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Contact 3-manifolds with Reeb flow invariant characteristic Jacobi operator

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Abstract We study contact Riemannian 3-manifolds whose characteristic Jacobi operator is invariant under Reeb flows.

Keywords Contact manifolds · characteristic Jacobi operator · 3-dimensional Lie groups

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Introduction

In a contact manifold (M, η), the Reeb vector field ξ of η generates a contact transformation, since $\pounds_\xi \eta = 0$. Here $\pounds_\xi$ is the Lie differentiation by ξ. An associated Riemannian metric g to (M, η) is said to be a K-*contact metric* if the Reeb flow is isometric with respect to it, that is, ξ is a Killing vector field with respect to g. A contact manifold (M, η) equipped with a K-contact metric is called a K-contact manifold.

Sasakian manifolds are defined as contact Riemannian manifolds which satisfy a normality condition. One can see that Sasakian manifolds are K-contact, but converse does not hold.

In case $\dim M = 3$, K-contact manifolds are Sasakian. In 3-dimensional contact Riemannian geometry, it is known that the local symmetry (parallelism of the Riemannian curvature R, *i.e.*, $\nabla R = 0$) is a strong restriction for associated metrics. In fact, Blair and Sharma showed that locally symmetric contact Riemannian 3-manifolds are of constant curvature 1 or 0 [5]. Note that local symmetry is equivalent to the

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parallelism of the Ricci operator S. Thus such a contact Riemannian 3-manifold is locally isomorphic to the unit 3-sphere $\mathbb{S}^3$ or the universal covering of the Euclidean motion group $\mathrm{E}(2)$. It should be remarked that $\mathrm{E}(2)$ is identified with the unit tangent sphere bundle $T_1\mathbb{E}^2$ of the Euclidean plane $\mathbb{E}^2$.

In the previous paper [8], the first named author investigated a milder condition on the Ricci operator S. He classified contact Riemannian 3-manifold whose Ricci operator S is invariant under the Reeb flow.

Theorem 0.1 ([8]) *Let M be a contact Riemannian 3-manifold. Then M satisfies $\pounds_\xi S = 0$ if and only if M is Sasakian or locally isomorphic to $\mathrm{SU}(2)$, $\mathrm{SL}_2\mathbb{R}$ or $\mathrm{E}(2)$ equipped with a left invariant contact Riemannian structure.*

On the other hand, in contact Riemannian geometry the self adjoint operator $\ell = R(\cdot,\xi)\xi$ plays an important role. In this paper we study Reeb flow invariance of the *characteristic Jacobi operator* ℓ on contact Riemannian 3-manifolds. We will show that the Reeb flow invariance $\pounds_\xi \ell = 0$ of ℓ is weaker than $\pounds_\xi S = 0$. The main result of this paper is the following theorem (Theorem 4.8):

Main Theorem *Let M be a contact Riemannian 3-manifold. Then M satisfies $S\xi = \sigma\xi$ for some function σ and $\pounds_\xi \ell = 0$ if and only if M is Sasakian or locally isomorphic to $\mathrm{SU}(2)$, $\mathrm{SL}_2\mathbb{R}$ or $\mathrm{E}(2)$ equipped with a left invariant contact Riemannian structure.*

In the main theorem, the condition $S\xi = \sigma\xi$ can not be removed. In fact, there exist non-unimodular Lie groups equipped with left invariant contact Riemannian structure which satisfy $\pounds_\xi \ell = 0$ but not $S\xi = \sigma\xi$.

Throughout this paper, all manifolds are assumed to be smooth and connected.

1 Preliminaries

1.1. Let (M, g) be a Riemannian manifold with Riemannian curvature

$$R(X,Y) = \nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]}, \quad X, Y \in \mathfrak{X}(M),$$

where $\mathfrak{X}(M)$ denotes the Lie algebra of all vector fields on M. Then (M, g) is said to be *pseudo-symmetric* if there exists a function L such that

$$R(X,Y)\cdot R = L(X\wedge Y)\cdot R$$

holds for all X and $Y \in \mathfrak{X}(M)$. The curvature-like tensor field $(X \wedge Y)$ is defined by

$$(X\wedge Y)Z = g(Y,Z)X - g(Z,X)Y.$$

In particular, a pseudo-symmetric Riemannian manifold is called a *pseudo-symmetric space of constant type* if L is constant.

On a Riemannian 3-manifold (M, g), the Riemannian curvature R is described by the Ricci tensor field Ric and corresponding *Ricci operator* S by

$$\begin{aligned} R(X,Y)Z =& \mathrm{Ric}(Y,Z)X - \mathrm{Ric}(Z,X)Y \\ &+ g(Y,Z)SX - g(Z,X)SY - \frac{\tau}{2}(X\wedge Y)Z \end{aligned}$$

for all vector fields X, Y and Z on M. Here $\tau = \mathrm{tr}\, S$ is the *scalar curvature*. From this, the following characterizations of pseudo-symmetry is deduced.

