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The Recurrence Sequences via Hurwitz Matrices

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Abstract In this paper, we define recurrence sequences by using the Hurwitz matrices which are defined by the characteristic polynomials of the 3-step Fibonacci, the generalized order-3 Pell and the generalized order-3 Jacobsthal sequences. Then we obtain miscellaneous properties of these sequences by aid of these matrices. Also, we consider the cyclic groups which are generated by reducing the Hurwitz matrices modulo m then we examine the orders of these groups. Furthermore, we extend the defined sequences to groups. Finally, we obtain the lengths of the periods of the extended sequences in the extended triangle group $E(2,q,2)$ for $q \geq 3$ as applications of the obtained results.

Keywords Hurwitz Matrix · Sequence · Group · Length

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1 Introduction and Preliminaries

Let an n^{th} degree real polynomial P be given by

$$P(x) = a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n.$$

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In [14], A. Hurwitz defined the Hurwitz matrix $H_n = [h_{ij}]_{n\times n}$associated to P as follows:

$$H_n = \begin{bmatrix} a_1 & a_3 & a_5 & \cdots & \cdots & \cdots & 0 & 0 & 0 \\ a_0 & a_2 & a_4 & & & & \vdots & \vdots & \vdots \\ 0 & a_1 & a_3 & & & & \vdots & \vdots & \vdots \\ \vdots & a_0 & a_2 & \ddots & & & 0 & \vdots & \vdots \\ \vdots & 0 & a_1 & & \ddots & & a_n & \vdots & \vdots \\ \vdots & \vdots & a_0 & & & \ddots & a_{n-1} & 0 & \vdots \\ \vdots & \vdots & 0 & & & & a_{n-2} & a_n & \vdots \\ \vdots & \vdots & \vdots & & & & a_{n-3} & a_{n-1} & 0 \\ 0 & 0 & 0 & \cdots & \cdots & \cdots & a_{n-4} & a_{n-2} & a_n \end{bmatrix}$$

The 3-step Fibonacci sequence $\left\{F_n^3\right\}$ is defined recursively by the equation

$$F_{n+2}^3 = F_{n+1}^3 + F_n^3 + F_{n-1}^3$$

for $n \geq 1$, where $F_0^3 = F_1^3 = 0$ and $F_2^3 = 1$. For more information on the k-step Fibonacci sequence $\left\{F_n^k\right\}$, see [15,20].

The generalized order-3 Pell sequence $\left\{P_n^3\right\}$ is defined recursively by the equation

$$P_n^3 = 2P_{n-1}^3 + P_{n-2}^3 + P_{n-3}^3$$

for $n > 0$, where $P_{-2}^3 = 1$ and $P_{-1}^3 = P_0^3 = 0$. For more information on the generalized order-k Pell sequence $\left\{P_n^k\right\}$, see [16].

The generalized order-3 Jacobsthal sequence $\left\{J_n^3\right\}$ is defined recursively by the equation

$$J_n^3 = J_{n-1}^3 + 2J_{n-2}^3 + J_{n-3}^3$$

for $n > 0$, where $J_{-2}^3 = 1$ and $J_{-1}^3 = J_0^3 = 0$. For more information on the generalized order-k Jacobsthal sequence $\left\{J_n^k\right\}$, see [23].

In Section 2, we define 3-step Fibonacci-Hurwitz, 3-step Pell-Hurwitz and 3-step Jacobsthal- Hurwitz numbers by using the recurrence relations such that these relations are obtained from the Hurwitz matrices which are defined by the characteristic polynomials of the 3-step Fibonacci, the generalized order-3 Pell and the generalized order-3 Jacobsthal sequences. Then we obtain their miscellaneous properties. In [8–11, 18,21], the authors obtained the cyclic groups via some special matrices. In Section 3, we consider the multiplicative orders of the Hurwitz matrices according to modulo m and then, we obtain the rules for the orders of the cyclic groups which are generated by reduced Hurwitz matrices modulo m. The study of recurrence sequences in groups began with the earlier work of Wall [22] where the ordinary Fibonacci sequences in cyclic groups were investigated. The concept extended to some special linear recurrence sequences by several authors; see for example, [2–4,8–13,17,19,21]. In Section 4 we define 3-step Fibonacci-Hurwitz, 3-step Pell-Hurwitz and 3-step Jacobsthal-Hurwitz sequences in 3-generated groups then we study these sequences in finite groups. Also

in this section, we obtain the lengths of the periods of 3-step Fibonacci-Hurwitz, 3-step Pell-Hurwitz and 3-step Jacobsthal-Hurwitz sequences in the extended triangle group $E(2,q,2)$, for $q \geq 3$.

2 3-Step Fibonacci-Hurwitz, 3-Step Pell-Hurwitz and 3-Step Jacobsthal-Hurwitz Numbers

It is easy to see that the characteristic polynomials of the 3-step Fibonacci, the generalized order-3 Pell and the generalized order-3 Jacobsthal sequencesare as follows, respectively:

$$P_1(x) = x^3 - x^2 - x - 1,$$

$$P_2(x) = x^3 - 2x^2 - x - 1$$

and

$$P_3(x) = x^3 - x^2 - 2x - 1.$$

Then we can write the following Hurwitz matrices for the polynomials P_1, P_2 and P_3, respectively:

$$H_3^1 = \begin{bmatrix} -1 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & -1 & -1 \end{bmatrix},$$

$$H_3^2 = \begin{bmatrix} -2 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & -2 & -1 \end{bmatrix}$$

and

$$H_3^3 = \begin{bmatrix} -1 & -1 & 0 \\ 1 & -2 & 0 \\ 0 & -1 & -1 \end{bmatrix}.$$

