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Some numerical results on finite non-abelian 2-generator p-groups of nilpotency class two

Mansour Hashemi · Mikhak Polkouei

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Abstract Let G be a finite group and n be a positive integer. The *n-th commutativity degree*, denoted by $P_n(G)$, is defined to be the probability of commuting the n-th power of a random element of G with another element. In this paper, we give explicit formulas for the n-th commutativity degree of all finite nonabelian 2-generator p-groups (p an odd prime) of nilpotency class two and also find integers n such that these groups are n-abelian for them.

Keywords p-groups · n-th commutativity degree · n-abelian groups

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1 Introduction

In 1968, Erdos and Turan [6] introduced the commutativity degree of a finite group, the definition which shows the probability that a random element of a group G commutes with another random element of it and is shown by $P(G)$. In [5], H. Doostie and M. Maghasedi give formulas for commutativity degree of some particular finite groups. The idea of commutativity degree was generalized to the *n-th commutativity degree* by Mohd. Ali and Sarmin [9] as follows.

Definition 1.1 *Let G be a finite group. The n-th commutativity degree of G, written $P_n(G)$, is defined as the ratio*

$$\frac{|(x,y) \in G \times G; x^n y = y x^n|}{|G|^2}.$$

Mansour Hashemi
Faculty of mathematical sciences,
University of Guilan,
Rasht, Iran
E-mail: m_hashemi@guilan.ac.ir

Mikhak Polkouei
Faculty of mathematical sciences,
University of Guilan,
Rasht, Iran
E-mail: mikhakp@yahoo.com

Let $n > 1$ be an integer. A group G is called n-abelian if $(xy)^n = x^n y^n$ for all x and y in G. Abelian groups and groups of exponent dividing n are clearly n-abelian. n-abelianity first defined by F. Levi [8]. This concept has been studied by several authors (see for example [1,3,4]).

Through this paper, the standard notations $[a, b] = a^{-1}b^{-1}ab$ (the commutator of elements a and b of a group) and $\rtimes$ (semidirect product of two groups) have been used. In Section 2, we state some lemmas and theorems which are needed for proving our main results. Section 3 is devoted to the n-th commutativity degree of all finite nonabelian 2-generator p-groups ($p \neq 2$) of nilpotency class two. In Section 4, we study n-abelianity of these groups.

2 Preliminary

In this section, we state some lemmas and theorems which will be used in other sections. First, we state a lemma without proof that establishes some properties of groups of nilpotency class two.

Lemma 2.1 *If G is a group and $G' \subseteq Z(G)$, then the following hold for every integer k and $u, v, w \in G$:*

(i) $[uv, w] = [u, w][v, w]$ *and* $[u, vw] = [u, v][u, w]$*;*
(ii) $[u^k, v] = [u, v^k] = [u, v]^k$*;*
(iii) $(uv)^k = u^k v^k [v, u]^{k(k-1)/2}$.

Theorem 2.2 [2] *Let G be a finite nonabelian 2-generator p-group of nilpotency class two (p an odd prime). Then G is isomorphic to exactly one group of the following three types:*

1. $G \cong (\langle c\rangle \times \langle a\rangle) \rtimes \langle b\rangle$, *where* $[a, b] = c$, $[a, c] = [b, c] = 1$, $|a| = p^\alpha$, $|b| = p^\beta$, $|c| = p^\gamma$, $\alpha, \beta, \gamma \in \mathbb{N}$, $\alpha \geq \beta \geq \gamma$*;*
2. $G \cong \langle a\rangle \rtimes \langle b\rangle$, *where* $[a, b] = a^{p^{\alpha-\gamma}}$, $|a| = p^\alpha$, $|b| = p^\beta$, $|[a, b]| = p^\gamma$, $\alpha, \beta, \gamma \in \mathbb{N}$, $\alpha \geq 2\gamma$, $\beta \geq \gamma$*;*
3. $G \cong (\langle c\rangle \times \langle a\rangle) \rtimes \langle b\rangle$, *where* $[a, b] = a^{p^{\alpha-\gamma}} c$, $[c, b] = a^{-p^{2(\alpha-\gamma)}} c^{-p^{\alpha-\gamma}}$, $|a| = p^\alpha$, $|b| = p^\beta$, $|c| = p^\sigma$, $|[a, b]| = p^\gamma$, $\alpha, \beta, \gamma, \sigma \in \mathbb{N}$, $\beta \geq \gamma > \sigma \geq 1$, $\alpha + \sigma \geq 2\gamma$.