Proposition 1.1 *A Riemannian* 3*-manifold is a pseudo-symmetric space of constant type with* $R(X,Y)\cdot R = L(X\wedge Y)\cdot R$ *if and only if the principal Ricci curvatures* (*eigenvalues of the Ricci tensor*) *locally satisfy the following relations* (*up to numeration*):

$$\rho_1 = \rho_2, \quad \rho_3 = 2L.$$

Proposition 1.2 *A Riemannian* 3*-manifold* (M,g) *is pseudo-symmetric if and only if it is quasi-Einstein. This means that there exists a one-form* ω *such that the Ricci tensor field* Ric *has the form*:

$$\mathrm{Ric} = a\,g + b\,\omega\otimes\omega.$$

Here a *and* b *are functions.*

2 Contact Riemannian 3-manifolds

2.1. A 3-dimensional manifold M is said to be a *contact manifold* if it admits a 1-form η satisfying $\eta\wedge(d\eta)\neq 0$. Such a 1-form is called a *contact form.* On a contact 3-manifold (M,η), there is a unique vector field ξ such that $\eta(\xi)=1$ and $d\eta(\xi,\cdot)=0$. The vector field ξ is called the *characteristic vector field* of (M,η). The characteristic vector field ξ is also called the *Reeb vector field* of (M,η).

Moreover (M,η) admits a Riemannian metric g, an endomorphism field φ such that

$$\eta(X) = g(X,\xi), \quad d\eta(X,Y) = g(X,\varphi Y), \quad \varphi^2 = -I + \eta\otimes\xi, \tag{2.1}$$

for all vector fields X and Y on M. From (2.1) it follows that

$$\varphi\xi = 0, \quad \eta\circ\varphi = 0, \quad g(\varphi X,\varphi Y) = g(X,Y) - \eta(X)\eta(Y), X,Y\in\mathfrak{X}(M). \tag{2.2}$$

A contact 3-manifold (M,η) equipped with the structure tensor (φ,ξ,g) satisfying (2.2) is called a *contact Riemannian* 3*-manifold* and denoted by $M=(M,\varphi,\xi,\eta,g)$.

2.2. Given a contact Riemannian 3-manifold M, following Blair [1], we define an endomorphism field h on M by $h = \pounds_\xi\varphi/2$. Then we observe that h is self-adjoint with respect to g and satisfies

$$h\xi = 0, \quad h\varphi = -\varphi h, \tag{2.3}$$

$$\nabla_X\xi = -\varphi X - \varphi hX, \quad X\in\mathfrak{X}(M). \tag{2.4}$$

Here ∇ is the Levi-Civita connection of g. From (2.3)–(2.4), we see that each trajectory of ξ is a geodesic. Along trajectories of ξ, the *characteristic Jacobi opeartor* ℓ defined by $\ell(X) = R(X,\xi)\xi$ is a self-adjoint endomorphism field, that is, $g(\ell X,Y) = g(X,\ell Y)$. We have

$$\mathrm{trace}\,\ell = \mathrm{Ric}\,(\xi,\xi) = 2 - \mathrm{trace}\,(h^2), \tag{2.5}$$

$$\nabla_\xi h = \varphi - \varphi\ell - \varphi h^2, \tag{2.6}$$

$$g(R(X,Y)\xi,Z) = g((\nabla_Z\varphi)X,Y) + g((\nabla_Y(\varphi h)X - (\nabla_X(\varphi h)Y,Z) \tag{2.7}$$

for all X, Y, $Z\in\mathfrak{X}(M)$.

2.3. A contact Riemannian 3-manifold is said to be *K-contact* if ξ is a Killing vector field. It is easy to see that a contact Riemannian 3-manifold is K-contact if and only if $h = 0$.

On the other hand, a contact Riemannian 3-manifold M is said to be a *Sasakian 3-manifold* if it is *normal*. Here a contact Riemannian 3-manifold M is said to be normal if it satisfies

$$[\varphi, \varphi] + 2d\eta \otimes \xi = 0,$$

where $[\varphi, \varphi]$ is the Nijenhuis torsion of φ. A Sasakian manifold is characterized by a condition

$$(\nabla_X \varphi)Y = g(X, Y)\xi - \eta(Y)X, \quad X, Y \in \mathfrak{X}(M). \tag{2.8}$$

It is also known that a contact Riemannian 3-manifold is Sasakian if and only if it satisfies

$$R(X, Y)\xi = \eta(Y)X - \eta(X)Y, \quad X, Y \in \mathfrak{X}(M). \tag{2.9}$$

2.4. It was proved in [22] that a contact Riemannian 3-manifold always satisfies

$$(\nabla_X \varphi)Y = g(X + hX, Y)\xi - \eta(Y)(X + hX). \tag{2.10}$$

Comparing (2.8) and (2.10), one can see that a contact Riemannian 3-manifold is Sasakian if and only if M is K-contact.