Now we define 3-step Fibonacci-Hurwitz, 3-step Pell-Hurwitz and 3-step Jacobsthal-Hurwitz numbers by using the matrices H_3^1, H_3^2 and H_3^3 as follows, respectively:

$$FH_3(n) = \begin{cases} -FH_3(n-3) - FH_3(n-2), & n \equiv 1 \bmod 3, \\ FH_3(n-4) - FH_3(n-3), & n \equiv 2 \bmod 3, \\ -FH_3(n-4) - FH_3(n-3), & n \equiv 0 \bmod 3 \end{cases} \quad \text{for } n > 3, \tag{2.1}$$

$$PH_3(n) = \begin{cases} -2PH_3(n-3) - PH_3(n-2), & n \equiv 1 \bmod 3, \\ PH_3(n-4) - PH_3(n-3), & n \equiv 2 \bmod 3, \\ -2PH_3(n-4) - PH_3(n-3), & n \equiv 0 \bmod 3, \end{cases} \quad \text{for } n > 3 \tag{2.2}$$

and

$$JH_3(n) = \begin{cases} -JH_3(n-3) - JH_3(n-2), & n \equiv 1 \bmod 3, \\ JH_3(n-4) - 2JH_3(n-3), & n \equiv 2 \bmod 3, \\ -JH_3(n-4) - JH_3(n-3), & n \equiv 0 \bmod 3, \end{cases} \quad \text{for } n > 3, \tag{2.3}$$

where $FH_3(1) = 0,\ FH_3(2) = FH_3(3) = 1,\ PH_3(1) = 0,\ PH_3(2) = PH_3(3) = 1$ and $JH_3(1) = 0,\ JH_3(2) = JH_3(3) = 1$.

For $n \geq 0$, by the inductive argument, we may write

$$FH_3(3n+3) = FH_3(3n+1) + (-1)^n, \tag{2.4}$$

$$PH_3(3n+3) = 2(PH_3(3n+2) + PH_3(3n+1)) - (-1)^n \tag{2.5}$$

and

$$JH_3(3n+3) = JH_3(3n+1) + (-1)^n. \tag{2.6}$$

It is easy to show that

$$\left(H_3^1\right)^n = \begin{bmatrix} FH_3(3n+2) & FH_3(3n+1) & 0 \\ -FH_3(3n+1) & FH_3(3n+2) & 0 \\ FH_3(3n+2) - (-1)^n & FH_3(3n+1) & (-1)^n \end{bmatrix}, \tag{2.7}$$

$$\left(H_3^2\right)^n = \begin{bmatrix} PH_3(3n+2) + PH_3(3n+1) & PH_3(3n+1) & 0 \\ -PH_3(3n+1) & PH_3(3n+2) & 0 \\ 2PH_3(3n+2) - 2(-1)^n & PH_3(3n+3) - (-1)^n & (-1)^n \end{bmatrix} \tag{2.8}$$

and

$$\left(H_3^3\right)^n = \begin{bmatrix} JH_3(3n+2) - JH_3(3n+1) & JH_3(3n+1) & 0 \\ -JH_3(3n+1) & JH_3(3n+2) & 0 \\ JH_3(3n+2) - JH_3(3n+1) - (-1)^n & JH_3(3n+1) & (-1)^n \end{bmatrix} \tag{2.9}$$

for $n \geq 0$, which can be proved by mathematical induction. Since $\det H_3^1 = -2$, $\det H_3^2 = -3$ and $\det H_3^3 = -3$, we can write the Simpson formulas for 3-step Fibonacci-Hurwitz, 3-step Pell-Hurwitz and 3-step Jacobsthal-Hurwitz numbers as:

$$(FH_3(3n+2))^2 + (FH_3(3n+1))^2 = 2^n, \tag{2.10}$$

$$(PH_3(3n+2))^2 + (PH_3(3n+1))^2 + PH_3(3n+2)\,PH_3(3n+1) = 3^n \tag{2.11}$$

and

$$(JH_3(3n+2))^2 + (JH_3(3n+1))^2 - JH_3(3n+2)\,JH_3(3n+1) = 3^n. \tag{2.12}$$

For more information on Simpson formula, see [7].

These definitions give us:

$$\begin{cases} FH_3(n) = 0 \text{ if } n \equiv 1, 8, 13, 20 \bmod 24, \\ FH_3(n) > 0 \text{ if } n \equiv 0, 2, 3, 7, 9, 11, 15, 16, 17, 18, 22 \bmod 24, \\ FH_3(n) < 0 \text{ if } n \equiv 4, 5, 6, 10, 12, 14, 19, 21, 23 \bmod 24, \end{cases}$$

$$\begin{cases} PH_3(n) = 0 \text{ if } n \equiv 1, 8, 19, 26 \bmod 36, \\ PH_3(n) > 0 \text{ if } n \equiv 2, 3, 7, 9, 11, 13, 17, 18, 22, 23, 24, 28, 30, 32, 34 \bmod 36, \\ PH_3(n) < 0 \text{ if } n \equiv 0, 4, 5, 6, 10, 12, 14, 15, 16, 20, 21, 25, 27, 29, 31, 33, 35 \bmod 36, \end{cases}$$

and

$$\begin{cases} JH_3(n) = 0 \text{ if } n \equiv 1, 14, 19, 32 \bmod 36, \\ JH_3(n) > 0 \text{ if } n \equiv 0, 2, 3, 7, 8, 9, 13, 15, 17, 21, 22, 23, 24, 28, 29, 30, 34 \bmod 36, \\ JH_3(n) < 0 \text{ if } n \equiv 4, 5, 6, 10, 11, 12, 16, 18, 20, 25, 26, 27, 31, 33, 35 \bmod 36. \end{cases}$$

