An. Ştiinţ. Univ. Al. I. Cuza Iaşi. Mat. (N.S.)
Tomul LXIII, 2017, f. 3

Approximation in Triangular Statistical Sense to B-Continuous Functions by Positive Linear Operators

Kamil Demirci · Fadime Dirik ·
Pınar Okçu

Received: 20.I.2015 / Accepted: 15.X.2015

Abstract The main object of this paper is to prove a Korovkin-type approximation theorem for sequences of positive linear operators defined on the space of all real valued B-continuous functions on a compact subset of the real line using a new type of statistical convergence of double sequences called triangular-A-statistical convergence for double real sequences. We give an illustrative example in support of our result. Finally, we investigate a rates of triangular-A-statistical convergence of positive linear operators.

Keywords Regularity for double sequences · Triangular A-statistical convergence for double sequences · the Korovkin theorem · B-continuity.

Mathematics Subject Classification (2010) 41A25 · 41A36

1 Introduction

The study of the Korovkin-type approximation theory is an area of active research, which deals with the problem of approximating a function by means of a sequence of positive linear operators (see, for instance, [2,19]). However, this theory was generalized by using the notion of B-continuity instead of the ordinary continuity [3–5].

Kamil Demirci
Department of Mathematics,
Faculty of Arts and Sciences,
Sinop University,
57000 Sinop, Turkey
E-mail: kamild@sinop.edu.tr

Fadime Dirik
Department of Mathematics,
Faculty of Arts and Sciences,
Sinop University,
57000 Sinop, Turkey
E-mail: fdirik@sinop.edu.tr

Pınar Okçu,
Department of Mathematics,
Faculty of Arts and Sciences,
Sinop University,
57000 Sinop, Turkey
E-mail: pinarokcu@sinop.edu.tr

Furthermore, in recent years, with the help of the concept of statistical convergence, various statistical approximation results have been proved [13–15, 18]. Recall that every convergent sequence (in the usual sense) is statistically convergent but its converse is not always true. Also, statistical convergent sequences do not need to be bounded. So, the usage of this method of convergence in the approximation theory provides us many advantages. Let $x = (x_{i,j})$ be a double sequence and suppose that $x = (x_{i,j})$ is neither A-statistical convergent nor convergent in the Pringsheim's sense. Then the question if there is any kind of statistical convergence which is different from A-statistical convergence and Pringsheim convergence naturally arises. To answer this question we consider two dimensional regular matrices for double sequences. Recently, Demirci et al. [10] have introduced an approximation theorem with the help of the notion of new statistical convergence called *triangular A-statistical convergence*. The main goal of this paper is to obtain a triangular statistical approximation theorem for sequences of positive linear operators defined on the space of all real valued B-continuous functions on a compact subset of the real line.

We now recall some basic definitions and notations used in the paper.

Let $A = (a_{ij})$ be a two-dimensional matrix transformation. For a double sequence $x = (x_{i,j})$ of real numbers, we put

$$(Ax)_i = \sum_{j=1}^{\infty} a_{i,j} x_{i,j},$$

if the series is convergent. We will say that A is regular if it maps every convergent sequence into a convergent sequence with the same limit. The well-known characterization of regularity for two dimensional matrix transformations is known as Silverman-Toeplitz conditions ([17]):

(i) $\|A\| = \sup_{i\to\infty} \sum_{j=1}^{\infty} |a_{i,j}| < \infty$,

(ii) $\lim_{i\to\infty} a_{i,j} = 0$ for each $j \in \mathbb{N}$,

(iii) $\lim_{i\to\infty} \sum_{j=1}^{\infty} a_{i,j} = 1$.

A double sequence $x = (x_{i,j})$, $i, j \in \mathbb{N}$, is convergent in Pringsheim's sense if, for every $\varepsilon > 0$, there exists $N = N(\varepsilon) \in \mathbb{N}$ such that $|x_{i,j} - L| < \varepsilon$ whenever $i, j > N$. Then, L is called the Pringsheim limit of x and is denoted by $P - \lim x = L$ (see [22]). In this case, we say that $x = (x_{i,j})$ is "P-convergent to L". Also, if there exists a positive number M such that $|x_{i,j}| \le M$ for all $(i, j) \in \mathbb{N}^2 = \mathbb{N} \times \mathbb{N}$, then $x = (x_{i,j})$ is said to be bounded. Recall that if a single sequence is convergent, then it is also bounded. But, this case does not hold for a double sequence, i.e. the convergence in Pringsheim's sense of a double sequence does not imply the boundedness of the double sequence.

Now let $A = (a_{n,m,i,j})$, $n, m, i, j \in \mathbb{N}$, be a four-dimensional summability matrix. For a given double sequence $x = (x_{i,j})$, the A-transform of x, denoted by $Ax := ((Ax)_{n,m})$, is given by

$$(Ax)_{n,m} = \sum_{(i,j)\in\mathbb{N}^2} a_{n,m,i,j} x_{i,j},$$

provided the double series converges in Pringsheim's sense for every $(n,m) \in \mathbb{N}^2$. In 1926, Robison [23] presented a four dimensional analog of the regularity by considering an additional assumption of boundedness. This assumption was made because a double P-convergent sequence is not necessarily bounded. The definition and the characterization of regularity for four dimensional matrices is known as Robison-Hamilton conditions, or briefly, RH-regularity (see, [16], [23]).

Recall that a four dimensional matrix $A = (a_{n,m,i,j})$ is said to be RH-regular if it maps every bounded P-convergent sequence into a P-convergent sequence with the same P-limit. The Robison-Hamilton conditions state that a four dimensional matrix $A = (a_{n,m,i,j})$ is RH-regular if and only if

(i) $P - \lim\limits_{n,m} a_{n,m,i,j} = 0$ for each i and j,

(ii) $P - \lim\limits_{n,m} \sum\limits_{i,j=1,1}^{\infty,\infty} a_{n,m,i,j} = 1$,

(iii) $P - \lim\limits_{n,m} \sum\limits_{i=1}^{\infty} |a_{n,m,i,j}| = 0$ for each $j \in \mathbb{N}$,

(iv) $P - \lim\limits_{n,m} \sum\limits_{j=1}^{\infty} |a_{n,m,i,j}| = 0$ for each $i \in \mathbb{N}$,

(v) $\sum\limits_{i,j=1,1}^{\infty,\infty} |a_{n,m,i,j}|$ is $P-$convergent for every $(n,m) \in \mathbb{N}^2$,

(vi) There exits finite positive integers A and B such that $\sum\limits_{i,j>B} |a_{n,m,i,j}| < A$ holds for every $(n,m) \in \mathbb{N}^2$.

