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Null 2-type Hypersurfaces in E^5 with all distinct principal curvatures

Ram Shankar Gupta ·
Deepika Kumari ·
Sharfuddin Ahmad

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Abstract In this paper, we prove that null 2-type hypersurfaces in Euclidean space E^5 with all distinct principal curvatures of constant scalar curvature must be of constant mean curvature.

Keywords Null 2-type submanifolds · Mean curvature vector

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1 Introduction

Let M^n be an n-dimensional, connected submanifold of the Euclidean space E^m. Denote by x, $\mathbf{H}$, and $\triangle$ respectively the position vector field of M^n, the mean curvature vector field of M^n, and the Laplace operator on M^n, with respect to the Riemannian metric g on M^n, induced from the Euclidean metric of the ambient space E^m. Then, as it is well known, (cf. [1])

$$\triangle x = -n\mathbb{H}. \tag{1.1}$$

Ram Shankar Gupta
University School of Basic and Applied Sciences,
Guru Gobind Singh Indraprastha University,
Sector-16C, Dwarka, New Delhi-110078, India,
E-mail: ramshankar.gupta@gmail.com

Deepika Kumari
University School of Basic and Applied Sciences,
Guru Gobind Singh Indraprastha University,
Sector-16C, Dwarka, New Delhi-110078, India,
E-mail: sdeep2007@gmail.com

Sharfuddin Ahmad
Department of Mathematics,
Faculty of Natural Sciences,
Jamia Millia Islamia (Central University),
New Delhi-110025, India,
E-mail: sharfuddin_ahmad12@yahoo.com

A submanifold of E^m is said to be of finite type [1,2,6] if the position vector x of M^n in E^m can be decomposed in the following form:

$$x = x_0 + x_1 + \cdots + x_k, \tag{1.2}$$

where x_0 is a constant vector and x_1,..., x_k are non-constant maps satisfying $\triangle x_q = \lambda_q x_q$, $q = 1, ..., k$. If all eigenvalues λ_q, $q = 1, ..., k$ are mutually different, then the submanifold is said to be of k-type. A submanifold is called null k-type if one of λ_q, $q = 1, ..., k$ is zero. By taking x_0 at origin, we have the following simple spectral decomposition of x for a null 2-type submanifold M:

$$x = x_1 + x_2, \triangle x_1 = 0, \triangle x_2 = \lambda x_2, \tag{1.3}$$

where λ is non-zero constant. From (1.1) and (1.3), we have

$$\triangle \mathbb{H} = \lambda \mathbb{H}. \tag{1.4}$$

In 1991, B. Y. Chen proposed the following interesting problem [2]:

"Determine all submanifolds of Euclidean spaces which are of null 2-type".

In [3], B. Y. Chen has given the classification of null 2-type surfaces in the Euclidean space E^3 and proved that they are circular cylinders. In [10], it was proved that a null 2-type Euclidean hypersurface in E^{n+1} with at most two distinct principal curvatures is a spherical cylinder $S^p \times \mathbb{R}^{n-p}$. Later, in [4], Chen proved that a surface M in the Euclidean space E^4 is of null 2-type with parallel normalized mean curvature vector if and only if M is an open portion of a circular cylinder in a hyperplane of E^4; and the only null 2-type surfaces of the Euclidean space E^4 with constant mean curvature are open portion of helical cylinders, by which we mean the product surfaces of a straight line and a helix.

In 1995, Hasanis and Vlachos [11] proved that null 2-type hypersurfaces in E^4 have constant mean curvature. Recently, Chen and Garray in [5] characterized $\delta(2)$-ideal null 2-type hypersurfaces in Euclidean space are spherical cylinders.

In [8], U. Dursun classified 3-dimensional null 2-type submanifolds of the Euclidean space E^5 with parallel normalized mean curvature vector. It was proved [9] that a 3-dimensional submanifold M of the Euclidean space E^5 such that M is not of 1-type is an open portion of a 3-dimensional helical cylinder if and only if M is flat and of null 2-type with constant mean curvature and non-parallel mean curvature vector. Recently, Yu Fu have proved that every null 2-type hypersurfaces with at most three distinct principal curvatures in a Euclidean space have constant mean curvature [12]. For more work in this field, see [7].

Therefore, in this paper, we have studied null 2-type hypersurfaces in E^5 with all distinct principal curvatures.

We denote

$$\begin{aligned} a_2 = \tfrac{(\lambda_2-\lambda_1)H}{e_1(H)}&[(3\lambda_4 - \lambda_2 - 2\lambda_1)(\lambda_3 - \lambda_2)^2(\omega_{44}^1 - \omega_{22}^1)^2 \\ &-(3\lambda_3 - \lambda_2 - 2\lambda_1)(\lambda_4 - \lambda_2)^2(\omega_{33}^1 - \omega_{22}^1)^2], \end{aligned}$$

$$\begin{aligned} b_2 = \tfrac{(\lambda_3-\lambda_1)H}{e_1(H)}&[(3\lambda_4 - \lambda_3 - 2\lambda_1)(\lambda_2 - \lambda_3)^2(\omega_{44}^1 - \omega_{33}^1)^2 \\ &-(3\lambda_2 - \lambda_3 - 2\lambda_1)(\lambda_4 - \lambda_3)^2(\omega_{22}^1 - \omega_{33}^1)^2], \end{aligned}$$

$$\begin{aligned} c_2 = \tfrac{(\lambda_4-\lambda_1)H}{e_1(H)}&[(3\lambda_2 - \lambda_4 - 2\lambda_1)(\lambda_3 - \lambda_4)^2(\omega_{22}^1 - \omega_{44}^1)^2 \\ &-(3\lambda_3 - \lambda_4 - 2\lambda_1)(\lambda_2 - \lambda_4)^2(\omega_{33}^1 - \omega_{44}^1)^2], \end{aligned}$$

then, we conclude that:

Theorem 1.1 *Let M be a null 2-type hypersurface in the Euclidean space E^5 with all distinct principal curvatures. If $a_2 = 0$, $b_2 = 0$ and $c_2 = 0$, then it has constant mean curvature.*

In the following, we replace the condition that $a_2 = 0$, $b_2 = 0$ and $c_2 = 0$ using the geometric condition that the hypersurface is of constant scalar curvature. We obtain

Theorem 1.2 *Let M be a null 2-type hypersurface in the Euclidean space E^5 with all distinct principal curvatures. If scalar curvature of the hypersurface is constant, then it has constant mean curvature.*

2 Preliminaries

Let (M, g) be a hypersurface isometrically immersed in a 5-dimensional Euclidean space $(E^5, \overline{g})$ and $g = \overline{g}_{|M}$.

Let $\overline{\nabla}$ and ∇ denote linear connections on E^5 and M, respectively. Then, the Gauss and Weingarten formulae are given by

$$\overline{\nabla}_X Y = \nabla_X Y + h(X, Y), \ \ \forall \ X, Y \in \Gamma(TM), \tag{2.1}$$

$$\overline{\nabla}_X \xi = -A_\xi X, \tag{2.2}$$

where ξ be the unit normal vector to M, h is the second fundamental form and A is the shape operator. It is well known that the second fundamental form h and shape operator A are related by

$$\overline{g}(h(X, Y), \xi) = g(A_\xi X, Y). \tag{2.3}$$

The mean curvature is given by

$$H = \frac{1}{4}\mathrm{trace}A. \tag{2.4}$$

The Gauss and Codazzi equations are given by

$$R(X, Y)Z = g(AY, Z)AX - g(AX, Z)AY, \tag{2.5}$$

$$(\nabla_X A)Y = (\nabla_Y A)X, \tag{2.6}$$

respectively, where R is the curvature tensor and

$$(\nabla_X A)Y = \nabla_X AY - A(\nabla_X Y) \tag{2.7}$$

for all $X, Y, Z \in \Gamma(TM)$.

In [4], it is obtained that if M is not 1-type, then necessary and sufficient conditions for M to be null 2-type in E^5 is

$$\triangle H + H\mathrm{trace}A^2 = \lambda H, \tag{2.8}$$

$$A\mathrm{grad}H + 2H\mathrm{grad}H = 0, \tag{2.9}$$

where H denotes the mean curvature. Also the Laplace operator $\triangle$ of a scalar valued function f is given by [5]

$$\triangle f = -\sum_{i=1}^{4}(e_i e_i f - \nabla_{e_i} e_i f), \tag{2.10}$$

where $\{e_1, e_2, e_3, e_4\}$ is an orthonormal local tangent frame on M.

