An. Ştiinţ. Univ. Al. I. Cuza Iaşi. Mat. (N.S.)
Tomul LXIII, 2017, f. 3

Nilpotency of Atomic Steenrod Squares

Özgür Ege · Ismet Karaca

Received: 17.VII.2014 / Accepted: 30.IV.2015

Abstract In this paper, we deal with nilpotency of atomic Steenrod squares. We prove that the nilpotence height of atomic square $Sq^{2(2^b-1)}$ for all integers $b \geq 2$ is $b+2$. Finally, we give a nilpotency table, a remark and an important conjecture related to the nilpotence height of atomic squares.

Keywords Steenrod Algebra · Steenrod square · nilpotence

Mathematics Subject Classification (2010) 55S05 · 55S10 · 57T05

1 Introduction

The Steenrod algebra, first constructed by N. Steenrod [12], is the algebra of the stable cohomology operations on mod 2 cohomology. It has a Hopf structure and its dual is a commutative Hopf algebra which is isomorphic to the polynomial algebra in generators ξ_k of degree 2^k-1 $(k \geq 0)$. The Steenrod algebra is considerably studied by researchers whose interests range from algebraic topology and cohomology theory, to manifold theory, homotopy theory, differential topology, and more. For more details and applications see [10, 12] and for a history of the study of the Steenrod algebra see [15].

The Steenrod algebra is one of the efficient tools in the hands of algebraic topologists since the Steenrod operations which generate it, have played a central role in homotopy theory. The structure of this algebra is important for applications to homotopy theory. The Steenrod square operations $Sq^i : H^n(X;\mathbb{Z}_2) \longrightarrow H^{n+i}(X;\mathbb{Z}_2)$ which help us to

Özgür Ege (Corresponding author)
Department of Mathematics,
Celal Bayar University,
Muradiye, Manisa, 45140, Turkey
E-mail: ozgur.ege@cbu.edu.tr

Ismet Karaca
Department of Mathematics,
Ege University,
Bornova, Izmir, 35100, Turkey
E-mail: ismet.karaca@ege.edu.tr

solve some questions in algebraic topology where $H^n(X;\mathbb{Z}_2)$ is the nth cohomology group of X with coefficient $\mathbb{Z}_2$.

The determining the nilpotence height of the positive dimensional Steenrod operations was thought first by John Milnor in 1960's. Important results related to this problem have been given by Davis, Wood, Walker, Monks and Karaca up to now. Nevertheless, due to the difficulty of structure of Steenrod algebra, a general solution for this problem has not been found yet. The nilpotence height of Sq^{2^n} was conjectured to be $2n+2$ by Steve Wilson in 1975. Donald M. Davis showed that $(Sq^{2^n})^{2n+1} \neq 0$. Moreover, Davis verified by computer calculation that

$$(Sq^{2^n})^{2n+2} = 0 \qquad \text{for} \quad n \leq 5.$$

This was also verified for $n = 6, 7$ by Kenneth Monks. This conjecture was proved by Walker and Wood [13]. They generalized this result for odd primes p in [14]. Monks [8] determined the nilpotence height of Milnor elements P_t^s when $p = 2$. Karaca used the method of Monks [8] and showed that [4] the nilpotence height of Milnor elements P_t^s is exactly $p\left(\left[\frac{s}{t}\right]+1\right)$. Monks [7] found the nilpotence of certain families of elements Sq^n. Karaca [5] adapted these results to odd primes. For example, Karaca [5] proved that the nilpotence heights of

$$P\left(\frac{p^k((p-1)p^m-1)+1}{p-1}\right) \qquad \text{and} \qquad P(p^m-1)$$

are $m+2$ and $m+1$, respectively. Karaca gave some nilpotence relations of Steenrod powers such as

$$Nil(P^{pr_1+p-1,\ldots,pr_n+p-1}) \leq \min\{k \mid r_n < p^{(k-1)n}-1\}$$

where $r_1, r_2, r_3, \ldots$ are arbitrary integers. Ege and Karaca [3] determined the structure of ideals for the mod p Steenrod algebra and calculated the nilpotence heights of some Steenrod operations such as P^{2p} and P^{2p+1}.