By these results we get:

Theorem 2.1 (i).

$$FH_3\left(3n+1\right)^2=\begin{cases}2^{n-1} & \text{if } n \text{ is odd}\\ 2^{n-1}\left(\left(-1\right)^{\frac{n}{2}+1}+1\right) & \text{if } n \text{ is even}\end{cases}$$

and

$$FH_3\left(3n+2\right)^2=\begin{cases}2^{n-1} & \text{if } n \text{ is odd,}\\ 2^{n-1}\left(\left(-1\right)^{\frac{n}{2}}+1\right) & \text{if } n \text{ is even.}\end{cases}$$

(ii).

$$PH_3\left(3n+1\right)^2=\begin{cases}0, & n\equiv 0 \bmod 6,\\ 3^{n-1}, & n\equiv 1,5 \bmod 6,\\ 3^n, & n\equiv 2,4 \bmod 6,\\ 4\cdot 3^{n-1}, & n\equiv 3 \bmod 6,\end{cases}$$

and

$$PH_3\left(3n+2\right)^2=\begin{cases}3^n, n\equiv 0 \bmod 6,\\ 2\cdot 3^{n-1}-PH_3\left(3n+1\right)\cdot PH_3\left(3n+2\right), n\equiv 1,5 \bmod 6,\\ 3^n, n\equiv 2,4 \bmod 6,\\ -3^{n-1}-PH_3\left(3n+1\right)\cdot PH_3\left(3n+2\right), n\equiv 3 \bmod 6.\end{cases}$$

(iii).

$$JH_3\left(3n+1\right)^2=\begin{cases}0, & n\equiv 0 \bmod 6,\\ 3^{n-1}, & n\equiv 1,5 \bmod 6,\\ 3^n, & n\equiv 2,4 \bmod 6,\\ 4\cdot 3^{n-1}, & n\equiv 3 \bmod 6,\end{cases}$$

and

$$JH_3\left(3n+2\right)^2=\begin{cases}3^n, & n\equiv 0 \bmod 6,\\ 2\cdot 3^{n-1}+JH_3\left(3n+1\right)\cdot JH_3(3n+2), & n\equiv 1,5 \bmod 6,\\ 3^n, & n\equiv 2,4 \bmod 6,\\ -3^{n-1}+JH_3\left(3n+1\right)\cdot JH_3\left(3n+2\right), & n\equiv 3 \bmod 6.\end{cases}$$

Proof. **(i).** By induction on n, we may write

$$per\left(H_3^1\right)^n=\begin{cases}FH_3\left(3n+1\right)^2-FH_3\left(3n+2\right)^2=0, & \text{if } n \text{ is odd,}\\ FH_3\left(3n+2\right)^2-FH_3\left(3n+1\right)^2=\left(-1\right)^{\frac{n}{2}}\cdot 2^n, & \text{if } n \text{ is even.}\end{cases}$$

Then we complete the proof by using (2.10). Since $per\left(H_3^2\right)^n$ and $per\left(H_3^3\right)^n$, respectively denoted by

$$\begin{cases}-PH_3\left(3n+1\right)^2+PH_3\left(3n+2\right)^2+PH_3\left(3n+1\right)PH_3\left(3n+2\right)=3^n,\\ \qquad\qquad n\equiv 0 \bmod 6,\\ PH_3\left(3n+1\right)^2-PH_3\left(3n+2\right)^2-PH_3\left(3n+1\right)PH_3\left(3n+2\right)=-3^{n-1},\\ \qquad\qquad n\equiv 1,5 \bmod 6,\\ -PH_3\left(3n+1\right)^2+PH_3\left(3n+2\right)^2+PH_3\left(3n+1\right)PH_3\left(3n+2\right)=-3^n,\\ \qquad\qquad n\equiv 2,4 \bmod 6,\\ PH_3(3n+1)^2-PH_3(3n+2)^2-PH_3(3n+1)PH_3(3n+2)=5\cdot 3^{n-1},\\ \qquad\qquad n\equiv 3 \bmod 6\end{cases}$$

and

$$\begin{cases} -JH_3(3n+1)^2 + JH_3(3n+2)^2 - JH_3(3n+1)\,JH_3(3n+2) = 3^n, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad n \equiv 0 \bmod 6, \\ JH_3(3n+1)^2 - JH_3(3n+2)^2 + JH_3(3n+1)\,JH_3(3n+2) = -3^{n-1}, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad n \equiv 1,5 \bmod 6, \\ -JH_3(3n+1)^2 + JH_3(3n+2)^2 - JH_3(3n+1)\,JH_3(3n+2) = -3^n, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad n \equiv 2,4 \bmod 6, \\ JH_3(3n+1)^2 - JH_3(3n+2)^2 + JH_3(3n+1)\,JH_3(3n+2) = 5\cdot 3^{n-1}, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad n \equiv 3 \bmod 6, \end{cases}$$

the proofs of (ii) and (iii) are obvious. □

3 The Cyclic Groups via the matrices H_3^1, H_3^2 and H_3^3

For a given matrix $A = [a_{ij}]$ of integers, $A \pmod m$ means that the entries of A are reduced modulo m. Let $\langle A\rangle_m = \{(A)^n \pmod m \mid n \geq 0\}$. If $(\det A, m) = 1$, $\langle A\rangle_m$ is the cyclic group. We denote cardinal of the set $\langle A\rangle_m$by $|\langle A\rangle_m|$. Since $\det H_3^1 = -2$, it is clear that the set $\left\langle H_3^1\right\rangle_m$ is a cyclic group if $2\nmid m$. Smilarly, the sets $\left\langle H_3^2\right\rangle_m$ and $\left\langle H_3^3\right\rangle_m$ are cyclic groups if $3\nmid m$.