Now let $A = (a_{n,m,i,j})$ be a nonnegative RH-regular summability matrix, and let $K \subset \mathbb{N}^2$. Then, a real double sequence $x = (x_{i,j})$ is said to be A-statistically convergent to a number L if, for every $\varepsilon > 0$,

$$P - \lim_{n,m} \sum_{(i,j)\in K(\varepsilon)} a_{n,m,i,j} = 0,$$

where

$$K(\varepsilon) := \{(i,j) \in \mathbb{N}^2 : |x_{i,j} - L| \geq \varepsilon\}.$$

In this case we write $st_A^2 - \lim\limits_{i,j} x_{i,j} = L$. Observe that, a P-convergent double sequence is A-statistically convergent to the same value but the converse is not always true.

We should note that if we take $A = C(1,1)$, which is the double Cesáro matrix, then $C(1,1)$-statistical convergence coincides with the notion of statistical convergence for double sequence, which was introduced in [20], [21], where $C(1,1) = (c_{n,m,i,j})$ is double Cesáro matrix defined by $c_{n,m,i,j} = \frac{1}{nm}$ if $1 \leq i \leq n$, $1 \leq j \leq m$, and $c_{n,m,i,j} = 0$ otherwise. Finally, if we replace the matrix A by the identity matrix for four-dimensional matrices, then A-statistical convergence reduces to the Pringsheim convergence.

We denote the set of all A-statistically convergent double sequences by st_A^2.

2 Triangular Statistical Convergence

Let $x = (x_{i,j})$ be a double sequence, neither A-statistical convergent or convergent in the Pringsheim sense. Then the question whether there is any kind of statistical convergence which is different from both A-statistical and Pringsheim convergence naturally arises. To answer this question we consider two-dimensional regular matrices for double sequences.

Let $A = (a_{i,j})$ be a nonnegative regular summability matrix, $K \subset \mathbb{N}^2$ be a nonempty set, and for every $i \in \mathbb{N}$, let $K_i = \{j \in \mathbb{N} : (i,j) \in K, j \leq i\}$. Let $|K_i|$ be the cardinality of K_i. We define *triangular A-density* of K by

$$\delta_A^T(K) := \lim_i \sum_{j \in K_i} a_{i,j}$$

provided that the limit on the right-hand side exists.

In a manner similar to the natural density, we can give some properties for the triangular A-density:

i) $\delta_A^T(\mathbb{N}^2) = 1$,

ii) if $K \subset L$ then $\delta_A^T(K) \leq \delta_A^T(L)$

iii) if K has triangular A-density then $\delta_A^T(\mathbb{N}^2/K) = 1 - \delta_A^T(K)$.

Definition 2.1 *Let $A = (a_{i,j})$ be a nonnegative regular summability matrix. The number sequence $x = (x_{i,j})$ is triangular A-statistically convergent to L provided that for every $\varepsilon > 0$*

$$\lim_i \sum_{j \in K_i(\varepsilon)} a_{i,j} = 0,$$

where $K_i(\varepsilon) = \{j \in \mathbb{N} : j \leq i, |x_{i,j} - L| \geq \varepsilon\}$ and this is denoted by $st_A^T - \lim_i x_{i,j} = L$.

We should note that if we take $A = C_1$ which is Cesáro matrix, then triangular C_1-statistical convergence coincides with the notion of triangular statistical convergence: If K denotes the set $\left\{(i,j) \in \mathbb{N}^2 : j \leq i\right\}$, then K_i is the i-section of K and $|K_i|$ denotes the cardinality of K_i.

Triangular density $\delta^T(K)$ is given by $\delta^T(K) = \lim_i \frac{1}{i}|K_i|$ or equivalently

$$\delta^T(K) = \lim_i (C_1 \chi_{K_i}(j))_i = \lim_i \sum_{j=1}^{\infty} c_{i,j} \chi_{K_i}(j),$$

if it exists. The number sequence $x = (x_{i,j})$ is triangular statistically convergent to L provided that for every $\varepsilon > 0$, the set $K := K_i(\varepsilon) := \{j \in \mathbb{N} : j \leq i, |x_{i,j} - L| \geq \varepsilon\}$ has triangular density zero; in that case we write $st^T - \lim_i x_{i,j} = L$.

We denote the set of all triangular A-statistically convergent sequences by st_A^T. Our preliminary result considers the problem of comparing triangular statistical convergence to statistical convergence.

Example 2.1 Take $A = C_1$ and the double sequence $x = (x_{i,j})$ given by

$$x_{i,j} = \begin{cases} 1, & i = j = k^2, \\ \frac{k}{4(k+1)}, & i = 2k,\ j = 2(k-1), \\ \frac{k}{3(k+1)}, & i = 2k+1,\ j = 2k+3, \\ 0, & \text{otherwise.} \end{cases} k \in \mathbb{N}$$

For every $\varepsilon \in \left(0, \frac{1}{8}\right]$,

$$\frac{1}{i} \left| \{ j \in \mathbb{N} : j \leq i, |x_{i,j}| \geq \varepsilon \} \right| = \begin{cases} 1, & i = 1, \\ \frac{2}{(2k)^2}, & i = (2k)^2, \\ \frac{1}{(2k+1)^2}, & i = (2k+1)^2, \\ \frac{1}{2k}, & i = 2k \text{ and } i \text{ is not square}, \\ 0, & \text{otherwise}, \end{cases} \qquad k \in \mathbb{N}$$

then clearly,

$$\lim_i \frac{1}{i} \left| \{ j \in \mathbb{N} : j \leq i, |x_{i,j}| \geq \varepsilon \} \right| = 0.$$

So, we obtain $st_{C_1}^T - \lim_i x_{i,j} = 0$. Nevertheless, $x = (x_{i,j})$ is not Pringsheim's and $C(1,1)$-statistically convergent.

Example 2.1 does not make it possible to characterize the concept of triangular $A-$statistical convergence as given below:

A double sequence $x = (x_{i,j})$ is triangular A-statistically convergent to L if and only if there exists a set $K \subset \mathbb{N}^2$ such that the triangular A-density of K is 1 and

$$P - \lim_{i,j \to \infty \text{ and } (i,j) \in K} \sum x_{i,j} = L \tag{2.1}$$

([21]). By 2.1 we mean that for every $\varepsilon > 0$ there exists an integer N such that

$$|x_{i,j} - L| \leq \varepsilon \text{ if } i, j \geq N \text{ and } (i,j) \in K.$$

Example 2.2 Take $A = C(1,1)$ and the double sequence $x = (x_{i,j})$ given by

$$x_{i,j} = \begin{cases} \sqrt{ij}, & \text{if } i \text{ and } j \text{ are squares}, \\ \frac{2}{ij}, & \text{otherwise.} \end{cases}$$

It is easy to see that $st_{C(1,1)}^2 - \lim_{i,j} x_{i,j} = 0$. Nevertheless, x is not Pringsheim's and triangular statistically convergent.