Note. Suppose that $H \lhd G$ and there is a subgroup K such that $G = HK$ and $H \cap K = \{e\}$, then G is said to be the semidirect product of H and K; in symbol $G = H \rtimes K$. Clearly if $K \lhd G$, then $H \rtimes K \cong H \times K$.

By the definition of semidirect product, the following Lemma can be simply obtained.

Lemma 2.3 *i. Each element of G has a unique presentation of the form ab where $a \in H$ and $b \in K$.*
ii. If H and K are finite, then $\mid G \mid = \mid H \mid\mid K \mid$.

Also we need the following lemmas [7].

Lemma 2.4 *For the integer β and variables x, y, z and u, the number of solutions of the equation $xy \equiv zu \pmod{p^\beta}$ is $p^{2\beta-1}(p^{\beta+1} + p^\beta - 1)$.*

Lemma 2.5 *Let m, n be integers and x, y, z and u be variables where $0 \leq x, z < n$ and $0 \leq y, u < m$. Then the number of solutions of the equation $xy \equiv zu \pmod d$ is*

$$\left(\frac{m}{d}\right)^2 \left(\frac{n}{d}\right)^2 \prod_{i=1}^{k} {p_i}^{2\alpha_i - 1} \left({p_i}^{\alpha_i + 1} + {p_i}^{\alpha_i} - 1\right)$$

where $d = g.c.d(m, n) = p_1^{\alpha_1} p_2^{\alpha_2} \ \ldots \ p_k^{\alpha_k}$.

3 The n-th commutativity degree of 2-generator p-groups of nilpotency class two

In this section, we find $P_n(G)$ for all finite nonabelian 2-generator p-groups ($p \neq 2$) of nilpotency class two.

Theorem 3.1 *Let t, s be integers where $t < s$. Also, let $i_1, i_2, j_1, j_2, k_1, k_2$ be integers where $0 \leq i_1, i_2, j_1, j_2, k_1, k_2 < p^s$ and p be a prime number. Then the number of solutions of the following congruence*

$$p^t(i_1 k_2 - i_2 k_1) + (k_1 j_2 - k_2 j_1) \equiv 0 \pmod{p^s}$$

is equal to $p^{4s-1}(p^{s+1} + p^s - 1)$.

Proof. The equation $p^t(i_1 k_2 - i_2 k_1) + (k_1 j_2 - k_2 j_1) \equiv 0 \pmod{p^s}$ can be rewritten to form

$$k_1(j_2 - p^t i_2) - k_2(j_1 - p^t i_1) \equiv 0 \pmod{p^s}.$$

Now, for every $0 \leq i_1, i_2 < p^s$ we put $s_1 = j_2 - p^t i_2$ and $s_2 = j_1 - p^t i_1$. Then we get the following equation

$$k_1 s_2 \equiv k_2 s_1 \pmod{p^s}$$

where $0 \leq k_1, k_2, s_1, s_2 < p^s$. So, by Lemma 2.4 the number of solutions is equal to $p^{2s-1}(p^{s+1} + p^s - 1)$ and since $0 \leq i_1, i_2 < p^s$, we get the desired result. □

We are now in position to prove the main results of this section. First for a finite group G, we consider

$$A_n = \{(x, y) \in G \times G \mid x^n y = y x^n\}$$

then we have

$$P_n(G) = \frac{\mid A_n \mid}{\mid G \mid^2}.$$

Theorem 3.2 *Let G be a finite nonabelian 2-generator p-group ($p \neq 2$) of nilpotency class two. Then by notations of Theorem 2.2 we have*

$$P_n(G) = \begin{cases} \frac{p^{\gamma+1} + p^{\gamma} - 1}{p^{2\gamma+1}} & if (n, p^{\gamma}) = 1; \\ \frac{p^{\gamma-t+1} + p^{\gamma-t} - 1}{p^{2(\gamma-t)+1}} & if (n, p^{\gamma}) \neq 1, n = mp^t, \gamma > t; \\ 1 & if (n, p^{\gamma}) \neq 1, n = mp^t, \gamma \leq t. \end{cases}$$

