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On almost pseudo-conharmonically symmetric Ricci-recurrent manifolds

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Abstract The object of the paper is to study almost pseudo-conharmonically symmetric Ricci-recurrent manifolds. Some geometric properties have been studied. We also study an almost pseudo-conharmonically symmetric Ricci-recurrent spacetime. We prove that such a spacetime with non-zero constant scalar curvature becomes a quasi-Einstein spacetime and we obtain Segre'characteristic of such a spacetime. Finally, we investigate the possibility of a fluid almost pseudo-conharmonically symmetric Ricci-recurrent spacetime to admit heat flux.

Keywords Almost pseudo-conharmonically symmetric manifold · Ricci-recurrent manifold · Einstein field equations · Quasi-Einstein manifold · Segre'characteristic

Mathematics Subject Classification (2010) 53C15 · 53C25 · 53B20 · 53B30 · 53B15

1 Introduction

Spaces admitting some sense of symmetry play an important role in general relativity. Without imposing a symmetry condition, solving the Einstein's field equations of gravitation becomes a difficult, if not impossible task. The study of Riemannian symmetric spaces was initiated in the late twenties by E. Cartan [4] who, in particular, obtained a classification of those spaces.

Let (M, g) be a Riemannian manifold of dimension n, $(n \geq 3)$, and let ∇ be its Levi-Civita connection. If a curvature tensor R of (M, g) is parallel with respect to its Levi-Civita connection ∇, that is, $\nabla R = 0$, then this manifold is called *locally symmetric* [4]. This condition of local symmetry is equivalent to the fact that at every point $p \in M$, the local geodesic symmetry $F(p)$ is an isometry [16]. The class of Riemannian symmetric manifolds is a very natural generalization of the class of manifolds of constant curvature. The same can be extended to the class of semi-Riemannian manifolds, where g is of arbitrary signature. The notion of a locally

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symmetric manifold has been weakened by many authors in several ways ([26], [27], [28], etc.)

A non-flat or semi-Riemannian manifold $(M, g), (n \geq 2)$ is said to be *a pseudo-symmetric manifold* [2], if its curvature tensor R satisfies the condition

$$\begin{aligned}(\nabla_X R)(Y,Z)W = {} & 2A(X)R(Y,Z)W + A(Y)R(X,Z)W \\ & + A(Z)R(Y,X)W \\ & + A(W)R(Y,Z)X + g(R(Y,Z)W,X)P, \end{aligned} \tag{1.1}$$

where A is a non-zero differential 1-form such that (1.1)

$$g(X,P) = A(X), \tag{1.2}$$

for all vector fields X; P being the vector field corresponding to the associated 1-form A and ∇ denotes the operator of covariant differentiation with respect to the metric tensor g. If $A = 0$, then the manifold reduces to a symmetric manifold in the sense of Cartan. An $n-$dimensional pseudo-symmetric manifold is denoted by $(PS)_n$. The class of pseudo symmetric manifolds arose during the study of conformally flat space of class one [23]. It is to be noted that the notion of a pseudo-symmetric manifold studied in particular by Deszcz [11] is different from that of Chaki [2]. To generalize the notion of a pseudo-symmetric manifold, De and Gazi [5] introduced the notion of an almost pseudo-symmetric manifold which is defined as follows:

A non-flat Riemannian manifold $(M, g)(n \geq 2)$ is called *an almost pseudo-symmetric* if the curvature tensor R satisfies the condition

$$\begin{aligned}(\nabla_X R)(Y,Z)W = {} & [A(X) + B(X)]\, R(Y,Z)W + A(Y)R(X,Z)W \\ & + A(Z)R(Y,X)W \\ & + A(W)R(Y,Z)X + g(R(Y,Z)W,X)P, \end{aligned} \tag{1.3}$$

where A and B are two non-zero 1-forms, defined by

$$g(X,P) = A(X) \text{ and } g(X,Q) = B(X) \tag{1.4}$$

for all X. Here the vector fields P and Q are called the basic vector fields of the manifold corresponding to the 1-forms A and B respectively. Such a manifold is denoted by $(APS)_n$. If $B = A$, then $(APS)_n$ reduces to $(PS)_n$.