Example 2.3 Take $A = C_1$ and the double sequence $x = (x_{i,j})$ given by

$$x_{i,j} = \begin{cases} 1, & i = j = k^2, \\ 0, & \text{otherwise.} \end{cases} k \in \mathbb{N}$$

Similarly, $st_{C_1}^T - \lim_i x_{i,j} = 0$ and $st_{C(1,1)}^2 - \lim_{i,j} x_{i,j} = 0$.

Example 2.4 Take $A = C_1$ and the double sequence $x = (x_{i,j})$ given by

$$x_{i,j} = \begin{cases} 1, & i = j = k^2, \\ \frac{k}{2k+1}, & i = 3k+1, \ j = 3k-1, \\ \frac{k}{4k+2}, & i = 3k, \ j = 3(k+1), \quad k \in \mathbb{N} \\ k, & i = k^2, \ j = k^2+1, \\ 0, & \text{otherwise.} \end{cases}$$

So, we can easily see that $st_{C_1}^{T} - \lim_{i} x_{i,j} = 0$. Neither $x = (x_{i,j})$ is Pringsheim's and $C(1,1)$-statistically convergent nor bounded.

Remark 2.1 *i*) Triangular statistical convergence and statistical convergence are incompatible; i.e., $st_A^T \nsubseteq st_A^2$ and $st_A^2 \nsubseteq st_A^T$.

ii) A $P-$convergent double sequence is A-statistically convergent and triangular A-statistically convergent to the same value but the inverse implications are not true, i.e., $st_A^2 \nsubseteq c^2$ and $st_A^T \nsubseteq c^2$.

3 A Korovkin-Type Theorem

The definition of B-continuity was introduced by Bögel [6–8] as follows:

Let X and Y be compact subsets of the real numbers, and let $D = X \times Y$. Then, a function $f : D \to \mathbb{R}$ is called a $B-$continuous at a point $(x, y) \in D$ if, for every $\varepsilon > 0$, there exists a positive number $\delta = \delta(\varepsilon)$ such that

$$|\Delta_{x,y}[f(u,v)]| < \varepsilon,$$

for any $(u, v) \in D$ with $|u - x| < \delta$ and $|v - y| < \delta$, where the symbol $\Delta_{x,y}[f(u,v)]$ denotes the mixed difference of f defined by

$$\Delta_{x,y}[f(u,v)] = f(u,v) - f(u,y) - f(x,v) + f(x,y).$$

By $C_b(D)$ we denote the space of all B-continuous functions on D. Recall that $C(D)$ and $B(D)$ denote the space of all continuous (in the usual sense) functions and the space of all bounded function on D, respectively. Then, notice that $C(D) \subset C_b(D)$. Moreover, one can find an unbounded $B-$continuous function, which follows from the fact that, for any function of the type $f(u,v) = g(u) + h(v)$, we have $\Delta_{x,y}[f(u,v)] = 0$ for all $(x, y), (u, v) \in D$.

We recall that the following lemma for B-continuous functions was proved by Badea et. al. [4].

Lemma 3.1 [4] *If $f \in C_b(D)$, then, for every $\varepsilon > 0$, there are two positive numbers $A(\varepsilon) = A(\varepsilon, f)$ and $B(\varepsilon) = B(\varepsilon, f)$ such that*

$$\Delta_{x,y}[f(u,v)] \leq \frac{\varepsilon}{3} + A(\varepsilon)(u-x)^2 + B(\varepsilon)(v-y)^2$$

holds for all (x, y), $(u, v) \in D$.

A function $f : D \to \mathbb{R}$ is said to be $B-$bounded on D if there exists $M > 0$ such that

$$|\Delta_{x,y}[f(u,v)]| \le M$$

holds for any (x, y), $(u, v) \in D$. By $B_b(D)$, we mean the set of all B-bounded functions on D. Since D is a compact subset of $\mathbb{R}^2$, it is not hard to see that $C_b(D) \subset B_b(D)$ (see, for instance, [3]). So, the usual supremum norm on the spaces $B(D)$ is given by

$$\|f\| := \sup_{(x,y)\in D} |f(x,y)| \quad \text{for } f \in B(D);$$

and also the norm on the space $B_b(D)$ is defined by

$$\|f\|_B := \sup_{(x,y),(u,v)\in D} |\Delta_{x,y}[f(u,v)]| \quad \text{for } f \in B_b(D).$$

Let L be a linear operator from $C_b(D)$ into $B(D)$. Then, as usual, we say that L is positive linear operator provided that $f \ge 0$ implies $Lf \ge 0$. Also, we denote the value of Lf at a point $(x, y) \in D$ by $L(f(u, v); x, y)$ or, briefly, $L(f; x, y)$.

Throughout the paper, for fixed $(x, y) \in D$ and $f \in C_b(D)$, we use the function $F_{x,y}$ defined as follows:

$$F_{x,y}(u,v) = f(u,y) + f(x,v) - f(u,v) \qquad \text{for } (u,v) \in D. \tag{3.1}$$

Since

$$\Delta_{x,y}[F_{x,y}(u,v)] = -\Delta_{x,y}[f(u,v)]$$

holds for all (x, y), $(u, v) \in D$, the B-continuity of f implies the B-continuity of $F_{x,y}$ for every fixed $(x, y) \in D$. We also use the following test functions

$$e_0(u,v) = 1, \; e_1(u,v) = u, \; e_2(u,v) = v \text{ and } e_3(u,v) = u^2 + v^2.$$

Now we have the following main result.

Theorem 3.2 *Let $(L_{i,j})$ be a sequence of positive linear operators acting from $C_b(D)$ into $B(D)$, and let $A = (a_{i,j})$ be a nonnegative regular summability matrix. Assume that the following conditions hold:*

$$\delta_A^T(\{j \in \mathbb{N} : j \le i, \; L_{i,j}(e_0; x, y) = 1 \; \textit{for all } (x,y) \in D\}) = 1 \tag{3.2}$$

and

$$st_A^T - \lim_i \left(\sup_{(x,y)\in D} |L_{i,j}(e_\ell; x, y) - e_\ell(x,y)| \right) = 0 \;\; \textit{for } \ell = 1,2,3. \tag{3.3}$$

Then, for all $f \in C_b(D)$, we have

$$st_A^T - \lim_i \left(\sup_{(x,y)\in D} |L_{i,j}(F_{x,y}; x, y) - f(x,y)| \right) = 0,$$

where $F_{x,y}$ is given by (3.1).