Now we consider the cyclic groups which are generated by the matrices H_3^1, H_3^2 and H_3^3.

Theorem 3.1 *Let M be any of the matrices H_3^1, H_3^2 and H_3^3. Suppose that u is the largest positive integer and p is a prime such that $(\det M, p) = 1$ and $\left|\langle M\rangle_p\right| = \left|\langle M\rangle_{p^u}\right|$. Then $\left|\langle M\rangle_{p^v}\right| = p^{v-u}\cdot\left|\langle M\rangle_p\right|$, for every $v \geq u$.*

Proof. Let us consider the cyclic group $\left\langle H_3^1\right\rangle_m$, then $2\nmid m$. Suppose that a is a positive integer and $\left|\left\langle H_3^1\right\rangle_m\right|$ is denoted by $k(m)$. If $\left(H_3^1\right)^{k\left(p^{a+1}\right)} \equiv I \pmod{p^{a+1}}$, then $\left(H_3^1\right)^{k\left(p^{a+1}\right)} \equiv I \pmod{p^a}$ where I is 3×3 identity matrix thus we get that $k(p^a)$ divides $k\left(p^{a+1}\right)$. Also, writing $\left(H_3^1\right)^{k(p^a)} = I + \left(h_{ij}^{(a)}\cdot p^a\right)$ we obtain

$$\left(H_3^1\right)^{k(p^a)\cdot p} = \left(I + \left(h_{ij}^{(a)}\cdot p^a\right)\right)^p = \sum_{i=0}^{p}\binom{p}{i}\left(h_{ij}^{(a)}\cdot p^a\right)^i \equiv I \pmod{p^{a+1}},$$

by the binomial expansion. This yields that $k\left(p^{a+1}\right)$ divides $k(p^a)\cdot p$. Thus, $k\left(p^{a+1}\right) = k(p^a)$ or $k\left(p^{a+1}\right) = k(p^a)\cdot p$. It is clear that $k\left(p^{a+1}\right) = k(p^a)\cdot p$ holds if and only if there exists an integer $h_{ij}^{(a)}$ which is not divisible by p. Since u is the largest positive integer such that $k(p) = k(p^u)$, $k(p^u) \neq k\left(p^{u+1}\right)$. There is an $h_{ij}^{(u+1)}$ which is not divisible by p. So we get that $k\left(p^{u+1}\right) \neq k\left(p^{u+2}\right)$. To complete the proof we may use an inductive method on u.

There are similar proofs for $\left\langle H_3^2\right\rangle_m$ and $\left\langle H_3^3\right\rangle_m$. □

Theorem 3.2 *Let G_m be any of the cyclic groups $\langle H_3^1\rangle_m$, $\langle H_3^2\rangle_m$ and $\langle H_3^3\rangle_m$ and let $m=\prod_{i=1}^{k} p_i^{e_i}$, $(k\geq 1)$ where p_i's are distinct primes, then*

$$|G_m| = \operatorname{lcm}\left[G_{p_1^{e_1}}, G_{p_2^{e_2}}, \cdots, G_{p_k^{e_k}}\right].$$

Proof. Let us consider the cyclic group $\langle H_3^2\rangle_m$, then $3\nmid m$. Let $\left|\langle H_3^2\rangle_{p_i^{e_i}}\right| = \lambda_i$ for $1\leq i\leq k$ and let $\left|\langle H_3^2\rangle_m\right| = \lambda$. Then by (2.8), we have

$$\begin{array}{c} PH_3(3\lambda_i+2)\equiv 1\ mod p_i^{e_i}, \\ PH_3(3\lambda_i+1)\equiv 0\ mod p_i^{e_i} \end{array}$$

and

$$\begin{array}{c} PH_3(3\lambda+2)\equiv 1\ mod m, \\ PH_3(3\lambda+1)\equiv 0\ mod m. \end{array}$$

Which implies that

$$PH_3(3\lambda+2) = c\cdot PH_3(3\lambda_i+2)$$

and

$$PH_3(3\lambda+1) = c\cdot PH_3(3\lambda_i+1),$$

where c is a positive integer, that is, $\left(H_3^2\right)^{\lambda}$ is of the form $c\cdot\left(H_3^2\right)^{\lambda_i}$ for all values of i. Thus it is verified that $\left|\langle H_3^2\rangle_m\right| = \operatorname{lcm}[\langle H_3^2\rangle_{p_1^{e_1}}, \langle H_3^2\rangle_{p_2^{e_2}}, ..., \langle H_3^2\rangle_{p_k^{e_k}}]$.