From (2.7) and (2.10) we have

$$R(X, Y)\xi = \eta(Y)(X + hX) - \eta(X)(Y + hY) + \varphi\{(\nabla_Y h)X - (\nabla_X h)Y\} \tag{2.11}$$

for all X, $Y \in \mathfrak{X}(M)$.

2.5. The curvature characterization (2.9) of Sasakian manifolds motivates the following definition.

Definition 2.1 ([17]) A contact Riemannian 3-manifold M is said to be a *contact generalized (κ, μ, ν)-space* if its Riemannian curvature R satisfies

$$R(X, Y)\xi = (\kappa\,\mathrm{I} + \mu h + \nu\varphi h)\{\eta(Y)X - \eta(X)Y\} \tag{2.12}$$

for all X, $Y \in \mathfrak{X}(M)$. Here κ, μ and ν are smooth functions. When κ, μ and ν are constants, then a contact generalized (κ, μ, ν)-space is called a *contact (κ, μ, ν)-space*. A contact $(\kappa, \mu, 0)$-space is called a *generalized (κ, μ)-space*.

Definition 2.2 ([3]) Let M be a contact generalized (κ, μ)-space. If both the functions κ and μ are constants, then M is called a *contact (κ, μ)-space*. A contact generalized (κ, μ)-space is said to be *proper* if $|d\kappa|^2 + |d\mu|^2 \neq 0$.

One can see that Sasakian manifolds are contact (κ, μ)-spaces with $\kappa = 1$ and $h = 0$.

2.6. To close this section, here we prove the following fundamental fact.

Lemma 2.3 *A contact Riemannian 3-manifold M is Sasakian if and only if $\ell = -\varphi^2$.*

Proof. Assume that M is Sasakian, then from (2.11), we get $\ell = -\varphi^2$. Conversely, if we assume that $\ell = -\varphi^2$ then from (2.5), trace $(h^2) = 0$. Since h is self-adjoint with respect to g, h vanishes identically on M, that is, M is Sasakian. $\square$

Remark 2.1 In [4], it is shown that every contact Riemannian 3-manifold satisfying $S\varphi = \varphi S$ has the characteristic Jacobi operator of the form $\ell = -\varphi^2 - h^2$. Moreover such a space satisfies $\nabla_\xi h = 0$.

For more details on contact Riemannian geometry, we refer to [1].

3 Contact 3-manifolds with Reeb-flow invariance

Here we recall the following lemma.

Lemma 3.1 ([16],[7]) *Let M be a contact Riemannian 3-manifold. Then there exists a local orthonormal frame field $\mathcal{E} = \{e_1, e_2, e_3\}$ such that :*

$$he_1 = \lambda e_1, \; e_2 = \varphi e_1, \; e_3 = \xi.$$

With respect to $\mathcal{E}$, the Levi-Civita connection ∇ is given by

$$\begin{array}{rrr} \nabla_{e_1} e_1 = be_2, & \nabla_{e_1} e_2 = -be_1 + (1+\lambda)\xi, & \nabla_{e_1}\xi = -(1+\lambda)e_2, \\ \nabla_{e_2} e_1 = -ce_2 + (\lambda - 1)\xi, & \nabla_{e_2} e_2 = ce_1, & \nabla_{e_2}\xi = (1-\lambda)e_1, \\ \nabla_{\xi} e_1 = \alpha e_2, & \nabla_{\xi} e_2 = -\alpha e_1, & \nabla_{\xi}\xi = 0. \end{array}$$

The Ricci operator S is given by

$$\begin{aligned} Se_1 &= \rho_{11} e_1 + \xi(\lambda) e_2 + (2b\lambda - e_2(\lambda))\xi, \\ Se_2 &= \xi(\lambda) e_1 + \rho_{22} e_2 + (2c\lambda - e_1(\lambda))\xi, \\ S\xi &= (2b\lambda - e_2(\lambda)) e_1 + (2c\lambda - e_1(\lambda)) e_2 + 2(1-\lambda^2)\xi, \end{aligned}$$

where

$$\rho_{11} = \frac{\tau}{2} + \lambda^2 - 2\alpha\lambda - 1, \quad \rho_{22} = \frac{\tau}{2} + \lambda^2 + 2\alpha\lambda - 1.$$

Here τ is the scalar curvature of M.