3 Null 2-type hypersurfaces with all distinct principal curvatures

In this section, we study null 2-type hypersurfaces M in E^5. Since in [12], the case that the hypersurfaces in arbitrary dimension of Euclidean space having number of distinct principal curvatures at most 3 was studied. Therefore, we restrict to the case that the principal curvatures are all distinct.

We assume that mean curvature is not constant and grad$H \neq 0$. Assuming non-constant mean curvature, implies the existence of an open connected subset U of M, with grad$_pH \neq 0$ for all $p \in U$. From (2.9), it is easy to see that grad$H$ is an eigenvector of the shape operator A with the corresponding principal curvature $-2H$. Without losing generality, we choose e_1 in the direction of grad$H$ and therefore shape operator A of hypersurfaces will take the following form with respect to a suitable frame $\{e_1, e_2, e_3, e_4\}$

$$A_{\mathbf{N}} = \begin{pmatrix} -2H & & & \\ & \lambda_2 & & \\ & & \lambda_3 & \\ & & & \lambda_4 \end{pmatrix} \tag{3.1}$$

where $\mathbf{N}$ is the unit direction of $\mathbf{H}$.
The grad$H$ can be expressed as

$$\mathrm{grad}H = \sum_{i=1}^{4} e_i(H)e_i. \tag{3.2}$$

As we have taken e_1 parallel to grad$H$, consequently

$$e_1(H) \neq 0, e_i(H) = 0, \qquad i = 2, 3, 4. \tag{3.3}$$

We express

$$\nabla_{e_i} e_j = \sum_{k=1}^{4} \omega_{ij}^k e_k, \qquad i, j = 1, 2, 3, 4. \tag{3.4}$$

Using (3.4) and the compatibility conditions $(\nabla_{e_k} g)(e_i, e_i) = 0$ and $(\nabla_{e_k} g)(e_i, e_j) = 0$, we obtain

$$\omega_{ki}^i = 0, \qquad \omega_{ki}^j + \omega_{kj}^i = 0, \tag{3.5}$$

for $i \neq j$, and $i, j, k = 1, 2, 3, 4$.

Taking $X = e_i, Y = e_j$ in (2.7) and using (3.1), (3.4), we get

$$(\nabla_{e_i} A)e_j = e_i(\lambda_j)e_j + \sum_{k=1}^{n} \omega_{ij}^k e_k(\lambda_j - \lambda_k).$$

Putting the value of $(\nabla_{e_i} A)e_j$ in (2.6), we find

$$e_i(\lambda_j)e_j + \sum_{k=1}^{n} \omega_{ij}^k e_k(\lambda_j - \lambda_k) = e_j(\lambda_i)e_i + \sum_{k=1}^{n} \omega_{ji}^k e_k(\lambda_i - \lambda_k),$$

whereby for $i \neq j = k$ and $i \neq j \neq k$, we obtain

$$e_i(\lambda_j) = (\lambda_i - \lambda_j)\omega_{ji}^j = (\lambda_j - \lambda_i)\omega_{jj}^i, \tag{3.6}$$

$$(\lambda_i - \lambda_j)\omega_{ki}^j = (\lambda_k - \lambda_j)\omega_{ik}^j, \tag{3.7}$$

respectively, for distinct $i, j, k = 1, 2, 3, 4$.

Since $\lambda_1 = -2H$, from (3.3), we get

$$e_1(\lambda_1) \neq 0, e_2(\lambda_1) = 0, e_3(\lambda_1) = 0, e_4(\lambda_1) = 0. \tag{3.8}$$

Using (3.3), (3.4) and the fact that $[e_i\ e_j](H) = 0 = \nabla_{e_i}e_j(H) - \nabla_{e_j}e_i(H) = \omega_{ij}^1 e_1(H) - \omega_{ji}^1 e_1(H)$, for $i \neq j$ and $i, j = 2, 3, 4$, we find

$$\omega_{ij}^1 = \omega_{ji}^1. \tag{3.9}$$

Now, we show that $\lambda_j \neq \lambda_1, j = 2, 3, 4$. In fact, if $\lambda_j = \lambda_1$ for $j \neq 1$, from (3.6), we find

$$e_1(\lambda_j) = (\lambda_1 - \lambda_j)\omega_{j1}^j = 0, \tag{3.10}$$

which contradicts the first expression of (3.8).

Since M has four distinct principal curvatures, from (2.4), we obtain that

$$\lambda_2 + \lambda_3 + \lambda_4 = 6H. \tag{3.11}$$

Putting $i \neq 1, j = 1$ in (3.6) and using (3.8) and (3.5), we find

$$\omega_{1i}^1 = 0, \qquad i = 1, 2, 3, 4. \tag{3.12}$$

Putting $k = 1, j \neq i$, and $i, j = 2, 3, 4$ in (3.7), and using (3.9), we get

$$\omega_{ij}^1 = \omega_{ji}^1 = \omega_{1i}^j = \omega_{i1}^j = 0, \qquad j \neq i, \text{ and } i, j = 2, 3, 4. \tag{3.13}$$

Now, using (3.4), (3.12) and (3.13), we have:

Lemma 3.1 *Let M be a null 2-type hypersurface of non-constant mean curvature with all distinct principal curvatures in Euclidean space E^5, having the shape operator given by (3.1) with respect to suitable orthonormal frame $\{e_1, e_2, e_3, e_4\}$. Then,*

$$\nabla_{e_1}e_l = 0,\ \ l = 1, 2, 3, 4, \qquad \nabla_{e_i}e_1 = -\omega_{ii}^1 e_i,\ \ i = 2, 3, 4,$$

$$\nabla_{e_i}e_i = \sum_{i\neq l, l=1}^{4} \omega_{ii}^l e_l,\ \ i = 2, 3, 4, \qquad \nabla_{e_i}e_j = \sum_{j\neq k, k=2}^{4} \omega_{ij}^k e_k,\ \ i, j = 2, 3, 4,$$

where ω_{ij}^i satisfy (3.5) and (3.6).

Evaluating $g(R(X,Y)Z,W)$, using Lemma 3.1, (2.5) and (3.1), we find the following (3.14)$\sim$(3.16):

Lemma 3.2 *Let M be a null 2-type hypersurface of non-constant mean curvature with all distinct principal curvatures in Euclidean space E^5, having the shape operator given by (3.1) with respect to suitable orthonormal frame $\{e_1, e_2, e_3, e_4\}$. Then, $g(R(e_1, e_i)e_1, e_i)$, $g(R(e_1, e_i)e_i, e_j)$ and $g(R(e_i, e_j)e_i, e_1)$ give the following:*

$$e_1(\omega_{ii}^1) - (\omega_{ii}^1)^2 = -2H\lambda_i,\ \ i = 2, 3, 4. \tag{3.14}$$

$$e_1(\omega_{ii}^j) - \omega_{ii}^j\omega_{ii}^1 = 0, \quad i \neq j,\ \ i, j = 2, 3, 4, \tag{3.15}$$

and

$$e_j(\omega_{ii}^1) + \omega_{ii}^j\omega_{jj}^1 - \omega_{ii}^j\omega_{ii}^1 = 0, \quad i \neq j,\ \ i, j = 2, 3, 4, \tag{3.16}$$

respectively.