The Riemannian manifolds with metrics g and $\widetilde{g}$ related through the relation $\widetilde{g} = e^{2\phi}g$, where ϕ is real valued differentiable function of coordinates, are said to be *conformal manifolds* and the correspondence between them is conformal transformation. It is well known that the conformal transformations do not change the angle between two vectors at a point. But a harmonic function, which is defined by vanishing Laplacian, is not transformed into a harmonic function by the conformal transformation in general. The condition under which a harmonic function remains invariant was studied by Ishii [14] who introduced the conharmonic transformation as a subgroup of conformal transformation satisfying $\phi^i_{,i} + \phi_{,i}\phi^i = 0$ and defined the conharmonic curvature tensor a geometrical invariant under conharmonic transformation.

The conharmonic curvature tensor $\widetilde{K}$ of type $(0,4)$ of a Riemannian manifold (M, g) $(n > 3)$ is given by [14],

$$\begin{aligned}\widetilde{K}(Y,Z,W,V) = {} & \widetilde{R}(Y,Z,W,V) - \tfrac{1}{n-2}\left[S(Z,W)g(Y,V)\right. \\ & \left. - S(Y,W)g(Z,V) + S(Y,V)g(Z,W) - S(Z,V)g(Y,W)\right], \end{aligned} \tag{1.5}$$

where $\widetilde{R}$ is the curvature tensor of type $(0,4)$ such that $\widetilde{R}(Y,Z,W,V)=g(R(Y,Z)W,V)$, S is the Ricci tensor of the manifold type of $(0,2)$ and L is the symmetric endomorphism corresponding to the Ricci tensor S, that is,

$$S(X,Y)=g(LX,Y). \tag{1.6}$$

Thus, from (1.5) and $\widetilde{K}(Y,Z,W,V)=g(K(Y,Z)W,V)$, the conharmonic curvature tensor K of type $(1,3)$ is the following form:

$$\begin{aligned} K(Y,Z)W = R(Y,Z)W - \frac{1}{n-2}\left[S(Z,W)Y - S(Y,W)Z\right. \\ \left. +g(Z,W)LY - g(Y,W)LZ\right]. \end{aligned} \tag{1.7}$$

Shaikh and Hui [24] showed the conharmonic curvature tensor K satisfies all the symmetry properties of the Riemannian curvature tensor R. The conharmonic curvature tensor K has many applications in physical field. In [1], Abdussattar showed its physical significance in general relativity. Kumar and Srivastava [18] studied FRW-cosmological model for conharmonically flat spacetime. This tensor has also been studied by Siddiqui and Ahsan [25], Ghosh, De and Taleshian [12], Kirichenko and Shinab [17] and many other.

A non-flat Riemannian manifold $(M,g)(n\geq 2)$ is called *an almost pseudo conharmonically symmetric manifold* [29] which is defined by the condition

$$\begin{aligned} \left(\nabla_X\widetilde{K}\right)(Y,Z,W,V) = {} & [A(X)+B(X)]\,\widetilde{K}(Y,Z,W,V) + A(Y)\widetilde{K}(X,Z,W,V) \\ & +A(Z)\widetilde{K}(Y,X,W,V) + A(W)\widetilde{K}(Y,Z,X,V) \\ & +A(V)\widetilde{K}(Y,Z,W,X). \end{aligned}$$

This condition for the conharmonic curvature tensor K of type $(1,3)$ is the following form:

$$\begin{aligned} (\nabla_X K)(Y,Z)W = {} & [A(X)+B(X)]\,K(Y,Z)W + A(Y)K(X,Z)W \\ & +A(Z)K(Y,X)W \\ & +A(W)K(Y,Z)X + g(K(Y,Z)W,X)P, \end{aligned} \tag{1.8}$$

where A and B have the meaning already mentioned. Such an n-dimensional manifold is denoted by $A(PCHS)_n$.