Proof. Let $(x,y) \in D$ and $f \in C_b(D)$ be fixed. Putting

$$K_i := \{j \in \mathbb{N} : j \leq i,\ L_{i,j}(e_0;x,y) = 1 \ \text{ for all } (x,y) \in D\}, \tag{3.4}$$

we get from (3.2) that

$$\delta_A^T\left(\mathbb{N}^2 \backslash K\right) = 0. \tag{3.5}$$

Using the $B-$continuity of the function $F_{x,y}$ given by (3.1), Lemma 3.1 and using the monotonicity and the linearity of the operators $L_{i,j}$ for all $j \in K_i$ implies that, for every $\varepsilon > 0$, there exist two positive numbers $A(\varepsilon)$ and $B(\varepsilon)$ such that

$$\begin{aligned} |L_{i,j}(F_{x,y};x,y) - f(x,y)| &= |L_{i,j}(\varDelta_{x,y}[F_{x,y}(u,v)];x,y)| \\ &\leq \frac{\varepsilon}{3} + C(\varepsilon)\{x^2+y^2+L_{i,j}(e_3;x,y) \\ &\quad -2xL_{i,j}(e_1;x,y) - 2yL_{i,j}(e_2;x,y)\}, \end{aligned} \tag{3.6}$$

where $C(\varepsilon) = \max\{A(\varepsilon), B(\varepsilon)\}$. The last inequality gives that

$$|L_{i,j}(F_{x,y};x,y) - f(x,y)| \leq \frac{\varepsilon}{3} + C(\varepsilon)\sum_{\ell=1}^{3}|L_{i,j}(e_\ell;x,y) - e_\ell(x,y)| \tag{3.7}$$

holds for all $j \in K_i$. Taking supremum over $(x,y) \in D$ on the both-sides of inequality (3.7) we obtain, for all $j \in K_i$, that

$$\begin{aligned} \sup\nolimits_{(x,y)\in D}|L_{i,j}(F_{x,y};x,y) - f(x,y)| \leq \tfrac{\varepsilon}{3} + C(\varepsilon)\textstyle\sum_{\ell=1}^{3}\sup\nolimits_{(x,y)\in D}|L_{i,j}(e_\ell;& \\ x,y) - e_\ell(x,y)|.& \end{aligned} \tag{3.8}$$

Now for a given $r > 0$, choose $\varepsilon > 0$ such that $\varepsilon < 3r$. Then consider the following sets:

$$\begin{aligned} U_i &:= \left\{j \in \mathbb{N} : j \leq i,\ \sup_{(x,y)\in D}|L_{i,j}(F_{x,y};x,y) - f(x,y)| \geq r\right\}, \\ U_i^\ell &:= \left\{j \in \mathbb{N} : j \leq i,\ \sup_{(x,y)\in D}|L_{i,j}(e_\ell;x,y) - e_\ell(x,y)| \geq \frac{3r-\varepsilon}{9C(\varepsilon)}\right\}, \quad \ell = 1,2,3, \end{aligned}$$

Hence, inequality (3.8) yields that

$$U_i \cap K_i \subseteq \bigcup_{\ell=1}^{3}(U_i^\ell \cap K_i),$$

which gives, for all $i \in \mathbb{N}$,

$$\sum_{j\in U_i\cap K_i} a_{i,j} \leq \sum_{\ell=1}^{3}\sum_{j\in U_i^\ell\cap K_i} a_{i,j} \leq \sum_{\ell=1}^{3}\sum_{j\in U_i^\ell} a_{i,j} \tag{3.9}$$

Letting $i \to \infty$ and also using (3.3), we see from (3.9) that

$$\lim_{i} \sum_{j \in U_i \cap K_i} a_{i,j} = 0. \tag{3.10}$$

Furthermore, if we use the inequality

$$\begin{aligned} \sum_{j \in U_i} a_{i,j} &= \sum_{j \in U_i \cap K_i} a_{i,j} + \sum_{j \in U_i \cap (\mathbb{N} \backslash K_i)} a_{i,j} \\ &\leq \sum_{j \in U_i \cap K_i} a_{i,j} + \sum_{j \in (\mathbb{N} \backslash K_i)} a_{i,j} \end{aligned}$$

and if we take limit as $i \to \infty$, then it follows from (3.5) and (3.10) that

$$\lim_{i} \sum_{j \in U_i} a_{i,j} = 0,$$

which means

$$st_A^T - \lim_{i} \left\{ \sup_{(x,y) \in D} \left| L_{i,j} \left(F_{x,y}; x, y \right) - f(x,y) \right| \right\} = 0.$$

The proof is completed. $\square$

Remark 3.1 We know that, for some $f \in C_b(D)$, the function f may be unbounded on the compact set D. However, we can say from the conditions (3.2), (3.6) and the $B-$continuity of the function $F_{x,y}$ given by (3.1) that the number

$$\sup_{(x,y) \in D} \left| L_{i,j} \left(F_{x,y}; x, y \right) - f(x,y) \right|$$

in Theorem 3.2 is finite for each $j \in K_i$, where K_i is given by (3.4).

Now we give the classical result which is given by Badea et. al. [4] and the statistical result which is given by Dirik et. al. [12].

Corollary 3.3 [4] *Let $\{L_{m,n}\}$ be a sequence of positive linear operators acting from $C_b(D)$ into $B(D)$. Assume that the following conditions hold:*
(i) $L_{m,n}(e_0; x, y) = 1$ for all $(x,y) \in D$ and $(m,n) \in \mathbb{N}^2$,
(ii) $L_{m,n}(e_1; x, y) = e_1(x) + u_{m,n}(x,y)$,
(iii) $L_{m,n}(e_2; x, y) = e_2(x) + v_{m,n}(x,y)$,
(iv) $L_{m,n}(e_3; x, y) = e_3(x) + w_{m,n}(x,y)$,
where $\{u_{m,n}(x,y)\}$, $\{v_{m,n}(x,y)\}$ and $\{w_{m,n}(x,y)\}$ converge to zero uniformly on D as m,n approach infinity in any manner whatsoever. Then, the sequence

$$\{L_{m,n} \left(F_{x,y}; x, y \right)\}$$

converges uniformly to $f(x,y)$ with respect to $(x,y) \in D$, where $F_{x,y}$ is given by (3.1).