Proof. From (2.5), (3.1) and using Lemma 3.1, for $i = 2, 3, 4$, we have

- $g(R(e_1, e_i)e_1, e_i) = g(Ae_i, e_1)g(Ae_1, e_i) - g(Ae_1, e_1)g(Ae_i, e_i) = 2H\lambda_i,$

or,
$2H\lambda_i = g(\nabla_{e_1}\nabla_{e_i}e_1 - \nabla_{e_i}\nabla_{e_1}e_1 - \nabla_{[e_1e_i]}e_1, e_i) = g(\nabla_{e_1}(-\omega^1_{ii}e_i) - \omega^1_{ii}\nabla_{e_i}e_1,\ e_i) = g(-e_1(\omega^1_{ii})e_i + (\omega^1_{ii})^2e_i, e_i) = -e_1(\omega^1_{ii}) + (\omega^1_{ii})^2.$

- $g(R(e_1, e_i)e_i, e_j) = g(Ae_i, e_i)g(Ae_1, e_j) - g(Ae_1, e_i)g(Ae_i, e_j) = 0,$

$i \neq j, i, j = 2, 3, 4$. Hence,

$$0 = g\left(\nabla_{e_1}\nabla_{e_i}e_i - \nabla_{e_i}\nabla_{e_1}e_i - \nabla_{[e_1e_i]}e_i, e_j\right) = g\left(\nabla_{e_1}\left(\sum_{i\neq l, l=1}^{4}\omega^l_{ii}e_l\right) - \omega^1_{ii}\nabla_{e_i}e_i, e_j\right)$$

$$= g\left(\left(\sum_{i\neq l, l=1}^{4} e_1(\omega^l_{ii})e_l\right) - \omega^1_{ii}\left(\sum_{i\neq l, l=1}^{4}\omega^l_{ii}e_l\right), e_j\right) = e_1\left(\omega^j_{ii}\right) - \omega^1_{ii}\omega^j_{ii}.$$

- $g(R(e_i, e_j)e_i, e_1) = g(Ae_j, e_i)g(Ae_i, e_1) - g(Ae_i, e_i)g(Ae_j, e_1) = 0,$

$i \neq j, i, j = 2, 3, 4$. Hence,

$$0 = g(\nabla_{e_i}\nabla_{e_j}e_i - \nabla_{e_j}\nabla_{e_i}e_i - \nabla_{[e_ie_j]}e_i, e_1) = g\left(\nabla_{e_i}\left(\sum_{j\neq k, k=2}^{4}\omega^k_{ji}e_k\right)\right.$$

$$\left. - \nabla_{e_j}\left(\sum_{i\neq l, l=1}^{4}\omega^l_{ii}e_l\right) - \nabla_{\nabla_{e_i}e_j - \nabla_{e_j}e_i}e_i, e_1\right) = g\left(\sum_{j\neq k, k=2}^{4}(e_i(\omega^k_{ji})e_k + \omega^k_{ji}\nabla_{e_i}e_k), e_1\right)$$

$$- g\left(\sum_{i\neq l, l=1}^{4}(e_j(\omega^l_{ii})e_l + \omega^l_{ii}\nabla_{e_j}e_l), e_1\right) - g(\nabla_{\nabla_{e_i}e_j - \nabla_{e_j}e_i}e_i, e_1).$$

Now,

$$g\left(\sum_{j\neq k, k=2}^{4}(e_i(\omega^k_{ji})e_k + \omega^k_{ji}\nabla_{e_i}e_k), e_1\right) = 0,$$

as $\nabla_{e_i}e_k$ is not having component along e_1 for $i \neq k$ and for $i = k, \omega^i_{ji} = 0$. Similarly,

$$g(\sum_{i\neq l, l=1}^{4}(e_j(\omega^l_{ii})e_l + \omega^l_{ii}\nabla_{e_j}e_l), e_1) = e_j(\omega^1_{ii}) + \omega^1_{jj}\omega^j_{ii},$$

$$g(\nabla_{\nabla_{e_i}e_j}e_i, e_1) = g(\sum_{k=2, k\neq j}^{4}\omega^k_{ij}\nabla_{e_k}e_i, e_1) = \omega^i_{ij}g(\nabla_{e_i}e_i, e_1) = -\omega^j_{ii}\omega^1_{ii},$$

and

$$g(\nabla_{\nabla_{e_j}e_i}e_i, e_1) = g(\sum_{k=2,k\neq j}^{4} \omega_{ji}^k \nabla_{e_k} e_i, e_1) = \omega_{ji}^i g(\nabla_{e_i} e_i, e_1) = 0.$$

Therefore,

$$e_j(\omega_{ii}^1) + \omega_{jj}^1 \omega_{ii}^j - \omega_{ii}^j \omega_{ii}^1 = 0,$$

whereby proof of Lemma 3.2 is complete. □

From (3.1), we find

$$\text{trace}A^2 = 4H^2 + \lambda_2^2 + \lambda_3^2 + \lambda_4^2.$$

Using Lemma 3.1, (2.10), (3.3) and putting the value of traceA^2 in (2.8), we obtain

$$-e_1e_1(H) + (\omega_{22}^1 + \omega_{33}^1 + \omega_{44}^1)e_1(H) + H(4H^2 + \lambda_2^2 + \lambda_3^2 + \lambda_4^2) = \lambda H. \tag{3.17}$$

Using (3.3), Lemma 3.1, and the fact that $[e_i\ e_1](H) = 0 = \nabla_{e_i}e_1(H) - \nabla_{e_1}e_i(H)$, for $i = 2, 3, 4$, we find

$$e_ie_1(H) = 0. \tag{3.18}$$

Also, from (3.18) and using the fact that $[e_ie_1](e_1(H)) = 0 = \nabla_{e_i}e_1(e_1(H)) - \nabla_{e_1}e_i(e_1(H))$, we obtain

$$e_ie_1e_1(H) = 0, \qquad i = 2, 3, 4. \tag{3.19}$$

Now, we have:

Lemma 3.3 *Let M be a null 2-type hypersurface with all distinct principal curvatures in Euclidean space E^5, having the shape operator given by* (3.1) *with respect to suitable orthonormal frame $\{e_1, e_2, e_3, e_4\}$. Then,*

$$\sum_{j\neq i,j=2}^{4} \omega_{jj}^i[(\omega_{jj}^1 - \omega_{ii}^1)(\lambda_j - \lambda_i) - \frac{H}{e_1(H)}(\lambda_j - \lambda_i)^2(\lambda_i - \lambda_1)] = 0, \tag{3.20}$$

$$\sum_{j\neq i,j=2}^{4} \omega_{jj}^i[(\omega_{jj}^1 - \omega_{ii}^1)^2(3\lambda_j - \lambda_i - 2\lambda_1) - 4H(\lambda_j - \lambda_i)^2] = 0. \tag{3.21}$$

Proof. The equation (3.11) can be written as

$$\sum_{j=2}^{4} \lambda_j = 6H. \tag{3.22}$$

Differentiating (3.22) with e_i and using (3.3), we get

$$e_i(\lambda_i) + \sum_{j\neq i,j=2}^{4} e_i(\lambda_j) = 0, \;\; i = 2, 3, 4. \tag{3.23}$$

Differentiating (3.22) with e_1, e_i successively and using (3.18), we have

$$e_i(e_1(\lambda_i)) + \sum_{j\neq i, j=2}^{4} e_i(e_1(\lambda_j)) = 0, \ \ i = 2, 3, 4. \tag{3.24}$$

Using (3.6) in (3.24), we find

$$e_i(\omega_{ii}^1(\lambda_i - \lambda_1)) + \sum_{j\neq i, j=2}^{4} e_i(\omega_{jj}^1(\lambda_j - \lambda_1)) = 0, \ \ i = 2, 3, 4,$$

or,

$$e_i(\omega_{ii}^1)(\lambda_i - \lambda_1) + \omega_{ii}^1 e_i(\lambda_i) + \sum_{j\neq i, j=2}^{4} [e_i(\omega_{jj}^1)(\lambda_j - \lambda_1) + \omega_{jj}^1 e_i(\lambda_j)] = 0, \tag{3.25}$$

as $e_i(\lambda_1) = 0$ from (3.3), for $i = 2, 3, 4$. Also, equation (3.17) can be written as

$$-e_1 e_1(H) + \sum_{j=2}^{4} \omega_{jj}^1 e_1(H) + \sum_{j=2}^{4} \lambda_j^2 H + 4H^3 = \lambda H. \tag{3.26}$$

Differentiating (3.26) with e_i and using (3.3), (3.18) and (3.19), we get

$$\sum_{j=2}^{4} e_i(\omega_{jj}^1) e_1(H) + \sum_{j=2}^{4} 2H\lambda_j e_i(\lambda_j) = 0,$$

or,

$$e_i(\omega_{ii}^1) e_1(H) + \sum_{i\neq j, j=2}^{4} e_i(\omega_{jj}^1) e_1(H) + 2H\lambda_i e_i(\lambda_i) + \sum_{i\neq j, j=2}^{4} 2H\lambda_j e_i(\lambda_j) = 0.$$

Using (3.23) in the above, we find

$$e_i(\omega_{ii}^1) = -\sum_{i\neq j, j=2}^{4} [e_i(\omega_{jj}^1) + \frac{2H}{e_1(H)}(\lambda_j - \lambda_i) e_i(\lambda_j)]. \tag{3.27}$$

Putting the value of $e_i(\lambda_i)$ and $e_i(\omega_{ii}^1)$ from (3.23) and (3.27) in (3.25), we obtain (3.20).