This paper is organized as follows. In the next section, we give the construction of the mod 2 Steenrod algebra and basic facts related to our study. In the section 3, we prove some propositions and a theorem related to the nilpotence height of atomic Steenrod square $Sq^{2(2^b-1)}$ for all integers $b \geq 2$. We also give a conjecture on atomic Steenrod squares. In the last section, we make a conclusion on this study.

2 Preliminaries

The Steenrod operation is a natural transformation

$$Sq^i : H^n(X;\mathbb{Z}_2) \longrightarrow H^{n+i}(X;\mathbb{Z}_2)$$

that has certain conditions such as instability. Serre [11] showed that the Steenrod squares Sq^k generate all stable operations in the cohomology theory. Adem [1,2] showed that all relations in the Steenrod algebra are generated by the set of Adem relations:

$$Sq^i Sq^j = \sum_{k=0}^{[\frac{i}{2}]} \binom{j-k-1}{i-2k} Sq^{i+j-k} Sq^k$$

for all $i, j > 0$ such that $i < 2j$.

The mod 2 Steenrod algebra $\mathcal{A}_2$ is the graded associative graded algebra over $\mathbb{Z}_2$ generated by the elements Sq^i of degree i, $i \geq 0$, subject to the Adem relations.

Serre [11] introduced the notion of admissible monomial. A monomial $Sq^{i_1} Sq^{i_2} ... Sq^{i_n}$ in the mod 2 Steenrod algebra is said to be *admissible* [12] if $i_h \geq 2i_{h+1}$ for all $h \geq 1$, $(i_{n+1} = 0)$.

Serre [11] also showed that the set of admissible monomials is an additive basis for the Steenrod algebra. Adem relations lead us to have a minimal algebraic generating set for the Steenrod algebra. An element in grading i is called *indecomposable* [12] if it cannot be written as a linear combination of products of elements of grading lower than i and the squaring operations Sq^{2^k} are indecomposable and the Steenrod algebra is generated as an algebra by Sq^0 and the Sq^{2^k} for $k \geq 0$.

Milnor [6] has shown that there is a unique algebra map $\psi : \mathcal{A}_2 \longrightarrow \mathcal{A}_2 \otimes \mathcal{A}_2$ such that for all $i \geq 0$

$$\psi(Sq^i) = \sum_{j+k=i} Sq^j \otimes Sq^k$$

which makes it into a Hopf algebra.

Axiomatic description of the mod 2 Steenrod algebra is given in the following [11]:

1. For all integers $i \geq 0$ and $q \geq 0$, there is a natural transformation of functors which is a homomorphism
$$Sq^i : H^q(X; \mathbb{Z}_2) \longrightarrow H^{q+i}(X; \mathbb{Z}_2).$$
2. $Sq^0 = 1$.
3. If $dim\ x = n$, $Sq^n x = x^2$.
4. If $i > dim\ x$, $Sq^i x = 0$.
5. (Cartan formula)
$$Sq^k(xy) = \sum_{i=0}^{k} Sq^i x . Sq^{k-i} y.$$
6. Sq^1 is the Bockstein homomorphism β of the coefficient sequence
$$0 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathbb{Z}_4 \longrightarrow \mathbb{Z}_2 \longrightarrow 0.$$
7. (Adem relations) If $0 < a < 2b$, then
$$Sq^a Sq^b = \sum_{j=0}^{[\frac{a}{2}]} \binom{b-1-j}{a-2j} Sq^{a+b-j} Sq^j.$$

The binomial coefficient is taken mod 2.

We would like to give some results from [6]. The mod 2 Steenrod algebra is a graded $\mathbb{Z}_2$-vector space with basis all $Sq(r_1, r_2, \ldots)$ where $r_i \geq 0$ and $r_i > 0$ for finitely many i. Let $R = (r_1, \ldots, r_m)$. We will write Sq^R for the Milnor basis element $Sq(r_1, \ldots, r_m)$. We recall that $Sq^{(r,0,\ldots)} = Sq^r$ where r is nonnegative integer.