The notion of Ricci-recurrent manifolds was introduced by Patterson [19] in 1952. A non-flat Riemannian manifold $(M,g)(n>2)$ is called Ricci-recurrent manifold if its Ricci tensor S of type $(0,2)$ is non-zero and satisfies the condition

$$(\nabla_X S)(Y,Z) = T(X)S(Y,Z), \tag{1.9}$$

where T is a non-zero 1-form, called *the 1-form of recurrence* and defined by

$$g(X,\mu)=T(X). \tag{1.10}$$

If $T=0$, then the manifold reduces to a Ricci symmetric manifold. In 1966, Roter [21] showed that in a Ricci-recurrent manifold the following relations hold:

$$S(R(X,Y)Z,W)+S(R(X,Y)W,Z)=0, \tag{1.11}$$

$$S(X, LY) = \frac{r}{2} S(X, Y). \tag{1.12}$$

In 2000, Chaki and Maity [3] introduced the notion of a quasi-Einstein manifold. A non-flat Riemannian manifold $(M, g)(n > 2)$ is called *a quasi-Einstein manifold* if its Ricci tensor S of type $(0, 2)$ is non-zero and satisfies the condition

$$S(X, Y) = ag(X, Y) + bD(X)D(Y), \tag{1.13}$$

where a and b are non-zero scalars, η is a unit vector and D is a non-zero 1-form such that $g(X, \eta) = D(X)$, for all vector fields X. Quasi-Einstein manifolds arouse during the study exact solutions of the Einstein field equations as well as during consideration of quasi-umbilical hypersurfaces of semi-Euclidean spaces. For instance, the Robertson-Walker spacetimes are quasi-Einstein manifolds. In 2004, De and Ghosh [6] showed that quasi-Einstein manifolds can be taken as a model of the perfect fluid spacetime in general relativity. So quasi-Einstein manifolds have some importance in the general relativity. The study of quasi-Einstein manifolds have been continued by De and Ghosh [7], [8], Debnath and Konar [9], Guha [13], Demirbağ, Yılmaz, Uysal and Zengin [10] and Jana and Shaikh [15] and many others.

In the present paper we have studied $A(PCHS)_n$ Ricci-recurrent manifolds, $n > 3$. The paper is organized as follows:

In section 2 it is shown that in an $A(PCHS)_n$ Ricci-recurrent manifold if $r \neq 0$ and $A = \varphi E$, $E = 2A + B$ and φ is a non-zero scalar function, then $\frac{r}{2}$ is an eigenvalue corresponding to the eigenvector ρ. Next we prove that at every point p of an $A(PCHS)_n$ Ricci-recurrent manifold, $n > 3$, if $A = \varphi E$, $E = 2A + B$ and φ is a non-zero scalar function, either the scalar curvature at that point is zero or $T = \beta E$, where $\beta = \frac{(n-3)(n-2\varphi-4)}{(n^2-7n+11)}$. Then we show that in an $A(PCHS)_n$ Ricci-recurrent manifold, $n > 3$, with non-zero scalar curvature and $A = \varphi E$, the vector field ρ defined by $g(X, \rho) = E(X)$ is irrotational and the integral curves of the vector field ρ are geodesics. In the next section we consider in an $A(PCHS)_n$ Ricci-recurrent spacetime, $n > 3$. we prove that an $A(PCHS)_n$ Ricci-recurrent perfect fluid spacetime of non-zero constant curvature becomes a quasi-Einstein spacetime and we get Segre' characteristic of such a spacetime. In the last section we investigate the possibility of a fluid $A(PCHS)_n$ Ricci-recurrent spacetime to admit heat flux.