Next, using (3.19) and (3.22), for $i = 2, 3, 4$, we get

$$\sum_{j=2}^{4} e_i e_1 e_1(\lambda_j) = 0,$$

$$\Rightarrow e_i e_1 e_1(\lambda_i) + \sum_{i\neq j, j=2}^{4} e_i e_1 e_1(\lambda_j) = 0,$$

which by use of (3.6),

$$\Rightarrow \quad e_i e_1[\omega_{ii}^1(\lambda_i - \lambda_1)] + \sum_{i\neq j, j=2}^{4} e_i e_1[\omega_{jj}^1(\lambda_j - \lambda_1)] = 0,$$

$$\Rightarrow \quad e_i[e_1(\omega_{ii}^1)(\lambda_i - \lambda_1) + \omega_{ii}^1 e_1(\lambda_i - \lambda_1)] + \sum_{i\neq j, j=2}^{4} e_i[e_1(\omega_{jj}^1)(\lambda_j - \lambda_1) + \omega_{jj}^1 e_1(\lambda_j - \lambda_1)] = 0,$$

which by use of (3.14) and $\lambda_1 = -2H$, gives

$$e_i[((\omega_{ii}^1)^2 - H\lambda_i)(\lambda_i - \lambda_1) + \omega_{ii}^1 e_1(H)] + \sum_{i\neq j, j=2}^{4} e_i[((\omega_{jj}^1)^2 - H\lambda_j)(\lambda_j - \lambda_1) + \omega_{jj}^1 e_1(H)] = 0,$$

$$\Rightarrow [(2\omega_{ii}^1 e_i(\omega_{ii}^1) - He_i(\lambda_i))(\lambda_i - \lambda_1) + e_i(\lambda_i)((\omega_{ii}^1)^2 - H\lambda_i) + e_i(\omega_{ii}^1)e_1(H)] + \sum_{i\neq j, j=2}^{4} [(2\omega_{jj}^1 e_i(\omega_{jj}^1) - He_i(\lambda_j))(\lambda_j - \lambda_1) + e_i(\lambda_j)((\omega_{jj}^1)^2 - H\lambda_j) + e_i(\omega_{jj}^1)e_1(H)] = 0,$$

$$e_i(\omega_{ii}^1)(2\omega_{ii}^1(\lambda_i - \lambda_1) + e_1(H)) + e_i(\lambda_i)((\omega_{ii}^1)^2 - 2H\lambda_i + H\lambda_1) + \sum_{i\neq j, j=2}^{4} [e_i(\omega_{jj}^1)(2\omega_{jj}^1(\lambda_j - \lambda_1) + e_1(H)) + e_i(\lambda_j)((\omega_{jj}^1)^2 - 2H\lambda_j + H\lambda_1)] = 0. \tag{3.28}$$

Using (3.23) in (3.28), we get

$$e_i(\omega_{ii}^1)(2\omega_{ii}^1(\lambda_i - \lambda_1) + e_1(H)) + \sum_{i\neq j, j=2}^{4} [e_i(\omega_{jj}^1)(2\omega_{jj}^1(\lambda_j - \lambda_1) + e_1(H)) + e_i(\lambda_j)((\omega_{jj}^1)^2 - (\omega_{ii}^1)^2 - 2H\lambda_j + 2H\lambda_i)] = 0. \tag{3.29}$$

Also, differentiating (3.23) with e_1, we have

$$e_1 e_i(\lambda_i) + \sum_{i\neq j, j=2}^{4} e_1 e_i(\lambda_j) = 0,$$

which by use of Lemma 3.1, $e_i e_1 - e_1 e_i = \nabla_{e_i} e_1 - \nabla_{e_1} e_i = -\omega_{ii}^1 e_i$,

$$\Rightarrow \quad e_i e_1(\lambda_i) + \omega_{ii}^1 e_i(\lambda_i) + \sum_{i\neq j, j=2}^{4} e_1 e_i(\lambda_j) = 0,$$

which by use of (3.6) and (3.23),

$$\Rightarrow \quad e_i(\omega_{ii}^1(\lambda_i - \lambda_1)) - \sum_{i\neq j, j=2}^{4} \omega_{ii}^1 e_i(\lambda_j) + \sum_{i\neq j, j=2}^{4} e_1(\omega_{jj}^i(\lambda_j - \lambda_i)) = 0,$$

or,

$$e_i(\omega^1_{ii})(\lambda_i-\lambda_1)+\omega^1_{ii}e_i(\lambda_i)-\sum_{i\neq j,j=2}^{4}\omega^1_{ii}e_i(\lambda_j)+\sum_{i\neq j,j=2}^{4}[e_1(\omega^i_{jj})(\lambda_j-\lambda_i)$$
$$+\omega^i_{jj}e_1(\lambda_j-\lambda_i)]=0,$$

which by use of (3.6), (3.23) and (3.15), gives

$$\begin{aligned}e_i(\omega^1_{ii})(\lambda_i-\lambda_1)-2\sum_{i\neq j,j=2}^{4}\omega^1_{ii}e_i(\lambda_j)+\sum_{i\neq j,j=2}^{4}[\omega^1_{jj}\omega^i_{jj}(\lambda_j-\lambda_i)\\ +\omega^i_{jj}(\omega^1_{jj}(\lambda_j-\lambda_1)-\omega^1_{ii}(\lambda_i-\lambda_1))]=0,\end{aligned}$$

whereby use of (3.6), we obtain

$$e_i(\omega^1_{ii})=-\frac{1}{(\lambda_i-\lambda_1)}\sum_{i\neq j,j=2}^{4}\omega^i_{jj}(\omega^1_{jj}-\omega^1_{ii})(2\lambda_j-\lambda_i-\lambda_1). \tag{3.30}$$

Using (3.15) and (3.30) in (3.29), we find

$$\begin{aligned}\sum_{j\neq i,j=2}^{4}\omega^i_{jj}[(\omega^1_{jj}-\omega^1_{ii})^2(3\lambda_j-\lambda_i-2\lambda_1)-\frac{2e_1(H)}{(\lambda_i-\lambda_1)}(\omega^1_{jj}-\omega^1_{ii})(\lambda_j-\lambda_i)\\ -2H(\lambda_j-\lambda_i)^2]=0.\end{aligned} \tag{3.31}$$

Using (3.20) to eliminate $e_1(H)$ from (3.31), we find (3.21) and proof of Lemma 3.3 is complete. □

Now, we have:

Lemma 3.4 *Let M be a null 2-type hypersurface with all distinct principal curvatures in Euclidean space E^5, having the shape operator given by (3.1) with respect to suitable orthonormal frame $\{e_1,e_2,e_3,e_4\}$. If $a_2=0$, $b_2=0$ and $c_2=0$, then*

$$\omega^2_{44}=0=\omega^2_{33},\quad \omega^3_{22}=0=\omega^3_{44},\quad \omega^4_{22}=0=\omega^4_{33}.$$

Proof. Putting $i=2,3,4$ in (3.20), we find following:

$$\begin{aligned}\omega^2_{33}[(\omega^1_{33}-\omega^1_{22})(\lambda_3-\lambda_2)-\tfrac{H}{e_1(H)}(\lambda_3-\lambda_2)^2(\lambda_2-\lambda_1)]\\ +\omega^2_{44}[(\omega^1_{44}-\omega^1_{22})(\lambda_4-\lambda_2)-\tfrac{H}{e_1(H)}(\lambda_4-\lambda_2)^2(\lambda_2-\lambda_1)]=0,\end{aligned} \tag{3.32}$$

$$\begin{aligned}\omega^3_{22}[(\omega^1_{33}-\omega^1_{22})(\lambda_3-\lambda_2)-\tfrac{H}{e_1(H)}(\lambda_3-\lambda_2)^2(\lambda_3-\lambda_1)]\\ +\omega^3_{44}[(\omega^1_{44}-\omega^1_{33})(\lambda_4-\lambda_3)-\tfrac{H}{e_1(H)}(\lambda_4-\lambda_3)^2(\lambda_3-\lambda_1)]=0,\end{aligned} \tag{3.33}$$

$$\begin{aligned}\omega^4_{33}[(\omega^1_{33}-\omega^1_{44})(\lambda_3-\lambda_4)-\tfrac{H}{e_1(H)}(\lambda_3-\lambda_4)^2(\lambda_4-\lambda_1)]\\ +\omega^4_{22}[(\omega^1_{44}-\omega^1_{22})(\lambda_4-\lambda_2)-\tfrac{H}{e_1(H)}(\lambda_4-\lambda_2)^2(\lambda_4-\lambda_1)]=0,\end{aligned} \tag{3.34}$$

respectively.