The Milnor product is given by

$$Sq(r_1, r_2, \ldots) \cdot Sq(s_1, s_2, \ldots) = \sum_X \beta(X) Sq(t_1, t_2, \ldots)$$

where the summation is taken over all matrices

$$X = (x_{ij})$$

satisfying:

$$\sum_i x_{ij} = s_j \tag{2.1}$$

$$\sum_j 2^j x_{ij} = r_i \tag{2.2}$$

$$\beta(X) = \prod_h (x_{h0}, x_{h-11}, \ldots, x_{0h}) \in \mathbb{Z}_2 \tag{2.3}$$

where

$$(n_1, \ldots, n_m) = \frac{(n_1 + \ldots + n_m)!}{n_1! \ldots n_m!}.$$

Such a matrix X is called $Sq(r_1, r_2, \ldots) \cdot Sq(s_1, s_2, \ldots)$-allowable. Each such allowable matrix yields a summand $Sq(t_1, t_2, \ldots)$ given by

$$t_h = \sum_{i+j=h} x_{ij}. \tag{2.4}$$

We note that x_{00} is never used and is assumed to be zero.

Let's give an example about Milnor product of any two Milnor basis elements.

Example 2.1 We compute the Milnor product of $Sq(2, 1, 3)$ and $Sq(1, 0, 2)$. Let

$$Sq(2, 1, 3) \cdot Sq(1, 0, 2) = \sum_X \beta(X) Sq(t_1, t_2, \ldots).$$

We now determine all coefficients $\beta(X)$ depending on all matrices X.

From the Figure 1, we get the following coefficients:

$$\begin{aligned}
\beta_1(X_1) &= \frac{3!}{2!1!} \cdot \frac{0!}{0!0!0!} \cdot \frac{4!}{2!2!0!0!} = 18 \equiv 0 \pmod 2 \\
\beta_2(X_2) &= \frac{2!}{2!0!} \cdot \frac{0!}{0!0!0!} \cdot \frac{2!}{2!0!0!0!} \cdot \frac{1!}{1!0!0!} \equiv 1 \pmod 2 \\
\beta_3(X_3) &= \frac{0!}{0!0!} \cdot \frac{1!}{1!0!0!} \cdot \frac{4!}{2!2!0!0!} = 6 \equiv 0 \pmod 2.
\end{aligned}$$

$$
\begin{array}{c|cccc}
 & 2^0 & 2^1 & 2^2 & 2^3 \\
 & & 1 & 0 & 2 \\
\hline
 & & 0 & 0 & 2 \\
2 & 0 & 1 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 \\
3 & 3 & 0 & 0 & 0
\end{array}
\qquad
\begin{array}{c|cccc}
 & 2^0 & 2^1 & 2^2 & 2^3 \\
 & & 1 & 0 & 2 \\
\hline
 & & 0 & 0 & 2 \\
2 & 2 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 \\
3 & 1 & 1 & 0 & 0
\end{array}
\qquad
\begin{array}{c|cccc}
 & 2^0 & 2^1 & 2^2 & 2^3 \\
 & & 1 & 0 & 2 \\
\hline
 & & 1 & 0 & 2 \\
2 & 2 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 \\
3 & 3 & 0 & 0 & 0
\end{array}
$$

Fig. 1 Matrices X_1, X_2 and X_3

As a result, we have

$$
\begin{aligned}
Sq(2,1,3)\cdot Sq(1,0,2) &= \beta_1(X)Sq(0,2,5,0,0)+\beta_2(X)Sq(2,1,3,1,0)\\
&\quad+\beta_3(X)Sq(3,1,5,0,0)\\
&= Sq(2,1,3,1).
\end{aligned}
$$

We recall that two important results about the nilpotence heights of some elements.

Theorem 2.1 **([7])** *For all integers $m, k \geq 1$,*

$$Nil(Sq^{2^m(2^k-1)+1}) = k+2.$$

Theorem 2.2 **([13])** *For every integers $n \geq 0$,*

$$Nil(Sq^{2^n}) = 2n+2.$$

3 Main Results

Let $x \in \mathcal{A}$. If $x^k = 0$ and $x^{k-1} \neq 0$ where $x^0 = 1$, then we say that x has nilpotence k and we write $Nil(x) = k$. In this section, we will determine the nilpotence height of $Sq^{2(2^b-1)}$ for all integers $b \geq 2$.