Proof. We find $P_n(G)$ for three types of p-groups which are stated in Theorem 2.2:

1. Let G be a p-group of first type in Theorem 2.2 and $z \in G$. Then z can be uniquely written as $c^i a^j b^k$ where $0 \le i < p^\gamma$, $0 \le j < p^\alpha$ and $0 \le k < p^\beta$. By Lemma 2.1, we get

$$z^n = c^{ni - \frac{n(n-1)}{2}kj} a^{nj} b^{nk}.$$

Now let $(x, y) \in A_n$. Then $x = c^{i_1} a^{j_1} b^{k_1}$, $y = c^{i_2} a^{j_2} b^{k_2}$ and $x^n y = y x^n$ where $0 \le i_1, i_2 < p^\gamma$, $0 \le j_1, j_2 < p^\alpha$ and $0 \le k_1, k_2 < p^\beta$. Hence by Lemma 2.1,

$$x^n y = c^{ni_1 + i_2 - \frac{n(n-1)}{2} j_1 k_1 - n j_2 k_1} a^{nj_1 + j_2} b^{nk_1 + k_2}$$

and

$$y x^n = c^{ni_1 + i_2 - \frac{n(n-1)}{2} j_1 k_1 - n j_1 k_2} a^{nj_1 + j_2} b^{nk_1 + k_2}.$$

Now, by uniqueness of presentation of $x^n y$ and $y x^n$ we get that $x^n y = y x^n$ if and only if

$$n(k_1 j_2 - k_2 j_1) \equiv 0 \pmod{p^\gamma}. \quad (1)$$

By the above facts, we consider two cases.

Case 1. Let $(n, p^\gamma) = 1$. Then the congruence (1) is simplified to the equivalent congruence

$$k_1 j_2 \equiv k_2 j_1 \pmod{p^\gamma}. \quad (2)$$

So, by Lemma 2.5, the number of solutions of congruence (2) is equal to $p^{2(\alpha-\gamma)} p^{2(\beta-\gamma)} p^{2\gamma-1}(p^{\gamma+1} + p^\gamma - 1)$. Also each of the integers i_1 and i_2 take p^γ possible values. Combining of all these, we obtain

$$\begin{aligned} |A_n| &= |\{(i_1, j_1, k_1, i_2, j_2, k_2);\ k_1 j_2 \equiv k_2 j_1 \pmod{p^\gamma}\}| \\ &= p^{2(\alpha-\gamma)} p^{2(\beta-\gamma)} p^{2\gamma} p^{2\gamma-1}(p^{\gamma+1} + p^\gamma - 1) \end{aligned}$$

and

$$\begin{aligned} P_n(G) = \frac{|A_n|}{|G|^2} &= \frac{p^{2(\alpha-\gamma)} p^{2(\beta-\gamma)} p^{2\gamma} p^{2\gamma-1}(p^{\gamma+1} + p^\gamma - 1)}{p^{2\alpha+2\beta+2\gamma}} \\ &= \frac{p^{\gamma+1} + p^\gamma - 1}{p^{2\gamma+1}}. \end{aligned}$$

Case 2. Let $(n, p^\gamma) \neq 1$. So, there exist integers m and t such that $n = mp^t$. We consider two subcases $\gamma > t$ and $\gamma \le t$:

First, let $\gamma > t$. Then $(n, p^\gamma) = p^t$ and congruence (1) is reduced to

$$k_1 j_2 - k_2 j_1 \equiv 0 \pmod{p^{\gamma-t}}. \quad (3)$$

By Lemma 2.5, the number of solutions of this congruence is

$$p^{2(\alpha-\gamma+t)} p^{2(\beta-\gamma+t)} p^{2(\gamma-t)-1}(p^{\gamma-t+1} + p^{\gamma-t} - 1).$$