Similarly, by putting $i = 2, 3, 4$ in (3.21), we get following:

$$\begin{aligned}\omega_{33}^2[(\omega_{33}^1 - \omega_{22}^1)^2(3\lambda_3 - \lambda_2 - 2\lambda_1) - 4H(\lambda_3 - \lambda_2)^2]\\ +\omega_{44}^2[(\omega_{44}^1 - \omega_{22}^1)^2(3\lambda_4 - \lambda_2 - 2\lambda_1) - 4H(\lambda_4 - \lambda_2)^2] = 0,\end{aligned} \tag{3.35}$$

$$\begin{aligned}\omega_{22}^3[(\omega_{22}^1 - \omega_{33}^1)^2(3\lambda_2 - \lambda_3 - 2\lambda_1) - 4H(\lambda_2 - \lambda_3)^2]\\ +\omega_{44}^3[(\omega_{44}^1 - \omega_{33}^1)^2(3\lambda_4 - \lambda_3 - 2\lambda_1) - 4H(\lambda_4 - \lambda_3)^2] = 0,\end{aligned} \tag{3.36}$$

$$\begin{aligned}\omega_{22}^4[(\omega_{22}^1 - \omega_{44}^1)^2(3\lambda_2 - \lambda_4 - 2\lambda_1) - 4H(\lambda_2 - \lambda_4)^2]\\ +\omega_{33}^4[(\omega_{33}^1 - \omega_{44}^1)^2(3\lambda_3 - \lambda_4 - 2\lambda_1) - 4H(\lambda_3 - \lambda_4)^2] = 0,\end{aligned} \tag{3.37}$$

respectively.

Eliminating ω_{33}^2 and ω_{44}^2 from (3.32) and (3.35), we have

$$4H(\lambda_3 - \lambda_2)(\lambda_4 - \lambda_2)a_1 + a_2 + a_3 = 0, \tag{3.38}$$

where

$$a_1 = (\lambda_4 - \lambda_2)(\omega_{33}^1 - \omega_{22}^1) - (\lambda_3 - \lambda_2)(\omega_{44}^1 - \omega_{22}^1),$$

$$\begin{aligned}a_2 = \tfrac{(\lambda_2-\lambda_1)H}{e_1(H)}&[(3\lambda_4 - \lambda_2 - 2\lambda_1)(\lambda_3 - \lambda_2)^2(\omega_{44}^1 - \omega_{22}^1)^2\\ &-(3\lambda_3 - \lambda_2 - 2\lambda_1)(\lambda_4 - \lambda_2)^2(\omega_{33}^1 - \omega_{22}^1)^2],\end{aligned}$$

$$\begin{aligned}a_3 = (\omega_{44}^1 - \omega_{22}^1)(\omega_{33}^1 - \omega_{22}^1)&[(3\lambda_3 - \lambda_2 - 2\lambda_1)(\lambda_4 - \lambda_2)(\omega_{33}^1 - \omega_{22}^1)\\ &-(3\lambda_4 - \lambda_2 - 2\lambda_1)(\lambda_3 - \lambda_2)(\omega_{44}^1 - \omega_{22}^1)].\end{aligned}$$

Similarly, eliminating ω_{22}^3 and ω_{44}^3 from (3.33) and (3.36) and eliminating ω_{22}^4 and ω_{33}^4 from (3.34) and (3.37), we obtain

$$4H(\lambda_2 - \lambda_3)(\lambda_4 - \lambda_3)b_1 + b_2 + b_3 = 0, \tag{3.39}$$

$$4H(\lambda_3 - \lambda_4)(\lambda_2 - \lambda_4)c_1 + c_2 + c_3 = 0, \tag{3.40}$$

respectively, where

$$b_1 = (\lambda_4 - \lambda_3)(\omega_{22}^1 - \omega_{33}^1) - (\lambda_2 - \lambda_3)(\omega_{44}^1 - \omega_{33}^1),$$

$$\begin{aligned}b_2 = \tfrac{(\lambda_3-\lambda_1)H}{e_1(H)}&[(3\lambda_4 - \lambda_3 - 2\lambda_1)(\lambda_2 - \lambda_3)^2(\omega_{44}^1 - \omega_{33}^1)^2\\ &-(3\lambda_2 - \lambda_3 - 2\lambda_1)(\lambda_4 - \lambda_3)^2(\omega_{22}^1 - \omega_{33}^1)^2],\end{aligned}$$

$$\begin{aligned}b_3 = (\omega_{44}^1 - \omega_{33}^1)(\omega_{22}^1 - \omega_{33}^1)&[(3\lambda_2 - \lambda_3 - 2\lambda_1)(\lambda_4 - \lambda_3)(\omega_{22}^1 - \omega_{33}^1)\\ &-(3\lambda_4 - \lambda_3 - 2\lambda_1)(\lambda_2 - \lambda_3)(\omega_{44}^1 - \omega_{33}^1)],\end{aligned}$$

$$c_1 = (\lambda_2 - \lambda_4)(\omega_{33}^1 - \omega_{44}^1) - (\lambda_3 - \lambda_4)(\omega_{22}^1 - \omega_{44}^1),$$

$$\begin{aligned}c_2 = \tfrac{(\lambda_4-\lambda_1)H}{e_1(H)}&[(3\lambda_2 - \lambda_4 - 2\lambda_1)(\lambda_3 - \lambda_4)^2(\omega_{22}^1 - \omega_{44}^1)^2\\ &-(3\lambda_3 - \lambda_4 - 2\lambda_1)(\lambda_2 - \lambda_4)^2(\omega_{33}^1 - \omega_{44}^1)^2],\end{aligned}$$

$$\begin{aligned}c_3 = (\omega_{22}^1 - \omega_{44}^1)(\omega_{33}^1 - \omega_{44}^1)&[(3\lambda_3 - \lambda_4 - 2\lambda_1)(\lambda_2 - \lambda_4)(\omega_{33}^1 - \omega_{44}^1)\\ &-(3\lambda_2 - \lambda_4 - 2\lambda_1)(\lambda_3 - \lambda_4)(\omega_{22}^1 - \omega_{44}^1)] = 0.\end{aligned}$$

Also, a_1, b_1, c_1 can be written as

$$a_1 = (\lambda_4 - \lambda_2)\omega_{33}^1 + (\lambda_2 - \lambda_3)\omega_{44}^1 + (\lambda_3 - \lambda_4)\omega_{22}^1,$$

$$b_1 = (\lambda_2 - \lambda_4)\omega_{33}^1 + (\lambda_3 - \lambda_2)\omega_{44}^1 + (\lambda_4 - \lambda_3)\omega_{22}^1,$$

$$c_1 = (\lambda_2 - \lambda_4)\omega_{33}^1 + (\lambda_3 - \lambda_2)\omega_{44}^1 + (\lambda_4 - \lambda_3)\omega_{22}^1,$$

and therefore, $a_1 = -b_1 = -c_1$.

Also, $a_2 = 0$ gives

$$\begin{aligned}&(3\lambda_4 - \lambda_2 - 2\lambda_1)(\lambda_3 - \lambda_2)^2(\omega_{44}^1 - \omega_{22}^1)^2\\ &= (3\lambda_3 - \lambda_2 - 2\lambda_1)(\lambda_4 - \lambda_2)^2(\omega_{33}^1 - \omega_{22}^1)^2.\end{aligned} \tag{3.41}$$

Using (3.41) in (3.38), we get

$$a_1[4H(\lambda_3 - \lambda_2)^2 + (3\lambda_3 - \lambda_2 - 2\lambda_1)(\omega_{33}^1 - \omega_{22}^1)^2] = 0. \tag{3.42}$$

We claim that $a_1 \neq 0$. In fact if $a_1 = 0$, then combining it with (3.41), we get $\lambda_4 = \lambda_3$, a contradiction to the fact that there are four distinct principal curvatures. Therefore, from (3.42), we get that

$$4H(\lambda_3 - \lambda_2)^2 + (3\lambda_3 - \lambda_2 - 2\lambda_1)(\omega_{33}^1 - \omega_{22}^1)^2 = 0. \tag{3.43}$$

Using (3.41) and (3.43), we find

$$4H(\lambda_4 - \lambda_2)^2 + (3\lambda_4 - \lambda_2 - 2\lambda_1)(\omega_{44}^1 - \omega_{22}^1)^2 = 0. \tag{3.44}$$

Using (3.43) and (3.44) in (3.35), we obtain

$$\omega_{33}^2(\lambda_3 - \lambda_2)^2 + \omega_{44}^2(\lambda_4 - \lambda_2)^2 = 0. \tag{3.45}$$

Now, eliminating ω_{33}^2 from (3.32) and (3.45), we get

$$\omega_{44}^2 a_1 = 0. \tag{3.46}$$

From (3.46), we find $\omega_{44}^2 = 0$ as $a_1 \neq 0$. Using it in (3.45), we get $\omega_{33}^2 = 0$.

