^2} \frac{\xi_{\gamma_n}^2}{\gamma_n^2} = 0,$$

where $F(\xi_{\gamma_n}) = \sup_{|\xi|\le \bar{S}\gamma_n\sqrt{K}} F(\xi)$. Hence, there exists $\overline{\gamma} > 0$ such that

$$\frac{\sup_{|\xi|\le \bar{S}\overline{\gamma}\sqrt{K}} F(\xi)}{\overline{\gamma}^2} < \min\left\{ \frac{\frac{1}{2}(1 - L\bar{S}^2)K(\hat{t} - \bar{t})F(\gamma)}{\frac{\gamma^2}{2}(1 + L\bar{S}^2)K + \int_0^{e(1)\gamma} h(x)dx}; \ \frac{\frac{1}{2}(1 - L\bar{S}^2)K}{\lambda} \right\}.$$

Applying Theorem 3.1 we have the conclusion. □

Moreover, the following result is a consequence of Theorem 3.2.

Theorem 3.4 *Let $f : \mathbb{R} \to \mathbb{R}$ be a nonnegative continuous function and $p : \mathbb{R} \to \mathbb{R}$ be a Lipschitz continuous function with the Lipschitz constant $L > 0$ such that*

$$L\bar{S}^2 < \frac{1}{43}\left(7 - \frac{18}{\alpha}\left(\frac{980^2}{99^4}\right)\right)$$

where $\alpha = 20(\frac{24\times 99^4 + 190^3}{19^3 \times 99^4})$. Assume that

$$\lim_{x\to 0^+} \frac{f(x)}{x} = 0$$

and

$$\int_0^{\frac{\bar{S}}{6}\sqrt{200\alpha}} f(\xi)d\xi < \frac{\alpha(1 - L\bar{S}^2)}{27(\alpha(1 + L\bar{S}^2) + \frac{980^2}{99^4})} \int_0^{\frac{1}{10}} f(\xi)d\xi.$$

Then, for every $\lambda \in \left]\frac{\frac{75}{2}(\alpha(1 + L\bar{S}^2) + (\frac{980^2}{99^4}))}{\int_0^{\frac{1}{10}} f(\xi)d\xi}, \frac{\frac{25}{18}\alpha(1 - L\bar{S}^2)}{\int_0^{\frac{\bar{S}}{5}\sqrt{200\alpha}} f(\xi)d\xi}\right[$ and for every continuous function $g : \mathbb{R} \to \mathbb{R}$, there exists $\delta > 0$ such that, for each $\mu \in [0, \delta[$, the problem (3.12) admits at least three generalized solutions.

Proof. Our aim is to employ Theorem 3.2 by choosing $\bar{t} = \frac{95}{100}$, $\hat{t} = \frac{99}{100}$, $\gamma = \frac{1}{10}, \gamma_2 = \frac{1}{6}$, $d(t) = \frac{3\times 20^2}{19^2}t^2 - \frac{2\times 20^3}{19^3}t^3$, $e(t) = \frac{200}{99}t - \frac{100^2}{99^2}t^2$ for all $t \in [0,1]$ and $h(s) = 2s$ for every $s \in \mathbb{R}$. Simple calculations show that

$$\frac{3}{2}\frac{\frac{\gamma^2}{2}(1+L\bar{S}^2)K + \int_0^{e(1)\gamma} h(x)dx}{\int_{\bar{t}}^{\hat{t}} F(t,\gamma)dt} = \frac{\frac{75}{2}(\alpha(1+L\bar{S}^2) + \frac{980^2}{99^4})}{\int_0^{\frac{1}{10}} f(\xi)d\xi}$$

and

$$\frac{1}{2}(1-L\bar{S}^2)K\frac{\gamma_2^2}{2\int_0^1 \sup_{|x|\leq \bar{S}\gamma_2\sqrt{K}} F(t,x)dt} = \frac{\frac{25}{18}\alpha(1-L\bar{S}^2)}{\int_0^{\frac{\bar{S}}{6}\sqrt{200\alpha}} f(\xi)d\xi}.$$

Moreover, since $\lim_{x\to 0^+} \frac{f(x)}{x} = 0$, one has

$$\lim_{x\to 0^+} \frac{\int_0^x f(\xi)d\xi}{x^2} = 0.$$

Then, there exists a positive constant $\gamma_1 < \sqrt{\frac{\alpha(1-L\bar{S}^2)+\frac{980^2}{99^4}}{200\alpha(1-L\bar{S}^2)}}$ such that

$$\frac{\int_0^{\bar{S}\gamma_1\sqrt{K}} f(\xi)d\xi}{\gamma_1^2} < \frac{100\alpha(1-L\bar{S}^2)\int_0^{\frac{1}{10}} f(\xi)d\xi}{\frac{75}{2}(\alpha(1+L\bar{S}^2) + \frac{980^2}{99^4})}$$

and

$$\frac{\gamma_1^2}{\int_0^{\bar{S}\gamma_1\sqrt{K}} f(\xi)d\xi} > \frac{1}{72\int_0^{\frac{\bar{S}}{10}\sqrt{200\alpha}} f(\xi)d\xi}.$$

Finally, we easily observe that all assumptions of Theorem 3.2 are satisfied, and it follows the result. □

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