Now since each of the integers i_1 and i_2 admit p^γ values, we have

$$\begin{aligned} P_n(G) &= \frac{\mid A_n \mid}{\mid G \mid^2} = \frac{|\{(i_1, j_1, k_1, i_2, j_2, k_2);\ k_1j_2 - k_2j_1 \equiv 0\ (mod\ p^{\gamma-t})\}|}{|G|^2} \\ &= \frac{p^{2(\alpha-\gamma+t)}p^{2(\beta-\gamma+t)}p^{2\gamma}p^{2(\gamma-t)-1}(p^{\gamma-t+1}+p^{\gamma-t}-1)}{p^{2(\alpha+\beta+\gamma)}} \\ &= \frac{p^{\gamma-t+1}+p^{\gamma-t}-1}{p^{2(\gamma-t)+1}}. \end{aligned}$$

Second, if $\gamma \leq t$, then the Congruence (1) holds for all integers $i_1, i_2, j_1, j_2, k_1, k_2$ and then $P_n(G) = 1$.

2. When G is a group of second type of classes of groups of Theorem 2.2, the result follows in a similar way as for the first type.

3. consider $G \cong (\langle c\rangle \times \langle a\rangle) \rtimes \langle b\rangle$, where $[a, b] = a^{p^{\alpha-\gamma}}c$, $[c, b] = a^{-p^{2(\alpha-\gamma)}}c^{-p^{\alpha-\gamma}}$, $|a| = p^\alpha$, $|b| = p^\beta$, $|c| = p^\sigma$, $|[a, b]| = p^\gamma$, $\alpha, \beta, \gamma, \sigma \in \mathbb{N}$, $\beta \geq \gamma > \sigma \geq 1$, $\alpha + \sigma \geq 2\gamma$. Also let $(x, y) \in A_n$. Then we can write $x = c^{i_1}a^{j_1}b^{k_1}$ and $y = c^{i_2}a^{j_2}b^{k_2}$, where $0 \leq i_1, i_2 < p^\sigma$, $0 \leq j_1, j_2 < p^\alpha$ and $0 \leq k_1, k_2 < p^\beta$. So, by Lemma 2.1

$$x^n = c^{ni+\frac{n(n-1)}{2}k(p^{\alpha-\gamma}i-j)}a^{nj+\frac{n(n-1)}{2}kp^{\alpha-\gamma}(p^{\alpha-\gamma}i-j)}b^{nk}.$$

Now, by Lemma 2.3 we have $x^ny = yx^n$ if and only if

$$np^{\alpha-\gamma}(i_1k_2 - i_2k_1) + n(k_1j_2 - k_2j_1) \equiv 0 \quad (mod\ p^\gamma). \quad (4)$$

We consider two subcases $(n, p^\gamma) = 1$ and $(n, p^\gamma) \neq 1$.
First, let $(n, p^\gamma) = 1$. Here, again we consider two subsubcases $\alpha-\gamma \geq \gamma$ and $\alpha-\gamma < \gamma$. If $\alpha - \gamma \geq \gamma$, then the congruence (4) is reduced to

$$k_1j_2 \equiv k_2j_1 \quad (mod\ p^\gamma).$$

Hence by Lemma 2.5, since parameters i_1, i_2 are free in this congruence,

$$\mid A_n \mid = p^{2(\alpha-\gamma)}p^{2(\beta-\gamma)}p^{2\sigma}p^{2\gamma-1}(p^{\gamma+1} + p^\gamma - 1)$$

and therefore

$$\begin{aligned} P_n(G) &= \frac{\mid A_n \mid}{\mid G \mid^2} = \frac{p^{2(\alpha-\gamma)}p^{2(\beta-\gamma)}p^{2\sigma}p^{2\gamma-1}(p^{\gamma+1}+p^\gamma-1)}{p^{2(\alpha+\beta+\sigma)}} \\ &= \frac{p^{\gamma+1}+p^\gamma-1}{p^{2\gamma+1}}. \end{aligned}$$

Now, let $\alpha - \gamma < \gamma$. Then

$$k_1(j_2 - p^{\alpha-\gamma}i_2) \equiv k_2(j_1 - p^{\alpha-\gamma}i_1) \quad (mod\ p^\gamma).$$