Proposition 3.1 *For all integers $b \geq 2$, we have*

$$Nil(Sq^{2(2^b-1)}) \leq \min\{k \mid 2(2^b-1) < 2^{k-1}-1\}.$$

Proof. It is enough to show that $(Sq^n)^k = 0$ if $n < 2^{k-1}-1$, where $n = 2^{b+1}-2$ and $b \geq 2$ is an integer. Let $Sq^T = Sq^{(t_1,t_2,\ldots,t_h)}$ be any summand of $(Sq^n)^{k-1}$. Let also $X = (x_{1j})$ be any Sq^nSq^T-allowable matrix. From the definition of Milnor product, we have $h = k-1$. If we combine this with (2.2), we get

$$n = \sum_{j=0}^{k-1} 2^j x_{1j} \geq \sum_{j=0}^{k-1} 2^j = 2^k - 1.$$

Thus if $n < 2^{k-1}-1$, then there are no Sq^nSq^T-allowable matrices. So we conclude that $(Sq^n)^k = 0$ and this completes the proof. $\square$

Proposition 3.2 *For all integers $b \geq 2$, we have*

$$Nil(Sq^{2(2^b-1)}) > \max\{k \mid 2(2^b-1) \equiv -2 \ (mod\ 2^k)\}.$$

Proof. Let k be the largest integer such that

$$n \equiv -2 \ (mod\ 2^k)$$

where $n = 2^{b+1} - 2$ and $b \geq 2$ is an integer. By Milnor product, for $1 \leq h \leq k$, $(Sq^n)^h$ always has at least a nonzero summand which is given by

$$Sq^{(a_1, a_2, \ldots, a_h)}$$

such that $a_1 + a_2 + \ldots + a_h = n$. So we have

$$(Sq^n)^k \neq 0.$$

As a result, we get

$$Nil(Sq^n) > \max\{k \mid n \equiv -2 \ (mod\ 2^k)\},$$

and this completes the proof. □

Theorem 3.3 $Nil(Sq^{2(2^b-1)}) = b + 2$ *for all integers* $b \geq 2$.

Proof. From Proposition 3.1, we have

$$Nil(Sq^{2(2^b-1)}) \leq b + 2,$$

and by Proposition 3.2, we get

$$Nil(Sq^{2(2^b-1)}) > b + 1.$$

We conclude that for all integers $b \geq 2$,

$$Nil(Sq^{2(2^b-1)}) = b + 2.$$

□

Using a Maple package program created by Ken Monks [9], we get some important results. For the nilpotence height of $Sq^{2^a(2^b-1)}$, see Figure 2. In this table, numbers with star are expectable nilpotence heights, dark-colored numbers are nilpotence heights of corresponding atomic squares.

Some results related to atomic squares can be infered from the Figure 2 as follows:

Remark 3.1 There are two main results for atomic squares $Sq^{2^a(2^b-1)}$ where integers $a \geq 0$ and $b > 0$:

(1) **i)** If $a < c$, then

$$Nil(Sq^{2^a(2^b-1)}) < Nil(Sq^{2^c(2^d-1)}),$$

ii) If $b > d$, then

$$Nil(Sq^{2^a(2^b-1)}) < Nil(Sq^{2^c(2^d-1)}),$$

where $a + b = n = c + d$, $c \geq 0$ and $d > 0$.

a\b	1	2	3	4	5	6	7
0	1-2	3-3	7-4	15-5	31-6	63-7	127-8
1	2-4	6-4	14-5	30-6	62-7	126-8	254-9
2	4-6	12-7	28-8	60-9	124-10	252-11	508-12
3	8-8	24-8	56-9	120-10*	248-11*	504-12*	1016-13*
4	16-10	48-11	112-12*	240-13*	496-14*	1008-15*	1032-16*
5	32-12	96-12	224-13*	480-14*	992-15*	2016-16*	2064-17*
6	64-14	192-15*	448-16*	960-17*	1984-18*	4032-19*	8128-20*
7	128-16	384-16*	896-17*	1920-18*	3968-19*	8064-20*	16256-21*
8	256-18	768-19*	1792-20*	3840-21*	7936-22*	16128-23*	32512-24*
9	512-20	1536-20*	3584-21*	7680-22*	15872-23*	32256-24*	65024-25*
10	1024-22	3072-23*	7168-24*	15360-25*	31744-26*	64512-27*	130048-28*
11	2048-24	6144-24*	14336-25*	30720-26*	63488-27*	129024-28*	260096-29*
12	4096-26	12288-27*	28672-28*	61440-29*	126976-30*	258048-31*	520192-32*
13	8192-28	24576-28*	57344-29*	122880-30*	253952-31*	516096-32*	1040384-33*
14	16384-30	49152-31*	114688-32*	245760-33*	507904-34*	1032192-35*	2080768-36*
15	32768-32	98304-32*	229376-33*	491520-34*	1015808-35*	2066384-36*	4161536-37*

Fig. 2 A table of the nilpotence height of $Sq^{2^a(2^b-1)}$

(2) $n+1 \leq Nil(Sq^{2^a(2^b-1)}) \leq 2n$, where $a+b=n$.