Similarly, we can show that $\omega_{22}^3 = 0 = \omega_{44}^3$, $\omega_{22}^4 = 0 = \omega_{33}^4$, whereby proving the Lemma. □

Now, we have:

Lemma 3.5 *Let M be a null 2-type hypersurface with all distinct principal curvatures in Euclidean space E^5, having the shape operator given by (3.1) with respect to suitable orthonormal frame $\{e_1, e_2, e_3, e_4\}$. If $a_2 = 0$, $b_2 = 0$ and $c_2 = 0$, then $\omega_{23}^4 = \omega_{32}^4 = \omega_{42}^3 = \omega_{24}^3 = \omega_{34}^2 = \omega_{43}^2 = 0$.*

Proof. From Lemma 3.4 and Lemma 3.1, we have

$$\nabla_{e_2} e_3 = \omega_{23}^4 e_4, \quad \nabla_{e_3} e_2 = \omega_{32}^4 e_4. \tag{3.47}$$

Now, evaluating $g(R(e_1, e_3)e_2, e_4)$ and $g(R(e_1, e_2)e_3, e_4)$, using (2.5), (3.1) and (3.47), we find

$$e_1(\omega_{32}^4) - \omega_{32}^4 \omega_{33}^1 = 0. \tag{3.48}$$

$$e_1(\omega_{23}^4) - \omega_{23}^4 \omega_{22}^1 = 0. \tag{3.49}$$

Putting $j = 4, k = 2, i = 3$ in (3.7), we get

$$(\lambda_2 - \lambda_4)\omega_{32}^4 = (\lambda_3 - \lambda_4)\omega_{23}^4. \tag{3.50}$$

Differentiating (3.50) with e_1, and using (3.48) and (3.49), we find

$$(e_1(\lambda_2) - e_1(\lambda_4))\omega_{32}^4 + (\lambda_2 - \lambda_4)\omega_{32}^4\omega_{33}^1 = (e_1(\lambda_3) - e_1(\lambda_4))\omega_{23}^4 + (\lambda_3 - \lambda_4)\omega_{23}^4\omega_{22}^1. \tag{3.51}$$

Putting the values of $e_1(\lambda_2), e_1(\lambda_3)$ and $e_1(\lambda_4)$ from (3.6) in (3.51), we obtain

$$\omega_{32}^4(\omega_{22}^1 - \omega_{44}^1) = \omega_{23}^4(\omega_{33}^1 - \omega_{44}^1). \tag{3.52}$$

Substituting the value of ω_{23}^4 from (3.50) in (3.52), we find

$$\omega_{32}^4 a_1 = 0. \tag{3.53}$$

As in the proof of Lemma 3.4, we have seen that assuming $a_1 = 0$ leads to contradiction, therefore from (3.53), we obtain $\omega_{32}^4 = 0$, which together with (3.50) gives $\omega_{23}^4 = 0$. Also, from (3.5), we get $\omega_{34}^2 = -\omega_{32}^4$ and $\omega_{23}^4 = -\omega_{24}^3$, therefore, we obtain $\omega_{34}^2 = 0$, and $\omega_{24}^3 = 0$, which together with (3.7) gives $\omega_{43}^2 = 0$, and $\omega_{42}^3 = 0$, whereby proof of Lemma 3.5 is complete. □

4 Proof of the theorem

In view of Lemma 3.1, Lemma 3.4 and Lemma 3.5, we have

$$\nabla_{e_i} e_1 = -\omega_{ii}^1 e_i, \ \ \nabla_{e_i} e_i = \omega_{ii}^1 e_1, \qquad \nabla_{e_i} e_j = 0, \ \ i \neq j, \ \ i, j = 2, 3, 4,$$

and evaluating $g(R(e_i, e_j)e_i, e_j)$, we find

$$-\omega_{ii}^1 \omega_{jj}^1 = \lambda_i \lambda_j, \quad i \neq j, \quad i, j = 2, 3, 4. \tag{4.1}$$

From (4.1), we get

$$-\omega_{22}^1\omega_{33}^1 = \lambda_2\lambda_3, \quad -\omega_{22}^1\omega_{44}^1 = \lambda_2\lambda_4, \quad -\omega_{33}^1\omega_{44}^1 = \lambda_3\lambda_4. \tag{4.2}$$

From (4.2), we get (4.3) and (4.4)

$$(\omega_{22}^1)^2 = -(\lambda_2)^2, \quad (\omega_{33}^1)^2 = -(\lambda_3)^2, \quad (\omega_{44}^1)^2 = -(\lambda_4)^2, \tag{4.3}$$

$$(\omega_{22}^1\omega_{33}^1\omega_{44}^1)^2 = -(\lambda_2\lambda_3\lambda_4)^2. \tag{4.4}$$

From (3.14), we have

$$\begin{array}{ll} e_1(\omega_{22}^1)-(\omega_{22}^1)^2=-2H\lambda_2, & e_1(\omega_{33}^1)-(\omega_{33}^1)^2=-2H\lambda_3, \\ e_1(\omega_{44}^1)-(\omega_{44}^1)^2=-2H\lambda_4. & \end{array} \tag{4.5}$$

Also, equation (3.17) can be written as

$$-e_1e_1(H)+Ke_1(H)+4H^3+PH=\lambda H, \tag{4.6}$$

where $K=\omega_{22}^1+\omega_{33}^1+\omega_{44}^1$, $P=\lambda_2^2+\lambda_3^2+\lambda_4^2$.

Moreover, using (4.2), (4.3) and (3.11), we get

$$\begin{array}{l} K^2=(\omega_{22}^1+\omega_{33}^1+\omega_{44}^1)^2=-(\lambda_2+\lambda_3+\lambda_4)^2=-36H^2, \\ \lambda_2\lambda_3+\lambda_4\lambda_2+\lambda_3\lambda_4=18H^2-\frac{P}{2}. \end{array} \tag{4.7}$$

We denote by $Q=\lambda_2\lambda_3\lambda_4$ and by $L=\omega_{22}^1\omega_{33}^1\omega_{44}^1$. Then, using (3.6), we can write

$$\begin{array}{l} e_1(Q)=e_1(\lambda_2\lambda_3\lambda_4)=e_1(\lambda_2)\lambda_3\lambda_4+e_1(\lambda_3)\lambda_2\lambda_4+e_1(\lambda_4)\lambda_3\lambda_2 \\ =\omega_{22}^1(\lambda_2-\lambda_1)\lambda_3\lambda_4+\omega_{33}^1(\lambda_3-\lambda_1)\lambda_2\lambda_4+\omega_{44}^1(\lambda_4-\lambda_1)\lambda_3\lambda_2 \\ =QK-\lambda_1(\lambda_3\lambda_4\omega_{22}^1+\lambda_2\lambda_4\omega_{33}^1+\lambda_3\lambda_2\omega_{44}^1), \end{array} \tag{4.8}$$

which on using (4.2) gives

$$e_1(Q)=QK-6HL \tag{4.9}$$

Now, using (4.5), we compute

$$\begin{array}{l} e_1(L)=e_1(\omega_{22}^1\omega_{33}^1\omega_{44}^1)=e_1(\omega_{22}^1)\omega_{33}^1\omega_{44}^1+e_1(\omega_{33}^1)\omega_{22}^1\omega_{44}^1 \\ +e_1(\omega_{44}^1)\omega_{33}^1\omega_{22}^1=[(\omega_{22}^1)^2-2H\lambda_2]\omega_{33}^1\omega_{44}^1+[(\omega_{33}^1)^2-2H\lambda_3]\omega_{22}^1\omega_{44}^1 \\ +[(\omega_{44}^1)^2-2H\lambda_4]\omega_{33}^1\omega_{22}^1, \end{array} \tag{4.10}$$

which on using (4.2) and (3.11) gives

$$e_1(L)=\lambda_2^2\lambda_3\lambda_4+\lambda_3^2\lambda_2\lambda_4+\lambda_4^2\lambda_3\lambda_2+6H\lambda_2\lambda_3\lambda_4=Q(\lambda_2+\lambda_3+\lambda_4+6H),$$