Therefore, by Theorem 3.1 the number of solutions is equal to

$$p^{4(\gamma-1)}(p^{\gamma+1} + p^\gamma - 1)$$

which results

$$\mid A_n \mid = p^{2(\alpha-\gamma)}p^{2(\beta-\gamma)}p^{4(\gamma-1)}(p^{\gamma+1}+p^{\gamma}-1).$$

Hence

$$P_n(G) = \frac{p^{\gamma+1}+p^{\gamma}-1}{p^{2\gamma+1}}.$$

The proof of subcase $(n, p^{\gamma}) \neq 1$ is similar to the proof for the first type when $(n, p^{\gamma}) \neq 1$. $\square$

These results can be checked for some small values by GAP [10]. The following table shows the number of solutions of congruence

$$n(k_1j_2 - k_2j_1) \equiv 0 \pmod{p^{\gamma}}$$

in first class of 2-generator p-groups for $p = 3$, $\alpha = 4$, $\beta = 3$, $\gamma = 2$ and some values of n.

Table 1.

n	The number of solutions	$P_n(G)$
2	688905	$\frac{35}{243}$
3	1948617	$\frac{11}{27}$
4	68895	$\frac{35}{243}$
5	68895	$\frac{35}{243}$
6	1948617	$\frac{11}{27}$
9	4782969	1
27	4782969	1

4 n-abelianity of 2-generator p-groups of nilpotency class two

Here, we consider all finite non-abelian 2-generator p-groups (p an odd prime) of nilpotency class two and investigate when these groups are n-abelian. By notations of Theorem 2.2 we have:

Theorem 4.1 *A finite non-abelian 2-generator p-group of nilpotency class two is n-abelian if and only if $p^{\gamma} \mid n^2 - n$.*

Proof. According to the Theorem 2.2, we consider three cases:
Case 1. Let $G \cong (\langle c\rangle \times \langle a\rangle) \rtimes \langle b\rangle$, where $[a, b] = c$, $[a, c] = [b, c] = 1$, $|a| = p^{\alpha}$, $|b| = p^{\beta}$, $|c| = p^{\gamma}$,$\alpha, \beta, \gamma \in \mathbb{N}$, $\alpha \geq \beta \geq \gamma$. Also by Lemma 2.3, let $x = c^{i_1}a^{j_1}b^{k_1}$ and $y = c^{i_2}a^{j_2}b^{k_2}$ be two elements of G where $0 \leq i_1, i_2 < p^{\gamma}$, $0 \leq j_1, j_2 < p^{\alpha}$ and $0 \leq k_1, k_2 < p^{\beta}$. Then by the Lemma 2.1 we get

$$x^n = c^{ni_1 - \frac{n(n-1)}{2}j_1k_1}a^{nj_1}b^{nk_1}$$

and

$$y^n = c^{ni_2 - \frac{n(n-1)}{2}j_2k_2}a^{nj_2}b^{nk_2}.$$

We observe immediately from Lemma 2.1 that

$$a^ub^v = b^va^uc^{uv}$$

for all integers u, v. Therefore

$$\begin{aligned} x^n y^n &= c^{n(i_1+i_2)-\frac{n(n-1)}{2}(j_1k_1+j_2k_2)} a^{nj_1} b^{nk_1} a^{nj_2} b^{nk_2} \\ &= c^{n(i_1+i_2)-\frac{n(n-1)}{2}(j_1k_1+j_2k_2)-n^2j_2k_1} a^{n(j_1+j_2)} b^{n(k_1+k_2)}. \end{aligned}$$

On the other hand we get

$$\begin{aligned} (xy)^n &= (c^{i_1+i_2-j_2k_1} a^{j_1+j_2} b^{k_1+k_2})^n \\ &= c^{n(i_1+i_2)-nj_2k_1-\frac{n(n-1)}{2}(j_1+j_2)(k_1+k_2)} a^{n(j_1+j_2)} b^{n(k_1+k_2)}. \end{aligned}$$

We know that G is n-abelian if and only if for all $x, y \in G$; $(xy)^n = x^n y^n$. Furthermore by uniqueness of presentation of $(xy)^n$ and $x^n y^n$, we see that $(xy)^n = x^n y^n$ if and only if

$$n^2 j_2k_1 + \frac{n(n-1)}{2}(j_1k_1 + j_2k_2) \equiv nj_2k_1 + \frac{n(n-1)}{2}(j_1 + j_2)(k_1 + k_2) \quad (mod\ p^\gamma).$$

This is equivalent to

$$\frac{n(n-1)}{2}(j_2k_1 - j_1k_2) \equiv 0 \ \ (mod\ p^\gamma).$$

Now since $(p, 2) = 1$; we obtain G is n-abelian if and only if $p^\gamma \mid n^2 - n$.