2 $A(PCHS)_n$ Ricci-recurrent manifolds, $n > 3$

Lemma 2.1 *In an $A(PCHS)_n$ Ricci-recurrent manifold if $r \neq 0$ and $A = \varphi E$, $E = 2A + B$ and φ is a non-zero scalar function, then $\frac{r}{2}$ is an eigenvalue corresponding to the eigenvector ρ.*

Proof. Contracting (1.8) with respect to X we obtain

$$(divK)(Y, Z)W = E(K(Y, Z)W) + \frac{r}{n-2}\{A(Z)g(Y, W) - A(Y)g(Z, W)\}, \tag{2.1}$$

where E is a non-zero 1-form and ρ is its corresponding vector field such that

$$E(X) = 2A(X) + B(X) \text{ and } E(X) = g(X, \rho), \tag{2.2}$$

for all vector fields X. From (1.7), it follows that

$$(divK)(Y,Z)W = \frac{n-3}{n-2}\left[\{(\nabla_Y S)(Z,W) - (\nabla_Z S)(Y,W)\} - \frac{1}{2(n-3)}\{dr(Y)g(Z,W) - dr(Z)g(Y,W)\}\right], \tag{2.3}$$

by virtue of (1.9), the equation (2.3) can be write

$$(divK)(Y,Z)W = \frac{n-3}{n-2}\left[\{T(Y)S(Z,W) - T(Z)S(Y,W)\} - \frac{1}{2(n-3)}\{dr(Y)g(Z,W) - dr(Z)g(Y,W)\}\right]. \tag{2.4}$$

Contracting (1.9) with respect to Y and Z we have

$$dr(X) = T(X)r, \tag{2.5}$$

and contracting (1.9) with respect to Z gives

$$(\nabla_X L)(Y) = T(X)LY. \tag{2.6}$$

Again contracting X in (2.6), we obtain

$$(divL)(Y) = T(LY). \tag{2.7}$$

Since $(divL)(Y) = \frac{1}{2}dr(Y)$ [20], with help of (2.5) and (2.7), we have

$$T(LY) = \frac{1}{2}dr(Y) = \frac{r}{2}T(Y). \tag{2.8}$$

Now, due to (2.1) and (2.4), we get

$$\begin{aligned} &E(K(Y,Z)W) + \frac{r}{n-2}\{A(Z)g(Y,W) - A(Y)g(Z,W)\} \\ &= \frac{n-3}{n-2}\left[\{T(Y)S(Z,W) - T(Z)S(Y,W)\} - \frac{1}{2(n-3)}\{dr(Y)g(Z,W) - dr(Z)g(Y,W)\}\right]. \end{aligned} \tag{2.9}$$

By virtue of (2.2), $W = \rho$ in (2.9) implies

$$\frac{r}{n-2}\{A(Z)E(Y) - A(Y)E(Z)\} = \frac{n-3}{n-2}\left[\{T(Y)S(Z,\rho) - T(Z)S(Y,\rho)\} - \frac{1}{2(n-3)}\{dr(Y)E(Z) - dr(Z)E(Y)\}\right], \tag{2.10}$$

since

$$E(K(Y,Z)\rho) = g(K(Y,Z)\rho,\rho) = 0. \tag{2.11}$$

If considering $A = \varphi E$, then the left hand of the equation (2.10) equals to zero. Thus, it follows from (2.10) that

$$\frac{n-3}{n-2}\left\{T(Y)S(Z,\rho)-T(Z)S(Y,\rho)\right\} = \frac{1}{2(n-3)}\left\{dr(Y)E(Z) - dr(Z)E(Y)\right\}. \quad (2.12)$$

and using (2.5) we get

$$\begin{aligned} &T(Y)\left\{\frac{(n-3)}{(n-2)}S(Z,\rho) - \frac{r}{2(n-3)}E(Z)\right\} = \\ &= T(Z)\left\{\frac{(n-3)}{(n-2)}S(Y,\rho) - \frac{r}{2(n-3)}E(Y)\right\}. \end{aligned} \quad (2.13)$$