From Lemma 3.1, one can deduce the following results.

Proposition 3.2 (*cf.* [12, Proposition 5.1], [14, Proposition 2.4]) *On a contact Riemannian 3-manifold with local orthonormal frame field $\mathcal{E}$ as in Lemma 3.1, $S\varphi = \varphi S$ holds if and only if*

$$2b\lambda - e_2(\lambda) = 0, \quad 2c\lambda - e_1(\lambda) = 0, \quad \xi(\lambda) = 0, \quad \rho_{11} = \rho_{22}.$$

Proposition 3.3 (*cf.* [12, Proposition 5.2]) *Let M be a contact Riemannian 3-manifold with local orthonormal frame field $\mathcal{E}$ as in Lemma 3.1. Then $\rho_{11} = \rho_{22}$ if and only if $\alpha = 0$ or M is Sasakian.*

Hence we obtain

Corollary 3.4 *Let M be a contact Riemannian 3-manifold with local orthonormal frame field $\mathcal{E}$ as in Lemma 3.1. Then $S\varphi = \varphi S$ holds if and only if M is Sasakian or*

$$\alpha = 0, \quad 2b\lambda - e_2(\lambda) = 0, \; 2c\lambda - e_1(\lambda) = 0, \; \xi(\lambda) = 0.$$

In particular when $b = c = 0$, $S\varphi = \varphi S$ holds if and only if $\alpha = 0$ and λ is constant.

4 Proof of the main theorem

4.1 The condition $S\xi = \sigma\xi$

First we recall the notion of an H-contact manifold.

Let $(M, \varphi, \xi, \eta, g)$ be a contact Riemannian 3-manifold with unit tangent sphere bundle T_1M. Denote by $\mathfrak{X}_1(M)$ the space of all sections of T_1M. Then M is said to be an *H-contact manifold* if its Reeb vector field ξ is a critical point of the energy functional restricted to $\mathfrak{X}_1(M)$.

In [21], Perrone proved the following fact.

Theorem 4.1 ([21]) *A contact Riemannian 3-manifold M is H-contact if and only if ξ is an eigenvector field of S.*

Moreover we know the following two facts.

Theorem 4.2 ([17]) *Let M be a contact Riemannian 3-manifold. If M is a contact generalized (κ, μ, ν)-space then M is an H-contact manifold. Conversely, if M is H-contact, then M satisfies the generalized (κ, μ, ν)-condition on an open dense subset of M.*

Proposition 4.3 ([17]) *Let M be a non-Sasakian 3-dimensional contact generalized (κ, μ, ν)-space. Then there exits a local orthonormal frame field $\mathcal{E} = \{e_1, e_2, e_3 = \xi\}$ as in Lemma 3.1 such that*

$$he_1 = \lambda e_1, \quad he_2 = -\lambda e_2, \quad e_2 = \varphi e_1,$$

where $\lambda = \sqrt{1-\kappa}$. The Ricci operator S is given by

$$S = AI + B\eta \otimes \xi + \mu\, h + \nu\, \varphi h$$

with

$$\begin{aligned} A =& \frac{1}{2}(\tau - 2\kappa) = \frac{\tau}{2} - (1-\lambda^2), \\ B =& \frac{1}{2}(6\kappa - \tau) = -\frac{\tau}{2} + 3(1-\lambda^2), \\ \mu =& -2\alpha, \quad \nu = \frac{\xi(\lambda)}{\lambda}. \end{aligned}$$

More specifically we have

$$\begin{aligned} Se_1 &= \frac{1}{2}(\tau - 2\kappa + 2\mu\sqrt{1-\kappa})e_1 + \nu\sqrt{1-\kappa}e_2, \\ Se_2 &= \nu\sqrt{1-\kappa}e_1 + \frac{1}{2}(\tau - 2\kappa - 2\mu\sqrt{1-\kappa})e_2, \\ Se_3 &= 2\kappa e_3. \end{aligned}$$

4.2. Now we start to prove our main theorem. Since we know that Sasakian 3-manifolds satisfy $S\xi = \sigma\xi$ and $\pounds_\xi \ell = 0$, it suffices to consider *non-Sasakian* contact Riemannian 3-manifolds.