Or,

$$e_1(L)=12HQ \tag{4.11}$$

Also, differentiating (4.4) along e_1, and using (4.11), we obtain

$$e_1(Q)=-12HL. \tag{4.12}$$

Using (4.12) and (4.9), we have

$$e_1(Q)=2KQ. \tag{4.13}$$

Differentiating (3.11) along e_1 and putting the values of $e_1(\lambda_2), e_1(\lambda_3)$, $e_1(\lambda_4)$, using (3.6), we find

$$(\lambda_2-\lambda_1)\omega_{22}^1+(\lambda_3-\lambda_1)\omega_{33}^1+(\lambda_4-\lambda_1)\omega_{44}^1=6e_1(H), \tag{4.14}$$

Again, differentiating (4.14) with e_1 and putting the values of $e_1(\omega_{22}^1)$, $e_1(\omega_{33}^1)$, $e_1(\omega_{44}^1)$, using (4.5) and (3.6), we obtain

$$\begin{aligned}2(\lambda_2-\lambda_1)(\omega_{22}^1)^2+2(\lambda_3-\lambda_1)(\omega_{33}^1)^2+2(\lambda_4-\lambda_1)(\omega_{44}^1)^2-Ke_1(\lambda_1)\\ -2H(P-\lambda_2\lambda_1-\lambda_3\lambda_1-\lambda_4\lambda_1)=6e_1e_1(H).\end{aligned}\tag{4.15}$$

Using (4.3), (3.11) and the fact that $\lambda_1=-2H$ in (4.15), we find

$$Ke_1(H)-(\lambda_2^3+\lambda_3^3+\lambda_4^3)-3HP-12H^3=3e_1e_1(H).\tag{4.16}$$

On the other hand
$\lambda_2^3+\lambda_3^3+\lambda_4^3-3\lambda_2\lambda_3\lambda_4=(\lambda_2+\lambda_3+\lambda_4)(\lambda_2^2+\lambda_3^2+\lambda_4^2-\lambda_2\lambda_3-\lambda_4\lambda_2-\lambda_3\lambda_4)$
$=6H(P-18H^2+\frac{P}{2})=9HP-108H^3$
or,

$$\lambda_2^3+\lambda_3^3+\lambda_4^3=9HP-108H^3+3Q,$$

therefore (4.16) can be written as

$$Ke_1(H)-12HP+96H^3-3Q=3e_1e_1(H).\tag{4.17}$$

Combining (4.6) and (4.17), we find

$$2Ke_1(H)=84H^3-15HP-3Q+3\lambda H.\tag{4.18}$$

Also, differentiating (4.7) with e_1, we find

$$Ke_1(K)=-36He_1(H).\tag{4.19}$$

Using (4.5) and (4.3), we find

$$\begin{aligned}e_1(K)&=(\omega_{22}^1)^2+(\omega_{33}^1)^2+(\omega_{44}^1)^2-2H(\lambda_2+\lambda_3+\lambda_4)\\ &=-\lambda_2^2-\lambda_3^2-\lambda_4^2-12H^2=-P-12H^2.\end{aligned}\tag{4.20}$$

Combining (4.19) and (4.20), we obtain

$$HP=-12H^3-Ke_1(H).\tag{4.21}$$

Eliminating P from (4.21) and (4.18), we have

$$13Ke_1(H)=3Q-3\lambda H-264H^3.\tag{4.22}$$

Also, using (4.21) in (4.6), we get

$$e_1e_1(H)=-8H^3-\lambda H.\tag{4.23}$$

Differentiating (4.22) with e_1 and using (4.23), (4.19) and (4.13), we find

$$13Ke_1e_1(H)+13e_1(K)e_1(H)=3e_1(Q)-3\lambda e_1(H)-792H^2e_1(H),$$

or,

$$13K(-8H^3-\lambda H)+13(-\frac{36He_1(H)}{K})e_1(H)=6KQ-3\lambda e_1(H)-792H^2e_1(H),$$

or,

$$-13[8K^2H^3+K^2\lambda H+36He_1^2(H)]=6K^2Q-3\lambda Ke_1(H)-792H^2Ke_1(H),$$

which by use of (4.7) and (4.22), gives

$$-13[8H^3(-36H^2) - 36\lambda H^3 + 36He_1^2(H)] = 6(-36H^2)Q$$
$$-(3\lambda + 792H^2)(\tfrac{3Q-3\lambda H-264H^3}{13}),$$

$$676He_1^2(H) = 169(32H^5 + 4\lambda H^3) + 312H^2Q + (\lambda + 264H^2)(Q - \lambda H - 88H^3). \quad (4.24)$$

Squaring (4.22) and using (4.7), we find

$$-676H^2e_1^2(H) = (Q - \lambda H - 88H^3)^2. \quad (4.25)$$

Eliminating $e_1^2(H)$ from (4.24) and (4.25), we obtain

$$(Q - \lambda H - 88H^3)^2 = -169H(32H^5 + 4\lambda H^3) - 312H^3Q - H(\lambda$$
$$+264H^2)(Q - \lambda H - 88H^3),$$

or,

$$(Q - \lambda H - 88H^3)(Q - \lambda H - 88H^3 + \lambda H + 264H^3) = -169H(32H^5 + 4\lambda H^3)$$
$$-312H^3Q,$$

or,

$$(Q - \lambda H - 88H^3)(Q + 176H^3) = -169(32H^5 + 4\lambda H^3) - 312H^3Q,$$

or,

$$Q^2 + Q(176H^3 - \lambda H - 88H^3) - 176\lambda H^4 - (88 \times 176)H^6 = -169(32H^6$$
$$+4\lambda H^4) - 312H^3Q$$
$$Q^2 + Q(400H^3 - \lambda H) + 500\lambda H^4 - 10080H^6 = 0. \quad (4.26)$$

Now, differentiating (4.26) with e_1, we find

$$2Qe_1(Q) + e_1(Q)(400H^3 - \lambda H) + Qe_1(H)(1200H^2 - \lambda)$$
$$+2000\lambda H^3e_1(H) - 60480H^5e_1(H) = 0,$$

which by using (4.13), gives

$$4Q^2K + 2KQ(400H^3 - \lambda H) + Qe_1(H)(1200H^2 - \lambda)$$
$$+2000\lambda H^3e_1(H) - 60480H^5e_1(H) = 0,$$

or,

$$4Q^2K^2 + 2K^2Q(400H^3 - \lambda H) + Ke_1(H)[Q(1200H^2 - \lambda)$$
$$+2000\lambda H^3 - 60480H^5] = 0,$$

which by use of (4.22) and (4.7), gives

$$4Q^2(-36H^2) + 2Q(-36H^2)(400H^3 - \lambda H) + \tfrac{(3Q-3\lambda H-264H^3)}{13}[Q(1200H^2 - \lambda)$$
$$+2000H^3\lambda - 60480H^5] = 0,$$

or,

$$48Q^2H^2 + 24QH^2(400H^3 - \lambda H) - \tfrac{(Q-\lambda H-88H^3)}{13}[Q(1200H^2 - \lambda)$$
$$+2000\lambda H^3 - 60480H^5] = 0,$$

or,

$$Q^2[624H^2-1200H^2+\lambda]+Q[312H^2(400H^3-\lambda H)-2000\lambda H^3+60480H^5 \\ +(1200H^2-\lambda)(\lambda H+88H^3)]+(\lambda H+88H^3)(2000\lambda H^3-60480H^5)=0,$$

or,

$$Q^2(\lambda-576H^2)+Q[290880H^5-1200\lambda H^3-\lambda^2 H] \\ +(\lambda H+88H^3)(2000\lambda H^3-60480H^5)=0. \quad (4.27)$$

In order to eliminate Q from (4.26) and (4.27) using Cramer's method, we get

$$Q^2=\frac{\triangle_{Q^2}}{\triangle}, \qquad Q=\frac{\triangle_Q}{\triangle},$$

or,

$$(\triangle_{Q^2})(\triangle)-(\triangle_Q)^2=0, \quad (4.28)$$

where

$$\triangle=\begin{vmatrix} 1 & 400H^3-\lambda H \\ \lambda-576H^2 & 290880H^5-1200\lambda H^3-\lambda^2 H \end{vmatrix},$$

$$\triangle_{Q^2}=\begin{vmatrix} (10080H^6-500\lambda H^4) & 400H^3-\lambda H \\ (\lambda H+88H^3)(60480H^5-2000\lambda H^3) & 290880H^5-1200\lambda H^3-\lambda^2 H \end{vmatrix},$$

$$\triangle_Q=\begin{vmatrix} 1 & (10080H^6-500\lambda H^4) \\ \lambda-576H^2 & (\lambda H+88H^3)(60480H^5-2000\lambda H^3) \end{vmatrix}.$$

Since λ is constant, therefore, (4.28) becomes an algebraic equation of degree 16 in H with constant coefficients. Thus, the real function H satisfies a polynomial equation $g(H)=0$ with constant coefficients, therefore it must be a constant, whereby proof of main Theorem 1.1 is complete.