Putting $Y = LY$ in (2.13) and using (1.12) and (2.8) we get

$$\begin{aligned} &\frac{r}{2}T(Y)\left\{\frac{(n-3)}{(n-2)}S(Z,\rho) - \frac{r}{2(n-3)}E(Z)\right\} = \\ &= \frac{r}{2}T(Z)\left\{\frac{(n-3)}{(n-2)}S(Y,\rho) - \frac{1}{(n-3)}E(LY)\right\}, \end{aligned} \quad (2.14)$$

which implies either $r = 0$, or

$$\begin{aligned} &T(Y)\left\{\frac{(n-3)}{(n-2)}S(Z,\rho) - \frac{r}{2(n-3)}E(Z)\right\} = \\ &= T(Z)\left\{\frac{(n-3)}{(n-2)}S(Y,\rho) - \frac{1}{(n-3)}E(LY)\right\}. \end{aligned} \quad (2.15)$$

Since $T \neq 0$, (2.13) and (2.15) together with the condition $r \neq 0$ give

$$E(LY) = \frac{r}{2}E(Y), \quad (2.16)$$

namely $g(LY,\rho) = \frac{r}{2}g(Y,\rho)$. Hence we have

$$S(Y,\rho) = \frac{r}{2}g(Y,\rho). \quad (2.17)$$

This completes the proof. □

Proposition 2.2 *At every point p of an $A(PCHS)_n$ Ricci-recurrent manifold, $n > 3$, if $A = \varphi E$, $E = 2A+B$ and φ is a non-zero scalar function, either the scalar curvature at that point is zero or $T = \beta E$, where $\beta = \frac{(n-3)(n-2\varphi-4)}{(n^2-7n+11)}$.*

Proof. Putting $V = \rho$ in (1.5) and using (2.17) we get

$$\begin{aligned} E(K(X,Y)Z) = E(R(X,Y)Z) - \frac{1}{n-2}\left[S(Y,Z)E(X) - S(X,Z)E(Y)\right. \\ \left. +\frac{r}{2}E(X)g(Y,Z) - \frac{r}{2}E(Y)g(X,Z)\right], \end{aligned} \quad (2.18)$$

with help of (2.9) this equation gives

$$\begin{aligned} E(R(X,Y)Z) = & S(Y,Z)\left[\frac{(n-3)}{(n-2)}T(X)+\frac{1}{(n-2)}E(X)\right] \\ & -S(X,Z)\left[\frac{(n-3)}{(n-2)}T(Y)+\frac{1}{(n-2)}E(Y)\right] \\ & +g(Y,Z)\left[\frac{r}{(n-2)}A(X)+\frac{r}{2(n-2)}E(X)-\frac{r}{2(n-3)}T(X)\right] \\ & -g(X,Z)\left[\frac{r}{(n-2)}A(Y)+\frac{r}{2(n-2)}E(Y)-\frac{r}{2(n-3)}T(Y)\right]. \end{aligned} \tag{2.19}$$

Using the symmetric and skew-symmetric properties of the curvature tensor and from the equations **(1.11)** and (2.17) we **obtain that**

$$E(R(X,Y)LZ) = \frac{r}{2}E(R(X,Y)Z). \tag{2.20}$$

Putting $Z = LZ$ in (2.19) and using (1.12) and (2.20) we get

$$\begin{aligned} rE(R(X,Y)Z) = & rS(Y,Z)\left[\frac{(n^2-7n+11)}{(n-2)(n-3)}T(X)+\frac{2}{(n-2)}\{E(X)+A(X)\}\right] \\ & -rS(X,Z)\left[\frac{(n^2-7n+11)}{(n-2)(n-3)}T(Y)+\frac{2}{(n-2)}\{E(Y)+A(Y)\}\right] \end{aligned} \tag{2.21}$$