Let M be a non-Sasakian contact Riemannian 3-manifold satisfying $S\xi = \sigma\xi$ for some function σ. Then as we have seen before, M is locally a contact generalized (κ, μ, ν)-space with $\sigma = 2\kappa$. In this case ℓ is computed as

$$\begin{aligned}\ell(X) = R(X, \xi)\xi &= (\kappa I + \mu h + \nu\varphi h)(X - \eta(X)\xi).\\ &= (1 - \lambda^2)(X - \eta(X)\xi) - 2\alpha hX + \frac{1}{\lambda}\xi(\lambda)\varphi hX.\end{aligned}$$

Hence we get

$$\ell(e_1) = (1 - \lambda^2 - 2\alpha\lambda)e_1 + \xi(\lambda)e_2, \quad \ell(e_2) = \xi(\lambda)e_1 + (1 - \lambda^2 + 2\alpha\lambda)e_2.$$

On the other hand, by using the definition of Ricci operator and the formula $S\xi = \sigma\xi$ we have

$$\begin{aligned}\ell(X) &= \rho(\xi, \xi)X - \rho(\xi, X)\xi + SX - \eta(X)S\xi - \frac{\tau}{2}(X - \eta(X)\xi)\\ &= \sigma X - \sigma\eta(X)\xi + SX - \sigma\eta(X)\xi - \frac{\tau}{2}(X - \eta(X)\xi)\\ &= SX + \left(\sigma - \frac{\tau}{2}\right)X - \left(2\sigma - \frac{\tau}{2}\right)\eta(X)\xi.\end{aligned}$$

From this equation we again get

$$\ell(e_1) = (1 - \lambda^2 - 2\alpha\lambda)e_1 + \xi(\lambda)e_2, \quad \ell(e_2) = \xi(\lambda)e_1 + (1 - \lambda^2 + 2\alpha\lambda)e_2.$$

Note that on contact generalized (κ, μ, ν)-spaces we have

$$\ell = -\kappa\varphi^2 + \mu h + \nu\varphi h,$$
$$\ell\varphi - \varphi\ell = 2\mu h\varphi + 2\nu h.$$

Let us compute $\pounds_\xi\ell$. First we observe that

$$[\xi, e_1] = (1 + \lambda + \alpha)e_2, \quad [\xi, e_2] = -(1 - \lambda + \alpha)e_1.$$

We put

$$\ell(e_i) = \ell_{i1}e_1 + \ell_{i2}e_2, \quad i = 1, 2.$$

Then

$$\begin{aligned}(\pounds_\xi\ell)e_1 =&[\xi, \ell(e_1)] - \ell[\xi, e_1] = \ell_{11}[\xi, e_1] + \ell_{12}[\xi, e_2] - (1 + \lambda + \alpha)\ell(e_2)\\ =&\ell_{11}(1 + \lambda + \alpha)e_2 - \ell_{12}(1 - \lambda + \alpha)e_1\\ &- (1 + \lambda + \alpha)(\ell_{21}e_1 + \ell_{22}e_2)\\ =& - 2\xi(\lambda)(1 + \alpha)e_1 + (\ell_{11} - \ell_{22})(1 + \lambda + \alpha)e_2\\ =& - 2\xi(\lambda)(1 + \alpha)e_1 - 4\alpha\lambda e_2.\end{aligned}$$

Since we assumed that M is *non-Sasakian, i.e.*, $\lambda \neq 0$, M satisfies $(\pounds_\xi\ell)e_1 = 0$ if and only if

$$\xi(\lambda) = 0, \quad \alpha = 0.$$

In a similar way we have $(\pounds_\xi\ell)e_2 = 0$ if and only if $\xi(\lambda) = 0$ and $\alpha = 0$. Hence M is a contact generalized $(\kappa, 0)$-space. In this case $\rho_{11} = \rho_{22}$, so M is pseudo-symmetric. Moreover since we assume that $\rho_{31} = \rho_{32} = 0$, we get

$$2b\lambda - e_2(\lambda) = 2c\lambda - e_1(\lambda) = 0.$$

Hence M satisfies $S\varphi = \varphi S$. Thus we obtain

Lemma 4.4 *Let M be an H-contact 3-manifold. If M satisfies $\pounds_\xi \ell = 0$, then M is a generalized $(\kappa, 0)$-space. Moreover M satisfies $S\varphi = \varphi S$.*

As we shall see in the next subsection, the condition $S\varphi = \varphi S$ implies that κ is constant.

4.2 The condition $S\varphi = \varphi S$

Here we recall the classification of contact Riemannian 3-manifolds satisfying $S\varphi = \varphi S$.

Theorem 4.5 ([4]) *Let M be a contact Riemannian 3-manifold. Then the following three conditions are mutually equivalent:*

1. *M is η-Einstein, that is, $S = a\mathrm{Id} + b\eta \otimes \xi$ for some functions a and b.*
2. *$S\varphi = \varphi S$;*
3. *M is a contact $(\kappa, 0)$-space, that is, M satisfies*

$$R(X, Y)\xi = \kappa\{\eta(Y)X - \eta(X)Y\}$$

for all vector fields X and Y on M. Here $\kappa \leq 1$ is a constant.