There are similar proofs for $\langle H_3^1\rangle_m$ and $\langle H_3^3\rangle_m$. □

It is well-known that a sequence is periodic if, after certain points, it consists only of repetitions of a fixed subsequence. The number of elements in the repeating subsequence is the period of the sequence. A sequence is simply periodic with period k if the first k elements in the sequence form a repeating subsequence.

Reducing 3-step Fibonacci-Hurwitz, 3-step Pell-Hurwitz and 3-step Jacobsthal-Hurwitz sequences by a modulus m, we can get a repeating sequences, respectively denoted by

$$\{FH_3^m(n)\} = \{FH_3^m(1),\ FH_3^m(2),\ FH_3^m(3),\cdots, FH_3^m(i),\cdots\},$$

$$\{PH_3^m(n)\} = \{PH_3^m(1),\ PH_3^m(2),\ PH_3^m(3),\cdots, PH_3^m(i),\cdots\}$$

and

$$\{JH_3^m(n)\} = \{JH_3^m(1),\ JH_3^m(2),\ JH_3^m(3),\cdots, JH_3^m(i),\cdots\}$$

where $FH_3^m(i) = FH_3(i) \pmod m$, $PH_3^m(i) = PH_3(i) \pmod m$ and $JH_3^m(i) = JH_3(i) \pmod m$. They have the same recurrences relation as in (2.1), (2.2) and (2.3), respectively.

Theorem 3.3 *If $2\nmid m$, the sequence $\{FH_3^m(n)\}$ is a simply periodic sequence. Similarly, the sequences $\{PH_3^m(n)\}$ and $\{JH_3^m(n)\}$ are simply periodic if $3\nmid m$.*

Proof. Let us consider the sequence $\{JH_3^m(n)\}$ and let $3 \nmid m$.

Let $X = \{(x_1, x_2, x_3) | \, 0 \leq x_i \leq m-1\}$.Then we have $|X| = m^3$. Thus, it is easy to see that the sequence repeats since there are only a finite number m^3 of triples of terms possible, and the recurrence of a triple results in recurrence of all following terms. So if $JH_3^m(i+3) \equiv JH_3^m(j+3)$, $JH_3^m(i+2) \equiv JH_3^m(j+2)$, $JH_3^m(i+1) \equiv JH_3^m(j+1)$ and $i > j$, then $i \equiv j \bmod 3$. From (2.3), we can easily derive

$$JH_3^m(i-1) \equiv JH_3^m(j-1), JH_3^m(i-2) \equiv JH_3^m(j-2), \cdots,$$
$$JH_3^m(i-j+1) \equiv JH_3^m(1),$$

which implies that the $\{JH_3^m(n)\}$is a simply periodic sequence.

The proofs for the sequences $\{FH_3^m(n)\}$ and $\{PH_3^m(n)\}$ are similar to the above and are omitted. □

We denote the periods of the sequences $\{FH_3^m(n)\}$, $\{PH_3^m(n)\}$ and $\{JH_3^m(n)\}$by $l_F(m)$, $l_P(m)$ and $l_J(m)$, respectively.

Then, we have the following useful results from (2.1) and (2.4), (2.2) and (2.5), (2.3) and (2.6), respectively.

Corollary 3.4 ***-(i).*** *If p is a prime such that $p \neq 2$, then $l_F(p) = 3 \cdot \left|\left\langle H_3^1 \right\rangle_p\right|$.*

(ii). *If p is a prime such that $p \neq 3$, then $l_P(p) = 3 \cdot \left|\left\langle H_3^2 \right\rangle_p\right|$ and $l_J(p) = 3 \cdot \left|\left\langle H_3^3 \right\rangle_p\right|$.*

4 The Fibonacci - Hurwitz, The Pell - Hurwitz and The Jacobsthal - Hurwitz Lengths of Groups

Let G be a finite j-generator group and let X be the subset of $\underbrace{G \times G \times G ... \times G}_{j}$ such that $(x_1, x_2, ..., x_j) \in X$ if, and only if, G is generated by x_1, $x_2, ...,$ x_j. We call $(x_1, x_2, \cdots, x_j)$ a generating $j-$tuple for G.

Definition 4.1 *Let G be 3-generator group. For a generating triple $(x_1, x_2, x_3) \in X$, Fibonacci-Hurwitz and co-Fibonacci-Hurwitz orbits of the group G are defined as follows:*

$$a_{i+3} = \begin{cases} (a_i)^{-1}(a_{i+1})^{-1}, i \equiv 1 \bmod 3, \\ (a_{i-1})(a_i)^{-1}, \quad i \equiv 2 \bmod 3, \quad \text{for } i \geq 1, \\ (a_{i-1})^{-1}(a_i)^{-1}, \ i \equiv 0 \bmod 3 \end{cases}$$

where $a_1 = x_1$, $a_2 = x_2x_3$, $a_3 = x_3$ for Fibonacci-Hurwitz orbit and $a_1 = x_1$, $a_2 = x_3x_2$, $a_3 = x_3$ for co-Fibonacci-Hurwitz orbit. The Fibonacci-Hurwitz orbit and the co-Fibonacci-Hurwitz orbit of a group G are denoted by $FH(G)_{(x_1,x_2,x_3)}$ and $c-FH(G)_{(x_1,x_2,x_3)}$, respectively.