Corollary 3.4 [12] *Let $(L_{i,j})$ a sequence of positive linear operators acting from $C_b(D)$ into $B(D)$, and let $A=(a_{n,m,i,j})$ be a nonnegative RH−regular summability matrix. Assume that the following conditions hold:*

$$\delta_A^2\left(\left\{(i,j)\in\mathbb{N}^2 : L_{i,j}(e_0;x,y)=1 \ \text{ for all } (x,y)\in D\right\}\right)=1$$

and

$$st_A^2-\lim_{i,j}\left(\sup_{(x,y)\in D}\left|L_{i,j}(e_\ell;x,y)-e_\ell(x,y)\right|\right)=0 \ \text{ for } \ell=1,2,3.$$

Then, for all $f\in C_b(D)$, we have

$$st_A^2-\lim_{i,j}\left(\sup_{(x,y)\in D}\left|L_{i,j}(F_{x,y};x,y)-f(x,y)\right|\right)=0,$$

where $F_{x,y}$ is given by (3.1).

We now present a remark. The first one shows that Corollary 3.3 and Corollary 3.4 do not work; but our approximation theorem works. The second one gives that our approximation theorem does not work but the Corollary 3.4 works.

Remark 3.2 To see this first consider the following the Bernstein-Stancu-type operators (see [2])

$$\begin{aligned}K_{i,j,\alpha,\beta,\gamma,\delta}(f;x,y)&=\textstyle\sum_{s=0}^{i}\sum_{t=0}^{j}F_{x,y}\left(\frac{\alpha+s}{\beta+i},\right.\\ &\left.\frac{\gamma+t}{\delta+j}\right)\binom{i}{s}\binom{j}{t}x^s y^t(1-x)^{i-s}(1-y)^{j-t},\end{aligned}\tag{3.11}$$

where $F_{x,y}$ is given by (3.1), $X=[0,1]\times[0,1]$, $(x,y)\in X$, $\alpha,\beta,\gamma,\delta$ are fixed real numbers. Also, observe that

$$\begin{aligned}K_{i,j,\alpha,\beta,\gamma,\delta}(1;x,y)&=1,\\ K_{i,j,\alpha,\beta,\gamma,\delta}(u;x,y)&=\frac{i}{i+\beta}x+\frac{i}{i+\beta},\\ K_{i,j,\alpha,\beta,\gamma,\delta}(v;x,y)&=\frac{j}{j+\delta}y+\frac{j}{j+\delta},\\ K_{i,j,\alpha,\beta,\gamma,\delta}\left(u^2+v^2;x,y\right)&=\frac{i(i-1)}{(i+\beta)^2}x^2+\frac{(2\alpha+1)i}{(i+\beta)^2}x+\frac{\alpha^2}{(i+\beta)^2}\\ &\quad+\frac{j(j-1)}{(j+\delta)^2}y^2+\frac{(2\gamma+1)j}{(j+\delta)^2}y+\frac{\gamma^2}{(j+\delta)^2}.\end{aligned}$$

(a) Now we take $A=C_1$ and define a double sequence $(\alpha_{i,j})$ by

$$\alpha_{i,j}=\begin{cases}1, & i=j=k^2,\\ \frac{k}{3(k+1)}, & i=2k+1,\ j=2k-1,\\ \frac{k}{2(k+1)}, & i=2k,\ j=2(k+1),\\ 0, & \text{otherwise.}\end{cases}\ k\in\mathbb{N}\tag{3.12}$$

It is clear that

$$st_{C_1}^{T}-\lim_{i}\alpha_{i,j}=0. \tag{3.13}$$

However, the sequence $(\alpha_{i,j})$ is not P-convergent and statistical convergent. Now using (3.11) and (3.12), we define the following positive linear operators on $C_b(X)$ as follows:

$$L_{i,j}(f;x,y)=(1+\alpha_{i,j})K_{i,j,\alpha,\beta,\gamma,\delta}\left(f;x,y\right). \tag{3.14}$$

Then, observe that the sequence of positive linear operators $(L_{i,j})$ defined by (3.14) satisfy all hypotheses of Theorem 3.2. So, by (3.13), we have

$$st_{C_1}^{T}-\lim_{i}\left\{\sup_{(x,y)\in X}\left|L_{i,j}(f;x,y)-f(x,y)\right|\right\}=0.$$

Also, since $(\alpha_{i,j})$ is not P-convergent and statistical convergent, we can say that the Korovkin theorem in classical (Corollary 3.3) and statistical sense (Corollary 3.4) do not work for our operators defined by (3.14).

(b) Now we take $A=C_1$ and define a double sequence $(\beta_{i,j})$ by

$$\beta_{i,j}=\begin{cases}\sqrt{ij}, & \text{if } i \text{ and } j \text{ are squares,}\\ \frac{1}{ij} & \text{otherwise.}\end{cases} \tag{3.15}$$

It is clear that

$$st_{C(1,1)}^{2}-\lim_{i,j}\beta_{i,j}=0. \tag{3.16}$$

However, the sequence $(\beta_{i,j})$ is not P-convergent and triangular-statistical convergent. Now using (3.11) and (3.15), we define the following positive linear operators on $C_b(X)$ as follows:

$$L_{i,j}(f;x,y)=\left(1+\beta_{i,j}\right)K_{i,j,\alpha,\beta,\gamma,\delta}\left(f;x,y\right). \tag{3.17}$$

So, by the Corollary 3.4 and (3.16), we see that

$$st_{C(1,1)}^{2}-\lim_{i,j}\left\{\sup_{(x,y)\in X}\left|L_{i,j}(f;x,y)-f(x,y)\right|\right\}=0.$$

Also, since $(\beta_{i,j})$ is not triangular-statistical convergent, we can say that the Korovkin theorem in triangular statistical sense (Theorem 3.2) does not work for operators defined by (3.17).

4 Rates of A-statistical convergence in Theorem 3.2

In this section using the same techniques as in [10], we study the rates of convergence of a sequence of positive linear operators and for summability matrices, we present four different ways to compute the corresponding rates of triangular A-statistical convergence in Theorem 3.2.