Theorem 4.6 ([4]) *Let M be a contact Riemannian 3-manifold. Then M satisfies $S\varphi = \varphi S$ if and only if M is either*

1. *a Sasakian 3-manifold,*
2. *a flat contact Riemannian 3-manifold, or*
3. *a non-Sasakian contact Riemannian space form of constant holomorphic sectional curvature $-\kappa$ and constant ξ-sectional curvature $\kappa < 1$.*

From Proposition 1.2, one can see that η-Einstein contact Riemannian 3-manifolds are pseudo-symmetric (see [9], [12]).

We have seen that if a contact Riemannian 3-manifold M satisfies $S\xi = \sigma\xi$ and $\pounds_\xi \ell = 0$, then M is a generalized (κ, μ, ν)-space. Moreover since $S\varphi = \varphi S$, M is η-Einstein and hence it is pseudo-symmetric. Moreover contact Riemannian 3-manifolds satisfying $S\varphi = \varphi S$ are pseudo-symmetric spaces of constant type (see [9]).

4.4. Now we return to *non-Sasakian* contact Riemannian 3-manifold M satisfying $S\xi = \sigma\xi$ and $\pounds_\xi \ell = 0$. Then combining Lemma 4.4 and Theorem 4.5, M is a contact $(\kappa, 0)$-space. Moreover $\lambda = \sqrt{1 - \kappa^2}$ is constant. Hence from Proposition 3.2, we have

$$[\xi, e_1] = (1 + \lambda)e_2, \ [e_1, e_2] = 2\xi, \ \ [e_2, \xi] = (1 - \lambda)e_1.$$

Thus M admits a local structure of a Lie group (see also [2]). Put $(c_1, c_2, c_3) := (1 - \lambda, 1 + \lambda, 2)$. Then we have the following classification.

- $\kappa > 0 \Longleftrightarrow \lambda^2 < 1$: In this case, the signature of (c_1, c_2, c_3) is $(+, +, +)$. Hence the possible Lie algebra is $\mathfrak{su}(2)$.
- $\kappa = 0 \Longleftrightarrow \lambda^2 = 1$: In this case the signature of (c_1, c_2, c_3) is $(0, +, +)$ or $(+, 0, +)$. The Lie algebra is $\mathfrak{e}(2)$ and M is flat.
- $\kappa < 0 \Longleftrightarrow \lambda^2 > 1$: In this case, the signature of (c_1, c_2, c_3) is $(-, +, +)$ or $(+, -, +)$. Hence the Lie algebra is $\mathfrak{sl}_2\mathbb{R}$.

Note that ℓ is given by $\ell(e_1) = \ell_1 e_1$, $\ell(e_2) = \ell_2 e_2$ with

$$\ell_1 = \frac{1}{4}(c_1 - c_2)(c_1 + 3c_2 - 4) + 1 = 1 - \lambda^2,$$
$$\ell_2 = \frac{1}{4}(c_1 - c_2)(3c_1 + c_2 - 4) + 1 = 1 - \lambda^2.$$

Thus we obtain the following theorem.

Theorem 4.7 *Let M be an H-contact 3-manifold. If M satisfies $\pounds_\xi \ell = 0$, then M is Sasakian or a homogeneous contact metric 3-manifold, locally isomorphic to* $\mathrm{SU}(2)$, $\mathrm{SL}_2\mathbb{R}$ *or* $\mathrm{E}(2)$.

As a result, we obtain the following main result of this paper:

Theorem 4.8 *Let M be a contact Riemannian 3-manifold. Then $S\xi = \sigma\xi$ for some function σ and at the same time $\pounds_\xi \ell = 0$ if and only if M is Sasakian or a homogeneous contact Riemannian 3-manifold, locally isomorphic to* $\mathrm{SU}(2)$ *or* $\mathrm{SL}_2\mathbb{R}$ *or* $\mathrm{E}(2)$.

As we have mentioned in Introduction, we *can not* remove the assumption H-contact in our main theorem.