Definition 4.2 *Let G be 3-generator group. For a generating triple $(x_1, x_2, x_3) \in X$, Pell-Hurwitz and co-Pell-Hurwitz orbits of the group G are defined as follows:*

$$a_{i+3} = \begin{cases} (a_i)^{-2}(a_{i+1})^{-1}, i \equiv 1 \bmod 3, \\ (a_{i-1})(a_i)^{-1}, \quad i \equiv 2 \bmod 3, \quad \text{for } i \geq 1, \\ (a_{i-1})^{-2}(a_i)^{-1}, \ i \equiv 0 \bmod 3 \end{cases}$$

where $a_1 = x_1$, $a_2 = x_2x_3$, $a_3 = x_3$ *for Pell-Hurwitz orbit and* $a_1 = x_1$, $a_2 = x_3x_2$, $a_3 = x_3$ *for co-Pell-Hurwitz orbit. The Pell-Hurwitz orbit and the co-Pell-Hurwitz orbit of a group* G *are denoted by* $PH(G)_{(x_1,x_2,x_3)}$ *and* $c-PH(G)_{(x_1,x_2,x_3)}$, *respectively.*

Definition 4.3 *Let* G *be 3-generator group. For a generating triple* $(x_1, x_2, x_3) \in X$, *Jacobsthal-Hurwitz and co-Jacobsthal-Hurwitz orbits of the group* G *are defined as follows:*

$$a_{i+3} = \begin{cases} (a_i)^{-1}(a_{i+1})^{-1}, & i \equiv 1 \bmod 3, \\ (a_{i-1})(a_i)^{-2}, & i \equiv 2 \bmod 3, \\ (a_{i-1})^{-1}(a_i)^{-1}, & i \equiv 0 \bmod 3 \end{cases} \quad \text{for } i \geq 1,$$

where $a_1 = x_1$, $a_2 = x_2x_3$, $a_3 = x_3$ *for Jacobsthal-Hurwitz orbit and* $a_1 = x_1$, $a_2 = x_3x_2$, $a_3 = x_3$ *for co-Jacobsthal-Hurwitz orbit. The Jacobsthal-Hurwitz orbit and the co-Jacobsthal-Hurwitz orbit of a group* G *are denoted by* $JH(G)_{(x_1,x_2,x_3)}$ *and* $c-JH(G)_{(x_1,x_2,x_3)}$, *respectively.*

Theorem 4.4 *Fibonacci-Hurwitz, co-Fibonacci-Hurwitz, Pell-Hurwitz, co-Pell-Hurwitz, Jacobsthal-Hurwitz and co-Jacobsthal-Hurwitz orbits of a finite group are periodic.*

Proof. Let us consider the Fibonacci-Hurwitz orbit $FH(G)_{(x_1,x_2,x_3)}$ and let n be the order of G. Since there n^3 distinct 3-tuples of elements of G, at least one of the 3-tuples appears twice in a Fibonacci-Hurwitz orbit of G. Thus, the subsequence following this 3-tuple repeats. Because of the repeating, the Fibonacci-Hurwitz orbit is periodic.

The proofs for other orbits are similar to the above and are omitted. □

We denote the lengths of the periods of the orbits

$$FH(G)_{(x_1,x_2,x_3)}, c-FH(G)_{(x_1,x_2,x_3)}, PH(G)_{(x_1,x_2,x_3)},$$
$$c-PH(G)_{(x_1,x_2,x_3)}, JH(G)_{(x_1,x_2,x_3)}, c-JH(G)_{(x_1,x_2,x_3)}$$

by

$$LFH(G)_{(x_1,x_2,x_3)}, Lc-FH(G)_{(x_1,x_2,x_3)}, LPH(G)_{(x_1,x_2,x_3)},$$
$$Lc-PH(G)_{(x_1,x_2,x_3)}, LJH(G)_{(x_1,x_2,x_3)}, Lc-JH(G)_{(x_1,x_2,x_3)},$$

respectively.

The triangle group (polyhedral group (l,m,n) for $l,m,n > 1$, is defined by the presentation

$$\left\langle x, y, z : x^l = y^m = z^n = xyz = e \right\rangle$$

or

$$\left\langle x, y : x^l = y^m = (xy)^n = e \right\rangle.$$

The triangle group (l,m,n) is finite if and only if the number

$$\mu = lmn\left(\frac{1}{l} + \frac{1}{m} + \frac{1}{n} - 1\right) = mn + nl + lm - lmn$$

is positive, i.e., in the case $(2,2,n)$, $(2,3,3)$, $(2,3,4)$ and $(2,3,5)$. Its order is $2lmn|\mu$. Using Tietze transformations we may show that $(l,m,n) \cong (m,n,l) \cong (n,l,m)$. For more information on these groups, see [6, pp. 67–68].

The extended triangle group $E(p,q,r)$, for $p,q,r>1$, is defined by the presentation

$$\left\langle x,y,z : x^2 = y^2 = z^2 = (xy)^p = (yz)^q = (zx)^r = e \right\rangle.$$

The extended triangle groups are a very important class of groups closely linked to the authomorphism groups of regular maps, see [5]. The triangle groups (p,q,r), are index two subgroups of extended triangle groups.

The group $E(p,q,r)$ is finite if and only if $\frac{1}{p}+\frac{1}{q}+\frac{1}{r}>1$.

For more information on these groups see [1,3] and [6, pp. 35–39].

Now we consider Fibonacci-Hurwitz, co-Fibonacci-Hurwitz, Pell-Hurwitz, co-Pell-Hurwitz, Jacobsthal-Hurwitz and co-Jacobsthal-Hurwitz orbits of the extended triangle group $E(2,q,2)$, for $q \geq 3$.