Case 2. Let $G \cong \langle a\rangle \rtimes \langle b\rangle$, where $[a, b] = a^{p^{\alpha-\gamma}}$, $|a| = p^\alpha$, $|b| = p^\beta$, $|[a, b]| = p^\gamma$, $\alpha, \beta, \gamma \in \mathbb{N}$, $\alpha \geq 2\gamma$, $\beta \geq \gamma$. Also by Lemma 2.3, let $x = a^{j_1} b^{k_1}$,$y = a^{j_2} b^{k_2}$ be elements of G, where $0 \leq j_1, j_2 < p^\alpha$ and $0 \leq k_1, k_2 < p^\beta$. By using Lemma 2.1 we get

$$x^n = a^{nj_1-\frac{n(n-1)}{2}p^{\alpha-\gamma}j_1k_1} b^{nk_1}$$

and

$$y^n = a^{nj_2-\frac{n(n-1)}{2}p^{\alpha-\gamma}j_2k_2} b^{nk_2}.$$

So

$$\begin{aligned} x^n y^n &= a^{nj_1-\frac{n(n-1)}{2}p^{\alpha-\gamma}j_1k_1} b^{nk_1} a^{nj_2-\frac{n(n-1)}{2}p^{\alpha-\gamma}j_2k_2} b^{nk_2} \\ &= a^{n(j_1+j_2)-\frac{n(n-1)}{2}p^{\alpha-\gamma}(j_1k_1+j_2k_2)} [b, a]^{n^2j_2k_1-\frac{n^2(n-1)}{2}p^{\alpha-\gamma}j_2k_1k_2} b^{n(k_1+k_2)}. \end{aligned}$$

Since $[a, b] = a^{p^{\alpha-\gamma}}$, we have

$$x^n y^n = a^{n(j_1+j_2)-\frac{n(n-1)}{2}p^{\alpha-\gamma}(j_1k_1+j_2k_2)-n^2p^{\alpha-\gamma}j_2k_1-\frac{n^2(n-1)}{2}p^{2(\alpha-\gamma)}j_2k_1k_2} b^{n(k_1+k_2)}.$$

Now since $\alpha \geq 2\gamma$, we can write

$$2(\alpha - \gamma) = \alpha + (\alpha - 2\gamma) \geq \alpha.$$

Hence

$$x^n y^n = a^{n(j_1+j_2)-\frac{n(n-1)}{2}p^{\alpha-\gamma}(j_1k_1+j_2k_2)-n^2p^{\alpha-\gamma}j_2k_1} b^{n(k_1+k_2)}.$$

On the other hand we obtain

$$xy = a^{j_1+j_2-p^{\alpha-\gamma}j_2k_1} b^{k_1+k_2}.$$

By induction on n, we get

$$(xy)^n = a^{n(j_1+j_2)-\frac{n(n-1)}{2}p^{\alpha-\gamma}(j_1+j_2)(k_1+k_2)-np^{\alpha-\gamma}j_2k_1}b^{n(k_1+k_2)}.$$

Now by uniqueness of presentation of $(xy)^n$ and x^ny^n; we obtain G is n-abelian if and only if

$$(n^2-n)p^{\alpha-\gamma}j_2k_1 - \frac{n(n-1)}{2}p^{\alpha-\gamma}(j_1k_2+j_2k_1) \equiv 0 \pmod{p^\alpha}.$$

This yields that

$$\frac{n^2-n}{2}(j_2k_1-j_1k_2) \equiv 0 \pmod{p^\gamma}.$$

Since $(p,2)=1$, we obtain the assertion.