Let G be a 3-dimensional non-unimodular Lie group equipped with a left invariant contact Riemannian structure. Then there exists an orthonormal basis $\{e_1, e_2 = \varphi e_1, e_3 = \xi\}$ of the Lie algebra $\mathfrak{g}$ of G such that

$$[e_1, e_2] = \alpha\, e_2 + 2e_3, \quad [e_2, e_3] = 0, \quad [e_3, e_1] = -\gamma\, e_2. \tag{4.1}$$

The Levi-Civita connection of G is given by the following table:

Proposition 4.9 ([20, p. 251])

$$\begin{array}{ccc}
\nabla_{e_1} e_1 = 0, & \nabla_{e_1} e_2 = -\frac{1}{2}(\gamma - 2)e_3, & \nabla_{e_1} e_3 = \frac{1}{2}(\gamma - 2)e_2 \\
\nabla_{e_2} e_1 = -\alpha e_2 - \frac{1}{2}(\gamma + 2)e_3, & \nabla_{e_2} e_2 = \alpha e_1, & \nabla_{e_2} e_3 = \frac{1}{2}(\gamma + 2)e_1 \\
\nabla_{e_3} e_1 = -\frac{1}{2}(\gamma + 2)e_2, & \nabla_{e_3} e_2 = \frac{1}{2}(\gamma + 2)e_1 & \nabla_{e_3} e_3 = 0.
\end{array}$$

Using this table, we get

$$S\xi = -\alpha\gamma e_2 + \left(2 - \frac{\gamma^2}{2}\right)\xi$$

and

$$\ell(e_1) = \ell_1 e_1, \quad \ell(e_2) = \ell_2 e_2,$$

where

$$\ell_1 = -\frac{1}{4}(\gamma + 2)(3\gamma - 2), \quad \ell_2 = \frac{1}{4}(\gamma + 2).$$

Proposition 4.10 *Let G be a 3-dimensional non-unimodular Lie group equipped with a left invariant contact Riemannian structure. Suppose that G satisfies $\pounds_\xi \ell = 0$. Then the Lie algebra $\mathfrak{g}$ is generated by the commutation relation (4.1) with $\gamma = 0$ (Sasakian) or $\gamma = -2$. In particular $\ell = 0$ if and only if $\gamma = -2$.*

Proof. We compute

$$\begin{aligned}(\pounds_\xi \ell)e_1 &= \pounds_\xi(\ell(e_1)) - \ell(\pounds_\xi e_1) = [\xi, \ell(e_1)] - \ell([\xi, e_1]) \\ &= \frac{1}{4}(\gamma+2)(3\gamma-2)\gamma e_2 + \frac{\gamma}{4}(\gamma+2)^2 e_2 \\ &= \gamma^2(\gamma+2)e_2\end{aligned}$$

and

$$(\pounds_\xi \ell)e_2 = \pounds_\xi(\ell(e_2)) - \ell(\pounds_\xi e_2) = [\xi, \ell(e_2)] - \ell([\xi, e_2]) = 0.$$

Since $(\pounds_\xi \ell)\xi = 0$, we have the required result. □

From this proposition, we see that the non-unimodular Lie group with $\gamma = -2$ satisfies $\pounds_\xi \ell = 0$ but $S\xi \neq \sigma\xi$. In fact, $S\xi = 2\alpha e_2 \neq 0$.

In our previous works, we studied pseudo-symmetry of contact Riemannian 3-manifolds [9,10,12]. In [15], the following result was obtained.

Theorem 4.11 ([15]) *Let M be contact Riemannian 3-manifold which is pseudo-symmetric and satisfies $S\xi = \sigma\xi$ for some function σ such that $\xi(\sigma) = 0$. Then there exists at most six open subsets of M for which their union is an open and dense subset inside of the closure of M and each of them as an open submanifold of M is either*

- *a Sasakian manifold,*
- *flat,*
- *locally isometric to one of the Lie groups* $\mathrm{SU}(2)$ *or* $\mathrm{SL}_2\mathbb{R}$ *equipped with a left invariant contact Riemannian structure,*
- *a pseudo-symmetric space of constant type with constant scalar curvature $\tau = 2(1 - \lambda^2 - 2\alpha)$,*
- *a semi K-contact manifold with $L = -3\alpha^2 + 4\alpha$,*
- *a semi K-contact manifold with $L = \alpha^2$.*

Theorem 4.12 ([15]) *Let M be a pseudo-symmetric contact Riemannian 3-manifold of constant type such that $S\xi = \sigma\xi$, where σ is a smooth function on M. Then σ is constant. If M is also complete then it is either a Sasakian manifold (meaning* $\mathrm{tr}\,\ell = 2$*) or locally isometric to one of the following Lie groups equipped with a left invariant metric:* $\mathrm{SU}(2)$, $\mathrm{SL}_2\mathbb{R}$, $\mathrm{E}(2)$ *or* $\mathrm{E}(1,1)$.

Theorem 4.13 ([15]) *A 3-dimensional generalized (κ, μ, ν)-space which is a pseudo-symmetric space of constant type is either a Sasakian manifold or a contact (κ, μ)-space. In the second case, if M is also complete, then it is locally isometric to one of the following Lie group with a left invariant metric:* $\mathrm{SU}(2)$, $\mathrm{SL}_2\mathbb{R}$, $\mathrm{E}(2)$ *or* $\mathrm{E}(1,1)$.