5 Null 2-type hypersurfaces in E^5 with constant scalar curvature

In section 3, we have proved Lemma 3.4 and Lemma 3.5 using the condition that $a_2=0$, $b_2=0$ and $c_2=0$. In fact, we tried to prove the results without any condition but it was not possible due to the fact that we were unable to show that the expressions (3.38), (3.39) and (3.40) were non zero. It will be interesting if the condition can be replaced by some geometric condition involving curvatures. In this section, we study null 2-hypersurfaces with constant scalar curvature in E^5.

Using (2.5) and (3.1), the scalar curvature ρ is given by

$$\rho=12H^2-\lambda_2^2-\lambda_3^2-\lambda_4^2. \quad (5.1)$$

Using (2.5), (2.8), (2.10), (3.1) and Lemma 3.1, we find

$$-e_1e_1(H)+(\omega_{22}^1+\omega_{33}^1+\omega_{44}^1)e_1(H)+H(16H^2-\rho)=0. \quad (5.2)$$

For constant scalar curvature ρ, using (5.2), (3.18) and (3.19), we get

$$e_i(\omega_{22}^1+\omega_{33}^1+\omega_{44}^1)=0, \qquad i=2,3,4. \quad (5.3)$$

Now, we have

Lemma 5.1 *Let M be a null 2-type hypersurface with all distinct principal curvatures in Euclidean space E^5 having shape operator given by (3.1) with respect to suitable orthonormal frame $\{e_1, e_2, e_3, e_4\}$. If scalar curvature of the hypersurface is constant, then*

$$\omega_{44}^2 = 0 = \omega_{33}^2, \quad \omega_{22}^3 = 0 = \omega_{44}^3, \quad \omega_{22}^4 = 0 = \omega_{33}^4.$$

Proof. Operating with e_2 on both sides of (5.1), (3.11) and using (3.6), we find

$$(\lambda_2 - \lambda_4)^2 \omega_{44}^2 + (\lambda_2 - \lambda_3)^2 \omega_{33}^2 = 0. \tag{5.4}$$

Differentiating (5.4) along e_1 and using (3.6), (3.15), we get

$$[-2(\lambda_1 - \lambda_2)(\lambda_2 - \lambda_4)\omega_{22}^1 + (2\lambda_1 + \lambda_2 - 3\lambda_4)(\lambda_2 - \lambda_4)\omega_{44}^1]\omega_{44}^2 + [-2(\lambda_1 - \lambda_2)(\lambda_2 - \lambda_3)\omega_{22}^1 + (2\lambda_1 + \lambda_2 - 3\lambda_3)(\lambda_2 - \lambda_3)\omega_{33}^1]\omega_{33}^2 = 0.$$

and using (5.4) in the above equation, we get

$$\begin{aligned} [2(\lambda_1 - \lambda_2)(\lambda_3 - \lambda_4)\omega_{22}^1 + (2\lambda_1 + \lambda_2 - 3\lambda_4)(\lambda_2 - \lambda_3)\omega_{44}^1 \\ -(2\lambda_1 + \lambda_2 - 3\lambda_3)(\lambda_2 - \lambda_4)\omega_{33}^1]\omega_{44}^2 = 0. \end{aligned} \tag{5.5}$$

Similarly, acting with e_1 and e_2 on (3.11), successively and using (3.6), (5.3), (3.16) and (5.4), subsequently, we obtain

$$[(\lambda_4 - \lambda_3)\omega_{22}^1 + (\lambda_3 - \lambda_2)\omega_{44}^1 + (\lambda_2 - \lambda_4)\omega_{33}^1]\omega_{44}^2 = 0. \tag{5.6}$$

Equations (5.5) and (5.6) show that either ω_{44}^2, or the expression between square brackets, has to vanish. We now prove that ω_{44}^2 has to be zero. In fact, if $\omega_{44}^2 \neq 0$, then the expressions between square brackets has to be zero:

$$\begin{aligned} 2(\lambda_1 - \lambda_2)(\lambda_3 - \lambda_4)\omega_{22}^1 + (2\lambda_1 + \lambda_2 - 3\lambda_4)(\lambda_2 - \lambda_3)\omega_{44}^1 \\ -(2\lambda_1 + \lambda_2 - 3\lambda_3)(\lambda_2 - \lambda_4)\omega_{33}^1 = 0. \end{aligned} \tag{5.7}$$

$$(\lambda_4 - \lambda_3)\omega_{22}^1 + (\lambda_3 - \lambda_2)\omega_{44}^1 + (\lambda_2 - \lambda_4)\omega_{33}^1 = 0. \tag{5.8}$$

Eliminating ω_{22}^1 from (5.7) and (5.8), we get

$$(\lambda_2 - \lambda_3)(\lambda_2 - \lambda_4)(\omega_{33}^1 + \omega_{44}^1) = 0, \tag{5.9}$$

which shows that

$$\omega_{33}^1 + \omega_{44}^1 = 0. \tag{5.10}$$

If (5.10) is true, then using it to eliminate ω_{33}^1, from (5.7) and (5.8), we find

$$\begin{aligned} 2(\lambda_1 - \lambda_2)(\lambda_3 - \lambda_4)\omega_{22}^1 + [(2\lambda_1 + \lambda_2 - 3\lambda_4)(\lambda_2 - \lambda_3) \\ +(2\lambda_1 + \lambda_2 - 3\lambda_3)(\lambda_2 - \lambda_4)]\omega_{44}^1 = 0. \end{aligned} \tag{5.11}$$

$$(\lambda_4 - \lambda_3)\omega_{22}^1 + (\lambda_3 + \lambda_4 - 2\lambda_2)\omega_{44}^1 = 0. \tag{5.12}$$

From (5.11) and (5.12), we obtain

$$(\lambda_2 - \lambda_3)(\lambda_2 - \lambda_4) = 0, \tag{5.13}$$

which is contradiction of the fact that principal curvatures are distinct. Therefore, $\omega_{44}^2 = 0$, which gives $\omega_{33}^2 = 0$ in view of (5.4).

In similar way, we can prove that $\omega_{22}^3 = \omega_{44}^3 = \omega_{22}^4 = \omega_{33}^4 = 0$, which completes the proof of Lemma 5.1. □

Next, we have

Lemma 5.2 *Let M be a null 2-type hypersurface with all distinct principal curvatures in Euclidean space E^5 having the shape operator given by (3.1) with respect to suitable orthonormal frame $\{e_1, e_2, e_3, e_4\}$. If scalar curvature of the hypersurface is constant, then $\omega_{23}^4 = \omega_{32}^4 = \omega_{42}^3 = \omega_{24}^3 = \omega_{34}^2 = \omega_{43}^2 = 0$.*

Proof. Using Lemma 5.1 and proceeding in a similar way as in the proof of Lemma 3.5, we obtain (3.53). Whereby a_1 cannot be zero in view of (5.9), therefore, we get $\omega_{32}^4 = 0$. Thereafter, proof follows as given in the proof of Lemma 3.5.

Using Lemma 3.1, Lemma 5.1 and Lemma 5.2 and using (2.5) and (3.1), evaluating $g(R(e_2, e_3)e_2, e_3)$, $g(R(e_2, e_4)e_2, e_4)$ and $g(R(e_3, e_4)e_3, e_4)$, we find (4.2). From (4.2), we obtain (4.3) which gives $\lambda_2 = \lambda_3 = \lambda_4 = 0$, a contradiction to all distinct principal curvatures. Whereby proof of the Theorem 1.2 is complete. □

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