Definition 4.1 [10] *Let $A=(a_{i,j})$ be a nonnegative regular summability matrix and let (α_i) be a positive non-increasing sequence. A double sequence $x=(x_{i,j})$ is triangular A-statistically convergent to a number L with the rate of $o(\alpha_i)$ if for every $\varepsilon>0$,*

$$\lim_i \frac{1}{\alpha_i} \sum_{j\in K_i(\varepsilon)} a_{i,j}=0,$$

where

$$K_i(\varepsilon):=\{\ j\in\mathbb{N}: j\leq i,\ \ |x_{i,j}-L|\geq\varepsilon\}.$$

In this case, we write

$$x_{i,j}-L=st_A^T-o(\alpha_i)\quad as\quad i\to\infty.$$

Definition 4.2 [10] *Let $A=(a_{i,j})$ and (α_i) be the same as in Definition 4.1. Then, a double sequence $x=(x_{i,j})$ is triangular A-statistically bounded with the rate of $O(\alpha_i)$ if for every $\varepsilon>0$,*

$$\sup_i \frac{1}{\alpha_i} \sum_{j\in L_i(\varepsilon)} a_{i,j}<\infty,$$

where

$$L_i(\varepsilon):=\{\ j\in\mathbb{N}: j\leq i,\ \ |x_{i,j}|\geq\varepsilon\ \}.$$

In this case, we write

$$x_{i,j}=st_A^T-O(\alpha_i)\quad as\quad i\to\infty.$$

We see from the above statements that, in two definitions the rate is directly effects the entries of the summability method rather than the terms of the sequence $x=(x_{i,j})$. For example, when one takes the identity matrix I, if $a_{i,i}=o(\alpha_i)$ then $x_{i,j}-L=st_A^T-o(\alpha_i)$ for any convergent sequence $(x_{i,j}-L)$ regardless of how slowly it goes to zero.

Definition 4.3 [10] *Let $A=(a_{i,j})$ and (α_i) be the same as in Definition 4.1. Then, a double sequence $x=(x_{i,j})$ is triangular A-statistically convergent to a number L with the rate of $o_k(\alpha_i)$ if for every $\varepsilon>0$,*

$$\lim_i \sum_{j\in M_i(\varepsilon)} a_{i,j}=0,$$

where

$$M_i(\varepsilon):=\{j\in\mathbb{N}: j\leq i,\ \ |x_{i,j}-L|\geq\varepsilon\ \alpha_j\}.$$

In this case, we write

$$x_{i,j}-L=st_A^T-o_k(\alpha_i)\quad as\quad i\to\infty.$$

Definition 4.4 [10] *Let $A = (a_{i,j})$ and (α_i) be the same as in Definition* 4.1. *Then, a double sequence $x = (x_{i,j})$ is triangular A-statistically bounded with the rate of $O_k(\alpha_i)$ if for every $\varepsilon > 0$,*

$$\lim_i \sum_{j \in N_i(\varepsilon)} a_{i,j} = 0,$$

where

$$N_i(\varepsilon) := \{j \in \mathbb{N} : j \le i, \ \ |x_{i,j}| \ge \varepsilon \ \alpha_j\}.$$

In this case, we write

$$x_{i,j} = st_A^T - O_k(\alpha_i) \quad as \quad i \to \infty.$$

Using these definitions, it is possible to obtain the following auxiliary results with an analogous technique in Demirci et al. [10].

Lemma 4.5 [10] *Let $(x_{i,j})$ and $(y_{i,j})$ be double sequences. Assume that $A = (a_{i,j})$ is a non-negative regular summability matrix, and let (α_i) and (β_i) be positive non-increasing sequences. If $x_{i,j} - L_1 = st_A^T - o(\alpha_i)$ and $y_{i,j} - L_2 = st_A^T - o(\beta_i)$, then we have*

(i) $(x_{i,j} - L_1) \mp (y_{i,j} - L_2) = st_A^T - o(\gamma_i)$ *as* $i \to \infty$, *where* $\gamma_i := \max\{\alpha_i, \beta_i\}$ *for each* $i \in \mathbb{N}$,

(ii) $\lambda(x_{i,j} - L_1) = st_A^T - o(\alpha_i)$ *as* $i \to \infty$ *for any real number* λ.

Furthermore, similar conclusions hold with the symbol "o" replaced by "O".

Lemma 4.6 [10] *Let $(x_{i,j})$ and $(y_{i,j})$ be double sequences. Assume that $A = (a_{i,j})$ is a nonnegative regular summability matrix, and let (α_i) and (β_i) be positive non-increasing sequences. If $x_{i,j} - L_1 = st_A^T - o_k(\alpha_i)$ and $y_{i,j} - L_2 = st_A^T - o_k(\beta_i)$, then we have*

(i) $(x_{i,j} - L_1) \mp (y_{i,j} - L_2) = st_A^T - o_k(\gamma_i)$ *as* $i \to \infty$, *where* $\gamma_i := \max\{\alpha_i, \beta_i\}$ *for each* $i \in \mathbb{N}$,

(ii) $\lambda(x_{i,j} - L_1) = st_A^T - o_k(\alpha_i)$ *as* $i \to \infty$ *for any real number* λ.

Furthermore, similar conclusions hold with the symbol "o_k" replaced by "O_k".

Now we remind the concept of mixed modulus of smoothness. For $f \in C_b(D)$, *the mixed modulus of smoothness* of f, denoted by $\omega_{mixed}(f; \delta_1, \delta_2)$, is defined to be

$$\omega_{mixed}(f; \delta_1, \delta_2) = \sup\{|\Delta_{x,y}[f(u,v)]| : |u - x| \le \delta_1, \ \ |v - y| \le \delta_2\}$$

for $\delta_1, \delta_2 > 0$. In order to obtain our result, we will make use of the elementary inequality

$$\omega_{mixed}(f; \lambda_1\delta_1, \lambda_2\delta_2) \le (1 + \lambda_1)(1 + \lambda_2)\,\omega_{mixed}(f; \delta_1, \delta_2)$$

for $\lambda_1, \lambda_2 > 0$. The modulus ω_{mixed} has been used by several authors in the framework of "Boolean sum type" approximation (see, for example, [9]). Elementary properties of ω_{mixed} can be found in [24] (see also [1]) and in particular for the case of B-continuous functions in [3].

Then we have the following result.

Theorem 4.7 *Let $(L_{i,j})$ be a sequence of positive linear operators from $C_b(D)$ into $B(D)$, and let $A=(a_{i,j})$ be a nonnegative regular summability matrix. Let (α_i) and (β_i) be a positive non-increasing sequence. Assume that the following conditions holds:*

$$\lim_i \frac{1}{\alpha_i}\sum_{j\in K_i} a_{i,j}=1, \tag{4.1}$$

where $K_i=\{j\in\mathbb{N}: j\le i,\ L_{i,j}(f_0;x,y)=f_0(x,y) \text{ for all } (x,y)\in D\}$; and

$$\omega_{mixed}(f;\gamma_{i,j},\delta_{i,j})=st_A^T-o(\beta_i) \quad as \quad i\to\infty, \tag{4.2}$$

where $\gamma_{i,j}:=\sqrt{\|L_{i,j}(\varphi)\|}$ and $\delta_{i,j}:=\sqrt{\|L_{i,j}(\Psi)\|}$ with $\varphi(u,v)=(u-x)^2$, $\Psi(u,v)=(v-y)^2$. Then we have, for all $f\in C_b(D)$,

$$\|L_{i,j}(F_{x,y})-f\|=st_A^T-o(c_i) \quad as \quad i\to\infty,$$

where $F_{x,y}$ is given by (3.1) and $c_i:=\max\{\alpha_i,\beta_i\}$ for each $i\in\mathbb{N}$. Furthermore, similar results holds when the symbol "o" is replaced by "O".