Comparing these with the main theorem of this paper, we notice that the Minkowski motion group $\mathrm{E}(1,1)$ is an interesting example. In fact

- $\mathrm{E}(1,1)$ is pseudo-symmetric,
- $\mathrm{E}(1,1)$ is (κ, μ)-space, hence $S\xi = \sigma\xi$
- but not $\pounds_\xi \ell = 0$.

The contact Riemannian structure on $\mathrm{E}(1,1)$ is given explicitly in our previous paper [9].

Definition 4.14 ([13]) *Let M be a contact Riemannian 3-manifold. Let $h = \lambda h_+ - \lambda h_-$ be the spectral decomposition of h on $U = \{p \in M \; |h_p \neq 0\}$. If $\nabla_{h_- X} h_- X = [\xi, h_+ X]$ holds for all vector field X on M and all points of an open subset W of U, and if $h = 0$ on the points of M that do not belong to W, then the manifold is said to be a semi K-contact manifold.*

Remark 4.1 Koufogiorgos and Tsichlias [18] studied contact Riemannian 3-manifolds with $\ell = 0$. In our previous paper [11], we studied model spaces for the class of contact Riemannian 3-manifolds with vanishing ℓ and constant norm $|S\xi|$.

Obviously the condition $\ell = 0$ is a special case of $\pounds_\xi \ell = 0$. In this paper we study contact Riemannian 3-manifolds satisfying both $S\xi = \sigma\xi$ and $\pounds_\xi \ell = 0$. However $S\xi = \sigma\xi$ can not occur in the subcase $\ell = 0$ except the flat case. See the following example due to Koufogiorgos and Tsichlias.

Example 4.1 [18] On the Cartesian 3-space $\mathbb{R}^3(x, y, z)$, define a contact 1-form η by

$$\eta = dx + 2ye^{-z}dz.$$

Next define a frame field $\{e_1, e_2, e_3\}$ by

$$e_1 = -2y\frac{\partial}{\partial x} + (2x - ye^z)\frac{\partial}{\partial y} + e^z\frac{\partial}{\partial z}, \quad e_2 = \frac{\partial}{\partial y}, \quad e_3 = \frac{\partial}{\partial x}.$$

Then we define a Riemannian metric g by the condition $\{e_1, e_2, e_3\}$ is orthonormal with respect to it. One can see that g is an associated metric to η. As usual, the endomorphism field φ is defined by $\varphi e_1 = e_2$, $\varphi e_2 = -e_1$ and $\varphi e_3 = 0$. The Reeb vector field is $\xi = e_3$. Direct computation shows that $\ell = 0$, $\tau = 0$. The components of the Ricci operator are $\rho_{23} = \rho_{32} = 2e^z$. Other components are zero. In particular $S\xi = 2e^z e_2$. Hence $S\xi$ is not parallel to ξ. Note that M is neither homogeneous nor flat.

This example also shows that we can not remove the assumption $S\xi = \sigma\xi$ in the main result of this paper.

Boeckx, Chun and the first named author of the present paper investigated unit tangent sphere bundles with η-parallel ℓ in [6]. In [19], K. Panagiotidou and P. J. Xenos studied (3-dimensional) real hypersurfaces in $\mathbb{C}P_2$ and $\mathbb{C}H_2$ with pseudo-parallel ℓ.

These studies lead to the following problems:

- Classify unit tangent sphere bundles with semi-parallel ℓ, *i.e.*, $R \cdot \ell = 0$.
- Classify contact Riemannian 3-manifolds with semi-parallel ℓ or more generally pseudo-parallel ℓ, *i.e.*,
  $$R(X, Y) \cdot \ell = L(X \wedge Y) \cdot \ell$$
  for some function L.

Remark 4.2 On the universal covering $\widetilde{\mathrm{E}}(2)$ of the Euclidean motion group $\mathrm{E}(2) = \mathrm{SO}(2) \times \mathbb{R}^2$, the standard contact form associated to the flat metric $(dx^2 + dy^2 + dz^2)/4$ is given by

$$\eta = \frac{1}{2}(\cos z \, dx + \sin z \, dy).$$

One can see that this contact structure is invariant under the action of discrete subgroup $\mathit{\Gamma} = 2\pi\mathbb{Z}^3$ of $\widetilde{\mathrm{E}}(2)$. Hence we obtain the flat torus $T^3 = \widetilde{\mathrm{E}}(2)/\mathit{\Gamma}$ equipped with left invariant contact Riemannian structure. This provide us a compact example of a homogeneous contact Riemannian 3-manifold with vanishing ℓ.

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