Case 3. Let $G \cong (\langle c\rangle \times \langle a\rangle) \rtimes \langle b\rangle$, where $[a,b]=a^{p^{\alpha-\gamma}}c$, $[c,b]=a^{-p^{2(\alpha-\gamma)}}c^{-p^{\alpha-\gamma}}$, $|a|=p^\alpha$, $|b|=p^\beta$, $|c|=p^\sigma$, $|[a,b]|=p^\gamma$, $\alpha,\beta,\gamma,\sigma \in \mathbb{N}$, $\beta \geq \gamma > \sigma \geq 1$, $\alpha+\sigma \geq 2\gamma$. By the presentation of G; for all $x,y \in G$ we have $x=c^{i_1}a^{j_1}b^{k_1}$ and $y=c^{i_2}a^{j_2}b^{k_2}$ where $0 \leq i_1,i_2 < p^\sigma$, $0 \leq j_1,j_2 < p^\alpha$ and $0 \leq k_1,k_2 < p^\beta$. Then

$$xy = c^{i_1+i_2+p^{\alpha-\gamma}i_2k_1-j_2k_1}a^{j_1+j_2+p^{2(\alpha-\gamma)}i_2k_1-p^{\alpha-\gamma}j_2k_1}b^{k_1+k_2}.$$

By an induction method, one can see

$$\begin{aligned}(xy)^n &= c^{n(i_1+i_2)+np^{\alpha-\gamma}i_2k_1-nj_2k_1+\frac{n(n-1)}{2}p^{\alpha-\gamma}(i_1+i_2)(k_1+k_2)-\frac{n(n-1)}{2}(j_1+j_2)(k_1+k_2)}\\ &\times a^{n(j_1+j_2)+np^{2(\alpha-\gamma)}i_2k_1-np^{\alpha-\gamma}j_2k_1+\frac{n(n-1)}{2}p^{2(\alpha-\gamma)}(i_1+i_2)(k_1+k_2)-\frac{n(n-1)}{2}p^{\alpha-\gamma}(j_1+j_2)(k_1+k_2)}\\ &\times b^{n(k_1+k_2)}.\end{aligned}$$

On the other hand, we obtain

$$\begin{aligned}x^ny^n &= c^{n(i_1+i_2)+n^2p^{\alpha-\gamma}i_2k_1-n^2j_2k_1+\frac{n(n-1)}{2}p^{\alpha-\gamma}(i_1k_1+i_2k_2)-\frac{n(n-1)}{2}(j_1k_1+j_2k_2)}\\ &\times a^{n(j_1+j_2)+n^2p^{2(\alpha-\gamma)}i_2k_1-n^2p^{\alpha-\gamma}j_2k_1+\frac{n(n-1)}{2}p^{2(\alpha-\gamma)}(i_1k_1+i_2k_2)-\frac{n(n-1)}{2}p^{\alpha-\gamma}(j_1k_1+j_2k_2)}\\ &\times b^{n(k_1+k_2)}.\end{aligned}$$

By the above facts, we see that for all $x,y \in G$; $(xy)^n = x^ny^n$ if and only if the following system holds

$$\begin{cases} \frac{n^2-n}{2}(p^{\alpha-\gamma}(i_2k_1-i_1k_2)+j_1k_2-j_2k_1) \equiv 0 \pmod{p^\sigma} \\ \frac{n^2-n}{2}(p^{\alpha-\gamma}(i_2k_1-i_1k_2)+j_1k_2-j_2k_1) \equiv 0 \pmod{p^\gamma}. \end{cases} \tag{5}$$

Now let $p^\gamma | n^2-n$, then surely $p^\sigma | n^2-n$ and the above congruence system holds. Hence G will be n-abelian.

Conversely let G be an m-abelian group. So the system (5) must hold for all $x,y \in G$ such as $x=c^2a^2b$ and $y=c^3ab$. Then we get

$$\begin{cases} \frac{m^2-m}{2}(p^{\alpha-\gamma}+1) \equiv 0 \pmod{p^\sigma} \\ \frac{m^2-m}{2}(p^{\alpha-\gamma}+1) \equiv 0 \pmod{p^\gamma}. \end{cases}$$

Hence $p^\gamma | m^2-m$. $\square$

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