Proof. Let $(x,y)\in D$ and $f\in C_b(D)$ be fixed. It follows from (4.1) that

$$\lim_i \frac{1}{\alpha_i}\sum_{j\in\mathbb{N}\setminus K_i} a_{i,j}=0. \tag{4.3}$$

Also, using the properties of ω_{mixed}, since

$$L_{i,j}(F_{x,y};x,y)-f(x,y)=L_{i,j}(\Delta_{x,y}[F_{x,y}(u,v)];x,y)$$

holds for all $j\in K_i$, we obtain

$$\begin{aligned}|\Delta_{x,y}[F_{x,y}(u,v)]| &\le \omega_{mixed}(f;|u-x|,|v-y|)\\ &\le \left(1+\frac{1}{\delta_1}|u-x|\right)\left(1+\frac{1}{\delta_2}|v-y|\right)\\ &\quad \times\omega_{mixed}(f;\delta_1,\delta_2).\end{aligned} \tag{4.4}$$

As in the proof of the Theorem 2.7 of [11], it follows from the monotonicity and the linearity of the operators $L_{i,j}$, for all $j\in K_i$ and the Cauchy-Schwarz inequality that

$$\begin{aligned}|L_{i,j}(F_{x,y};x,y)-f(x,y)| \le &\left\{1+\frac{1}{\delta_1}\sqrt{L_{i,j}(\varphi;x,y)}+\frac{1}{\delta_2}\sqrt{L_{i,j}(\Psi;x,y)}\right.\\ &\left.\frac{1}{\delta_1\delta_2}\sqrt{L_{i,j}(\varphi;x,y)}\sqrt{L_{i,j}(\Psi;x,y)}\right\}\\ &\times\omega_{mixed}(f;\delta_1,\delta_2).\end{aligned} \tag{4.5}$$

Taking supremum over $(x,y)\in D$ on the both-sides of inequality (4.5) we obtain, for all $j\in K_i$, that

$$\|L_{i,j}(F_{x,y})-f\|\le 4\omega_{mixed}(f;\gamma_{i,j},\delta_{i,j}) \tag{4.6}$$

where $\delta_1 := \gamma_{i,j} := \sqrt{\|L_{i,j}(\varphi)\|}$ and $\delta_2 := \delta_{i,j} := \sqrt{\|L_{i,j}(\Psi)\|}$. Now, given $\varepsilon > 0$, define the following sets:

$$\begin{aligned} D_i :&= \left\{ j \in \mathbb{N} : j \leq i, \ \sup_{(x,y)\in D} |L_{i,j}(F_{x,y}; x, y) - f(x,y)| \geq \varepsilon \right\}, \\ D_i^1 :&= \left\{ j \in \mathbb{N} : j \leq i, \ \omega_{mixed}(f; \gamma_{i,j}, \delta_{i,j}) \geq \frac{\varepsilon}{4} \right\}. \end{aligned}$$

Hence, it follows from (4.6) that

$$D_i \cap K_i \subseteq D_i^1 \cap K_i,$$

which gives, for all $i \in \mathbb{N}$,

$$\begin{aligned} \frac{1}{c_i} \sum_{j \in D_i \cap K_i} a_{i,j} &\leq \frac{1}{c_i} \sum_{j \in D_i^1 \cap K_i} a_{i,j} \\ &\leq \frac{1}{c_i} \sum_{j \in D_i^1} a_{i,j} \\ &\leq \frac{1}{\beta_i} \sum_{j \in D_i^1} a_{i,j}. \end{aligned} \tag{4.7}$$

where $c_i := \max\{\alpha_i, \beta_i\}$. Letting $i \to \infty$ in (4.7) and from (4.2), we conclude that

$$\lim_i \frac{1}{c_i} \sum_{j \in D_i \cap K_i} a_{i,j} = 0. \tag{4.8}$$

Furthermore, if we use the inequality

$$\begin{aligned} \sum_{j \in D_i} a_{i,j} &= \sum_{j \in D_i \cap K_i} a_{i,j} + \sum_{j \in D_i \cap (\mathbb{N} \setminus K_i)} a_{i,j} \\ &\leq \sum_{j \in D_i \cap K_i} a_{i,j} + \sum_{j \in (\mathbb{N} \setminus K_i)} a_{i,j} \end{aligned}$$

which gives,

$$\frac{1}{c_i} \sum_{j \in D_i} a_{i,j} \leq \frac{1}{c_i} \sum_{j \in D_i \cap K_i} a_{i,j} + \frac{1}{\alpha_i} \sum_{j \in (\mathbb{N} \setminus K_i)} a_{i,j}. \tag{4.9}$$

Letting $i \to \infty$ in (4.9) and from (4.8) and (4.3), we conclude that

$$\lim_i \frac{1}{c_i} \sum_{j \in D_i} a_{i,j} = 0.$$

The proof is completed. □

The following similar result holds.

Theorem 4.8 *Let $(L_{i,j})$ be a sequence of positive linear operators from $C_b(D)$ into $B(D)$, and let $A = (a_{i,j})$ be a nonnegative regular summability matrix. Let (α_i) be a positive non-increasing sequence. Assume that the following conditions holds:*

$$\lim_i \sum_{j \in K_i} a_{i,j} = 1,$$

where $K_i = \{j \in \mathbb{N} : j \leq i,\ L_{i,j}(f_0; x, y) = f_0(x, y) \text{ for all } (x, y) \in D\}$; *and*

$$\omega_{mixed}(f; \gamma_{i,j}, \delta_{i,j}) = st_A^T - o_k(\alpha_i) \quad as \quad i \to \infty,$$

where $\gamma_{i,j} := \sqrt{\|L_{i,j}(\varphi)\|}$ *and* $\delta_{i,j} := \sqrt{\|L_{i,j}(\Psi)\|}$ *with* $\varphi(u, v) = (u - x)^2$, $\Psi(u, v) = (v - y)^2$. *Then we have, for all* $f \in C_b(D)$,

$$\|L_{i,j}(F_{x,y}) - f\| = st_A^T - o_k(\alpha_i) \quad as \quad i \to \infty,$$

where $F_{x,y}$ is given by (3.1). Furthermore, similar results holds when the symbol "o_k" is replaced by "O_k".