The following conjecture is related to the nilpotence height of atomic square $Sq^{2^a(2^b-1)}$.

Conjecture 3.4 For all integers $a \geq 1$ and $b > 1$,

$$Nil(Sq^{2^a(2^b-1)}) = \begin{cases} b+2a, & a \text{ is odd} \\ b+2a+1, & a \text{ is even.} \end{cases}$$

4 Conclusion

The goal of this work is to determine the nilpotence height of atomic square $Sq^{2(2^b-1)}$ for all integers $b \geq 2$. For this reason, we use a Maple package program and make some calculations. An important conjecture is given in the last section about nilpotence heights of all atomic squares $Sq^{2^a(2^b-1)}$. This results will be useful in the study of nilpotence in mod 2 Steenrod algebra.

Acknowledgements We would like to express our gratitude to the anonymous referees for their helpful suggestions and corrections. This research was partially supported by Ege University Fund Treasurer (Project No: 13FEN044).

References

1. Adem, J. – *The iteration of the Steenrod squares in algebraic topology*, Proc. Nat. Acad. Sci. USA, 38 (1952), 720–726.
2. Adem, J. – *The relations on Steenrod powers of cohomology classes. Algebraic geometry and topology.* A symposium in honour of S. Lefschetz, Princeton University Press, Princeton, N. J., (1957), 191–238.
3. Ege, Ö.; Karaca, I. – *Some results of the nilpotence in the mod p Steenrod algebra*, Comptes rendus de l'Academie bulgare des Sciences, 67 (2014), no. 12, 1611–1620.

4. Karaca, I. – *The nilpotence height of P_t^s for odd primes*, Trans. Amer. Math. Soc., 351 (1999), no. 2, 547–558.
5. Karaca, I. – *Nilpotence relations in the mod p Steenrod Algebra*, Journal of Pure and Applied Algebra, 171 (2002), no. 2-3, 257–264.
6. Milnor, J. – *The Steenrod algebra and its dual*, Annals of Math. (2), 67 (1958), 150–171.
7. Monks, K.G. – *Nilpotence in the Steenrod algebra. Papers in honor of José Adem (Spanish)*, Bol. Soc. Mat. Mexicana (2), 37 (1992), no. 1-2, 401–416.
8. Monks, K.G. – *The nilpotence height of P_t^s*, Proc. Amer. Math. Soc., 124 (1996), no. 4, 1297–1303.
9. Monks, K.G. – Steenrod: a maple package for computing with the Steenrod algebra, (1995), http://mathweb.scranton.edu/monks/software/Steenrod/steen.html.
10. Mosher, R.E.; Tangora, M.C. – *Cohomology operations and applications in homotopy theory*, Harper and Row, Publishers, New York - London, (1968), x+214 pp.
11. Serre, J.P. – *Cohomologie modulo 2 des complexes d'Eilenberg-MacLane(French)*, Comment. Math. Helv., 27 (1953), 198–232.
12. Steenrod, N.E.; Epstein, D.B.A. – *Cohomology operations. Lectures by N. E. Steenrod written and revised by D.B.A. Epstein.*, Annals of Mathematics Studies, no. 50, Princeton University Press, Princeton, N.J., (1962), vii+139 pp.
13. Walker, G.; Wood, R.M.W. – *The nilpotence height of Sq^{2^n}*, Proc. Amer. Math. Soc., 124 (1996), no. 4, 1291–1295.
14. Walker, G.; Wood, R.M.W. – *The nilpotence height of P^{p^n}*, Math. Proc. Cambridge Philos. Soc. 123 (1998), no. 1, 85–93.
15. Wood, R.M.W. – *Problems in the Steenrod algebra*, Bull. London Math. Soc., 30 (1998), no. 5, 449–517.