From the assumption, we know that

$$A(X) = \varphi E(X). \tag{2.22}$$

Then, contracting Y and Z in (2.21) and using (2.8), (2.17), (2.22), we get after simple calculation

$$r^2\left[\frac{(n^2-7n+11)}{2(n-2)(n-3)}T(X) - \frac{(n-2\varphi-4)}{2(n-2)}E(X)\right] = 0, \tag{2.23}$$

which implies that either $r = 0$ or $T(X) = \frac{(n-3)(n-2\varphi-4)}{(n^2-7n+11)}E(X)$. Hence, we can write

$$T(X) = \beta E(X), \tag{2.24}$$

where $\beta = \frac{(n-3)(n-2\varphi-4)}{(n^2-7n+11)}$. This completes the proof. □

Proposition 2.3 *In an $A(PCHS)_n$ Ricci-recurrent manifold, $n > 3$, with non-zero scalar curvature and $A(X) = \varphi E(X)$, the vector field ρ defined by $g(X,\rho) = E(X)$ is irrotational and the integral curves of the vector field ρ are geodesics.*

Proof. From Proposition 2.2, it follows that $T = \beta E$, where T is the 1-form of recurrent manifold. Since in a Ricci-recurrent manifold with non-zero scalar curvature the 1-form of recurrence T is closed, we obtain from (2.24) that 1-form E is also closed, namely $dE(X,Y) = 0$, which implies

$$(\nabla_X E)(Y) = (\nabla_Y E)(X). \tag{2.25}$$

From (2.25), it follows that

$$g(Y, \nabla_X \rho) = g(X, \nabla_Y \rho) \tag{2.26}$$

for all vector fields X, Y, which means that the vector field ρ is irrotational.

Let us now put $Y = \rho$ in (2.26). Then we get

$$g(\rho, \nabla_X \rho) = g(X, \nabla_\rho \rho). \tag{2.27}$$

Suppose ρ is a unit vector field. Then, $g(\rho, \nabla_X \rho) = 0$. Hence, from (2.27), it follows that $g(X, \nabla_\rho \rho) = 0$ for all vector fields X, which implies $\nabla_\rho \rho = 0$. This means that the integral curves of vector field ρ are geodesics. □

3 $A(PCHS)_n$ Ricci-recurrent spacetime, $n > 3$

In general relativity the matter content of spacetime is described the energy momentum tensor T which is to be determined by physical considerations dealing with the distribution of the matter and energy. Since the matter content of the universe is assumed to behave like a perfect fluid in the standard cosmological models, the physical motivation for studying Lorentzian manifolds is the assumption that a gravitational field may be effectively modeled by some defined on a suitable four-dimensional manifold M.

In this section we want to deal with the study of $A(PCHS)_n$, in general relativity by the coordinate free method of differential geometry. In this method of study the spacetime of general relativity is regarded as a connected four-dimensional semi-Riemannian manifold (M, g) with Lorentz metric g with signature $(-, +, +, +)$.

Here we consider a perfect fluid $A(PCHS)_n$ spacetime of non-zero scalar curvature and having the basic vector ρ as the timelike vector field of the fluid, that is, $g(\rho, \rho) = -1$.

For the perfect fluid spacetime, we have to the Einstein equation without cosmological constant as

$$S(X, Y) - \frac{r}{2} g(X, Y) = kT(X, Y), \tag{3.1}$$

where k is the gravitational constant, T is the energy momentum tensor of type $(0, 2)$ given by [16]

$$T(X, Y) = (\sigma + p)E(X)E(Y) + pg(X, Y), \tag{3.2}$$

with σ and p the energy density and isotropic pressure of the fluid respectively, E is a non-zero 1-form defined by $g(X, \rho) = E(X)$ for all vector fields X, ρ being the velocity vector field of the fluid.