5 Possible further developments

Here we introduce a further extension of the notion of A-statistical convergence, which includes the triangular A-statistical convergence as a particular case. The idea is to consider a general infinite subset $H \subset \mathbb{N}^2$, defined by means a relation which links the indexes i and j. Put

$$H := \left\{(i, j) \in \mathbb{N}^2 : \Phi(i, j) \geq 0\right\}$$

where $\Phi : \mathbb{N} \times \mathbb{N} \to \mathbb{Z}$ is a function satisfying suitable assumptions. We then define the *Φ-A-density* of H as

$$\delta_A^{\Phi}(H) := \lim_i \sum_{j \in H_i} a_{i,j}$$

and the corresponding notion of Φ-A-statistical convergence as follows. Let $A = (a_{i,j})$ be a nonnegative regular summability matrix. A sequence $x = (x_{i,j})$ is said to be Φ-A-statistical convergent to a real number L provided that for every $\varepsilon > 0$

$$\lim_i \sum_{j \in K_i(\varepsilon)} a_{i,j} = 0,$$

where $K_i(\varepsilon) := \{i \in \mathbb{N} : \Phi(i, j) \geq 0,\ |x_{i,j} - L| \geq \varepsilon\}$.

Note that, when we choose $\Phi(i, j) = i - j$, we obtain to the case of triangular A-statistical convergence.

References

1. Anastassiou, G.A.; Gal, S.G. – *Approximation theory. Moduli of continuity and global smoothness preservation*, Birkhäuser, Boston, Inc., Boston, MA, 2000, xiv+525 pp.
2. Altomare, F.; Campiti, M. – *Korovkin-type approximation theory and its applications. Appendix A by Michael Pannenberg and Appendix B by Ferdinand Beckhoff*, De Gruyter Studies in Mathematics, 17. Walter de Gruyter & Co., Berlin, 1994, xii+627 pp.
3. Badea, I. – *Modulus of continuity in Bögel sense and some applications for approximation by a Bernstein-type operator*, Studia Univ. Babeş-Bolyai Ser. Math.-Mech. **18** (1973), no. 2, 69–78 (in Romanian).
4. Badea, C.; Badea, I.; Gonska, H.H. – *A test function theorem and approximation by pseudopolynomials*, Bull. Austral. Math. Soc. **34** (1986), no. 1, 53–64.
5. Badea, C.; Cottin, C. – *Korovkin-type theorems for generalized Boolean sum operators*, Approximation theory (Kecskemét, 1990), 51–68, Colloq. Math. Soc. János Bolyai, **58**, North-Holland, Amsterdam, 1991.
6. Bögel, K. – *Mehrdimensionale differentiation von funktionen mehrerer reeller veränderlichen* (German), J. Reine Angew. Math. **170** (1934), 197–217.
7. Bögel, K. – *Über mehrdimensionale differentiation, integration und beschränkte variation* (German), J. Reine Angew. Math. **173** (1935), 5–30.
8. Bögel, K. – *Über die mehrdimensionale differentiation* (German), Jahresber. Deutsch. Mat.-Verein. **65** (1962/1963), Abt. 1, 45–71.
9. Cottin, C. – *Approximation by bounded pseudo-polynomials*, in Function Spaces, J. Musielak et al (Eds.), Teubner-Texte zur Mathematik **120** (1991), 152–160.
10. Bardaro, C.; Boccuto, A.; Demirci, K.; Mantellini, I.; Orhan, S. – *Triangular A-statistical approximation by double sequences of positive linear operators*, Results Math. 68 (2015), no. 3–4, 271–291.
11. Dirik, F.; Demirci, K. – *Four-dimensional matrix transformation and rate of A-statistical convergence of Bögel-type continuous functions*, Stud. Univ. Babeş-Bolyai Math. 56 (2011), no. 3, 95–104.
12. Dirik, F.; Duman, O.; Demirci, K. – *Approximation in statistical sense to B-continuous functions by positive linear operators*, Studia Sci. Math. Hungarica, **47** (2010), no. 3, 289–298.
13. Duman, O.; Khan, M.K.; Orhan, C. – *A-Statistical convergence of approximating operators*, Math. Inequal. Appl. **6** (2003), no. 4, 689–699.
14. Duman, O.; Erkuş, E.; Gupta, V. – *Statistical rates on the multivariate approximation theory*, Math. Comput. Modelling **44** (2006), no. 9–10, 763–770.
15. Erkuş, E.; Duman, O.; Srivastava, H.M. – *Statistical approximation of certain positive linear operators constructed by means of the Chan-Chyan-Srivastava polynomials*, Appl. Math. Comput. **182** (2006), no. 1, 213–222.
16. Hamilton, H.J. – *Transformations of multiple sequences*, Duke Math. J. **2** (1936), no. 1, 29–60.
17. Hardy, G.H. – *Divergent Series*, Oxford at the Clarendon Press, London, 1949, xvi+396 pp.
18. Karakuş, S.; Demirci, K.; Duman, O. – *Equi-statistical convergence of positive linear operators*, J. Math. Anal. Appl. **339** (2008), no. 2, 1065–1072.
19. Korovkin, P.P. – *Linear Operators and Approximation Theory.* Translated from the Russian ed. (1959). Russian Monographs and Texts on Advanced Mathematics and Physics, Vol. III. Gordon and Breach Publishers, Inc., New York; Hindustan Publishing Corp. (India), Delhi 1960, vii+222 pp.
20. Móricz, F. – *Statistical convergence of multiple sequences*, Arch. Math. (Basel) **81** (2003), no. 1, 82–89.
21. Mursaleen, M.; Edely, O.H.H. – *Statistical convergence of double sequences*, J. Math. Anal. Appl. **288** (2003), no. 1, 223–231.
22. Pringsheim, A. – *Zur theorie der zweifach unendlichen Zahlenfolgen (German)*, Math. Ann. **53** (1900), no. 3, 289–321.
23. Robison, G.M. – *Divergent double sequences and series*, Trans. Amer. Math. Soc **28** (1926), no. 1, 50–73.
24. Schumaker, L.L. – *Spline Functions: Basic Theory.* Pure and Applied Mathematics, A Wiley-Interscience Publication, John Wiley & Sons, Inc., New York, 1981, xiv+553 pp.