Theorem 4.5 *For $q \geq 3$, $LFH(E(2,q,2))_{(x,y,z)} = 24\cdot\lambda_1$ and $Lc-FH(E(2,q,2))_{(x,y,z)} = 24\cdot\lambda_2$, where $\lambda_1, \lambda_2 \in \mathbb{N}$.*

Proof. Firstly, let us consider the Fibonacci-Hurwitz orbit $FH(E(2,q,2))_{(x,y,z)}$. Then we have the sequence

$$\begin{array}{l} x, yz, z, xzy, xzy, zyz, (yz)^2, e, xz, (zy)^2, (yz)^2, zx, e, (zy)^4, \\ (zy)^2 zx, (yz)^4, (yz)^4, xyzy, (zy)^8, e, (zy)^2 zx, (yz)^8, (zy)^8, \\ xz(yz)^2, e, (yz)^{16}, (yz)^5 yx, (zy)^{16}, (zy)^{16}, (zy)^{12} zx, (yz)^{32}, \\ e, x(yz)^3 y, (zy)^{32}, (yz)^{32}, y(zy)^3 x, e, (zy)^{64}, (zy)^{28} zx, (yz)^{64}, \\ (yz)^{64}, xy(zy)^{35}, (zy)^{128}, e, (zy)^{28} zx, (yz)^{128}, (zy)^{128}, xz(yz)^{28}, \\ e, (yz)^{256}, (yz)^{99} yx, (zy)^{256}, (zy)^{256}, (zy)^{156} zx, (yz)^{512}, e, x(yz)^{99} y, \cdots. \end{array}$$

Using the above, the sequence becomes:

$$\begin{array}{l} a_7 = (yz)^2,\ a_8 = e,\ a_9 = xz, \\ a_{7+24i} = (yz)^{2^{4i+1}},\ a_{8+24i} = e,\ a_{9+24i} = x(yz)^{\frac{\left(2^{4i+1}-12\right)}{5}-1} y, \end{array}$$

where $i \geq 1$. So we get $LFH(E(2,q,2))_{(x,y,z)} = 24\cdot\lambda_1$ such that $q \geq 3$ and $\lambda_1 \in \mathbb{N}$. Secondly, let us consider the Fibonacci-Hurwitz orbit $c-FH(E(2,q,2))_{(x,y,z)}$. Since $c-FH(E(2,q,2))_{(x,y,z)}$ is

$$\begin{array}{l} x, zy, z, xyz, xyz, y, (zy)^2, e, zx, (yz)^2, (zy)^2, xz, e, (yz)^2, yzyx, \\ (zy)^2, (zy)^2, zx, (yz)^4, e, yzyx, (zy)^4, (yz)^4, xyzy, e, (zy)^8, \\ (zy)^2 zx, (yz)^8, (yz)^8, x(yz)^5 y, (zy)^{16}, e, (zy)^2 zx, (yz)^{16}, \\ (zy)^{16}, xz(yz)^2, e, (yz)^{32}, (yz)^{13} yx, (zy)^{32}, (zy)^{32}, (zy)^{18} zx, \\ (yz)^{64}, e, (yz)^{13} yx, (zy)^{64}, (yz)^{64}, xy(zy)^{13}, e, (zy)^{128}, (zy)^{50} zx, \\ (yz)^{128}, (yz)^{128}, (yz)^{77} yx, (zy)^{256}, e, (zy)^{50} zx, \cdots, \end{array}$$

we obtain

$$\begin{array}{l} a_7 = (zy)^2,\ a_8 = e,\ a_9 = zx, \\ a_{7+24i} = (zy)^{2^{4i}},\ a_{8+24i} = e,\ a_{9+24i} = (zy)^{\frac{\left(2^{4i}-6\right)}{5}} zx, \end{array}$$

where $i \geq 1$. Thus, it is clear that $Lc - FH\left(E\left(2,q,2\right)\right)_{(x,y,z)} = 24 \cdot \lambda_2$ such that $q \geq 3$ and $\lambda_2 \in \mathbb{N}$. $\square$

Theorem 4.6 *For $q \geq 3$, $LPH(E(2,q,2))_{(x,y,z)} = 18 \cdot \lambda_3$, $Lc - PH(E(2,q,2))_{(x,y,z)} = 18 \cdot \lambda_4$, $LJH(E(2,q,2))_{(x,y,z)} = 18 \cdot \lambda_5$, $Lc - JH\left(E\left(2,q,2\right)\right)_{(x,y,z)} = 18 \cdot \lambda_6$ such that $\lambda_3, \lambda_4, \lambda_5, \lambda_6 \in \mathbb{N}$.*

Proof. We process as similar to the proof of the Theorem (4.5). we obtain the orbits $PH\left(E\left(2,q,2\right)\right)_{(x,y,z)}$, $c - PH\left(E\left(2,q,2\right)\right)_{(x,y,z)}$, $JH\left(E\left(2,q,2\right)\right)_{(x,y,z)}$ and $c - JH\left(E\left(2,q,2\right)\right)_{(x,y,z)}$ as follows, respectively:

$$a_{1+36i} = x,$$
$$a_{2+36i} = (yz)^{3^{6i}},$$
$$a_{3+36i} = (yz)^{2^4 \cdot 13\left(3^{6(i-1)+1}+3^{6(i-2)+1}+\cdots+3\right)-1}\, y,$$
$$a_{10+36i} = (zy)^{2\cdot 3^{6i+1}}\, x,$$
$$a_{11+36i} = (yz)^{3^{6i+1}},$$
$$a_{12+36i} = (yz)^{2^4 \cdot 13\left(3^{6(i-1)+1}+3^{6(i-2)+1}+\cdots+3\right)-1}\, y,$$
$$a_{19+36i} = x,$$
$$a_{20+36i} = (zy)^{3^{6i+3}},$$
$$a_{21+36i} = (zy)^{2^4 \cdot 13\left(3^{6(i-1)+4}+3^{6(i-2)+4}+\cdots+3^4\right)+24}\, z,$$
$$a_{28+36i} = (yz)^{2\cdot 3^{6i+4}}\, x,$$
$$a_{29+36i} = (zy)^{3^{6i+4}},$$
$$a_{30+36i} = (zy)^{2^4 \cdot 13\left(3^{6(i-1)+4}+3^{6(i-2)+4}+\cdots+3^4\right)+24}\, z, \cdots,$$

$$a_{1+36i} = x,$$
$$a_{2+36i} = (zy)^{3^{6i}},$$
$$a_{3+36i} = (zy)^{2^4 \cdot 13\left(3^{6(i-1)+1}+3^{6(i-2)+1}+\cdots+3\right)}\, z,$$
$$a_{10+36i} = (yz)^{2\cdot 3^{6i+1}}\, x,$$
$$a_{11+36i} = (zy)^{3^{6i+1}},$$
$$a_{12+36i} = (zy)^{2^4 \cdot 13\left(3^{6(i-1)+1}+3^{6(i-2)+1}+\cdots+3\right)}\, z,$$
$$a_{19+36i} = x,$$
$$a_{20+36i} = (yz)^{3^{6i+3}},$$
$$a_{21+36i} = (yz)^{2^4 \cdot 13\left(3^{6(i-1)+4}+3^{6(i-2)+4}+\cdots+3^4\right)+23}\, y,$$

$$
\begin{aligned}
a_{28+36i} &= (zy)^{2\cdot 3^{6i+4}}\, x,\\
a_{29+36i} &= (yz)^{3^{6i+4}},\\
a_{30+36i} &= (yz)^{2^4\cdot 13\left(3^{6(i-1)+4}+3^{6(i-2)+4}+\cdots+3^4\right)+23}\, y,\cdots,
\end{aligned}
$$

$$
\begin{aligned}
a_{1+36i} &= x,\\
a_{2+36i} &= (yz)^{3^{6i}},\\
a_{3+36i} &= (yz)^{2^4\cdot 13\left(3^{6(i-1)}+3^{6(i-2)}+\cdots+1\right)-1}\, y,\\
a_{10+36i} &= (zy)^{2\cdot 3^{6i+1}}\, x,\\
a_{11+36i} &= (zy)^{3^{6i+1}},\\
a_{12+36i} &= (zy)^{2^5\cdot 13\left(3^{6(i-1)+1}+3^{6(i-2)+1}+\cdots+3\right)+2}\, z,\\
a_{19+36i} &= x,\\
a_{20+36i} &= (yz)^{3^{6i+3}},\\
a_{21+36i} &= (zy)^{2^4\cdot 13\left(3^{6(i-1)+3}+3^{6(i-2)+3}+\cdots+3^3\right)+8}\, z,\\
a_{28+36i} &= (yz)^{2\cdot 3^{6i+4}}\, x,\\
a_{29+36i} &= (yz)^{3^{6i+4}},\\
a_{30+36i} &= (yz)^{2^5\cdot 13\left(3^{6(i-1)+4}+3^{6(i-2)+4}+\cdots+3^4\right)+45}\, y,\cdots
\end{aligned}
$$

and

$$
\begin{aligned}
a_{1+36i} &= x,\\
a_{2+36i} &= (zy)^{3^{6i}},\\
a_{3+36i} &= (zy)^{2^4\cdot 13\left(3^{6(i-1)}+3^{6(i-2)}+\cdots+1\right)}\, z,\\
a_{10+36i} &= (yz)^{2\cdot 3^{6i+1}}\, x,\\
a_{11+36i} &= (zy)^{3^{6i+1}},\\
a_{12+36i} &= (yz)^{2^5\cdot 13\left(3^{6(i-1)+1}+3^{6(i-2)+1}+\cdots+3\right)+1}\, y,\\
a_{19+36i} &= x,\\
a_{20+36i} &= (yz)^{3^{6i+3}},\\
a_{21+36i} &= (yz)^{2^4\cdot 13\left(3^{6(i-1)+3}+3^{6(i-2)+3}+\cdots+3^3\right)+7}\, y,\\
a_{28+36i} &= (zy)^{2\cdot 3^{6i+4}}\, x,\\
a_{29+36i} &= (zy)^{3^{6i+4}},\\
a_{30+36i} &= (zy)^{2^5\cdot 13\left(3^{6(i-1)+4}+3^{6(i-2)+4}+\cdots+3^4\right)+46}\, z,\cdots.
\end{aligned}
$$

Which implies that the lengths of the periods of Pell-Hurwitz, co-Pell-Hurwitz, Jacobsthal-Hurwitz and co-Jacobsthal-Hurwitz orbits of the extended triangle group $E(2,q,2)$, $(q \geq 3)$ are $18 \cdot \lambda_3$, $18 \cdot \lambda_4$, $18 \cdot \lambda_5$, $18 \cdot \lambda_6$, repectively, where $\lambda_3, \lambda_4, \lambda_5, \lambda_6 \in \mathbb{N}$. □

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