The Einstein equations are fundamental in construction of cosmological models which imply that the matter determines the geometry of the spacetime and conversely the motion of matter is determined by the metric tensor of space which is non-flat.

It is noted that the basic geometric features of $A(PCHS)_n$ are also being maintained in the Lorenzian manifold which is necessarily a semi-Riemannian manifold. Thus Lemma 2.1 is also true for an $A(PCHS)_n$ spacetime. From (3.1) and (3.2), we get

$$S(X, Y) - \frac{r}{2} g(X, Y) = k\left[(\sigma + p)E(X)E(Y) + pg(X, Y)\right]. \tag{3.3}$$

Putting $Y = \rho$ in (3.3) and using Lemma 2.1 and $E(\rho, \rho) = -1$ yields

$$-k\sigma E(X) = 0. \tag{3.4}$$

Since $k \neq 0$ and $E \neq 0$ due to (2.2), from (3.4), it follows that $\sigma = 0$. We know that pure matter exists in case of $\sigma > 0$. Hence for non-zero scalar curvature, the spacetime under consideration can not contain pure matter.

Let us now deal with an $A(PCHS)_4$ Ricci-recurrent perfect fluid spacetime of non-zero scalar curvature with velocity vector fluid ρ obeying the Einstein's equation with cosmological constant and $A = \varphi E$. The Einstein's equation is given by

$$S(X,Y) - \frac{r}{2}g(X,Y) + \lambda g(X,Y) = k\left[(\sigma + p)E(X)E(Y) + pg(X,Y)\right], \tag{3.5}$$

where λ is the cosmological constant.

Putting $Y = \rho$ in (3.5) and using Lemma 2.1 and $E(\rho) = g(\rho, \rho) = -1$, we obtain

$$\left[\lambda + k\sigma\right] E(X) = 0. \tag{3.6}$$

Thus, it follows from $E(X) \neq 0$ that

$$\sigma = -\frac{\lambda}{k}. \tag{3.7}$$

Again taking a frame field and contracting (3.5) over X and Y we get by using (3.7)

$$p = \frac{3\lambda - r}{3k}. \tag{3.8}$$

Thus we have the following:

Theorem 3.1 *If an $A(PCHS)_4$ Ricci-recurrent perfect fluid spacetime of non-zero constant scalar curvature and $A = \varphi E$ obeys Einstein's equation with cosmological constant, then the pressure and density of the fluid are constant.*

By virtue of (3.7) and (3.8) we get

$$\sigma + p = -\frac{r}{3k}. \tag{3.9}$$

This leads the following:

Theorem 3.2 *If an $A(PCHS)_4$ Ricci-recurrent perfect fluid spacetime of non-zero scalar curvature and $A = \varphi E$ obeys Einstein's equation with cosmological constant, then the matter content can be a perfect fluid with $\sigma + p \neq 0$.*

Theorem 3.3 *If an $A(PCHS)_4$ Ricci-recurrent perfect fluid spacetime of non-zero constant scalar curvature and $A = \varphi E$ becomes a quasi-Einstein spacetime.*

Proof. We consider an $A(PCHS)_4$ Ricci-recurrent perfect fluid spacetime of non-zero constant scalar curvature and $A = \varphi E$ as a perfect fluid spacetime without cosmological constant λ in which the associated vector field ρ is the velocity vector field of the fluid, namely $g(\rho, \rho) = -1$. Now taking a frame field and contracting (3.3) over X and Y, we obtain

$$r = k(\sigma - 3p). \tag{3.10}$$

Since here $\lambda = 0$, it follows from (3.7) that

$$\sigma = 0. \tag{3.11}$$

Hence from (3.3), (3.10) and (3.11), we have

$$S(X,Y) = \frac{r}{6}g(X,Y) - \frac{r}{3}E(X)E(Y), \tag{3.12}$$

which implies that such a manifold is a quasi-Einstein manifold. □

Let us now take another eigenvalue ξ of S different from ρ. Then ξ must be orthogonal to ρ. Hence $g(\rho, \xi) = 0$, namely

$$E(\xi) = 0. \tag{3.13}$$

Putting $Y = \xi$ in (3.5) and using (3.13) we obtain

$$S(X,\xi) = \left(\frac{1}{2}r + kp - \lambda\right) g(X,\xi). \tag{3.14}$$

Using (3.8) in (3.14) we get

$$S(X,\xi) = \frac{r}{6}g(X,\xi). \tag{3.15}$$

From (3.15), it follows that $\frac{r}{6}$ is another eigenvalue of S and ξ is an eigenvector corresponding to this eigenvalue. Since for a given eigenvector there is only one eigenvalue and $\frac{r}{2}$ and $\frac{r}{6}$ are different, it follows that the Ricci tensor has only distinct eigenvalues, namely $\frac{r}{2}$ and $\frac{r}{6}$.

Let the multiplicity of $\frac{r}{2}$ be m. Since the dimension of the spacetime is 4, the multiplicity of $\frac{r}{6}$ is $(4-m)$. Hence $m\left(\frac{r}{2}\right) + (4-m)\frac{r}{6} = 0$, which gives $m = 1$. Therefore, the multiplicity of $\frac{r}{2}$ is 1 and the multiplicity of $\frac{r}{6}$ is 3. $m = 4$ implies that is only one eigenvalue $\frac{r}{2}$ of multiplicity 4. But we have proved that there exist two eigenvalues $\frac{r}{2}$ and $\frac{r}{6}$. So we cannot take $m = 4$. Hence the Segre'characteristic of S is $[(111), 1]$. This leads to the following result:

Theorem 3.4 *If an $A(PCHS)_4$ Ricci-recurrent perfect fluid spacetime of non-zero constant scalar curvature and $A = \varphi E$ is of Segre' characteristic* $[(111), 1]$.

4 Possibility of a fluid $A(PCHS)_4$ Ricci-recurrent spacetime to admit heat flux

This section deals with an $A(PCHS)_4$ Ricci-recurrent spacetime in which the matter distribution is fluid with the characteristic vector field ρ of the spacetime as the flow vector field the fluid. If possible, suppose that the matter distribution is described by the energy momentum tensor $T(X,Y)$ given by the following form

$$T(X,Y) = (\sigma + p)E(X)E(Y) + pg(X,Y) + E(X)W(Y) + E(Y)W(X), \tag{4.1}$$

where $W(X) = g(X,\eta)$ for all vector fields X, η being the heat flux vector field of the fluid orthogonal to ρ, namely $g(\rho,\eta) = 0$ [22]. Thus, we have

$$g(\rho,\eta) = E(\eta) = W(\rho) = 0. \tag{4.2}$$

From (3.6), it follows that

$$T(X,\rho) = -\sigma E(X) - W(X). \tag{4.3}$$

Putting $Y = \rho$ in (3.5) and using (2.17) and (4.3), we obtain

$$kW(X) = -(\lambda + k\sigma)E(X). \tag{4.4}$$

Since $W(\rho) = 0$, the equation (4.4) yields

$$(\lambda + k\sigma) = 0. \tag{4.5}$$

By virtue of (4.4) and (4.5) we obtain $kW(X) = 0$. Since $k \neq 0$, it follows that $W(X) = 0$ for all vector fields X. Hence in an $A(PCHS)_4$ Ricci-recurrent spacetime the matter distribution cannot be described by the energy momentum tensor of the form (4.1). Thus we can state the following:

Theorem 4.1 *If in an $A(PCHS)_4$ Ricci-recurrent spacetime the matter distribution is fluid with the characteristic vector field ρ of the spacetime as the flow vector field the fluid, then such a fluid cannot admit heat flux.*

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