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# Positive integer solutions of some second-order Diophantine equations

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**Abstract** Let $k \geqslant 3$ be a positive integer and $A = k \mp 2$. In this paper, we give all positive integer solutions of the second-order diophantine equations $x^2 - kxy + y^2 = \mp A$, $x^2 - (k^2-4)y^2 = \mp 4A$, $x^2 - kxy + y^2 = \mp(k^2-4)A$, $x^2-(k^2+2)xy+y^2 = -k^2$, $x^2-(k^2 \mp 2)xy+y^2 = k^2$, and $x^2+4xy-[(k^2-2)y]^2 = 4k^2$ in terms of generalized Fibonacci and Lucas sequences. Moreover, we find necessary and sufficient condition for the equations $x^2 - kxy + y^2 = -(k+2)$ and $x^2 - kxy + y^2 = k-2$ to have integer solutions.

**Keywords** Fibonacci numbers · Lucas numbers · Generalized Fibonacci numbers · Generalized Lucas numbers · Diophantine equations

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## 1 Introduction

Let $k$ and $s$ be two nonzero integers with $k^2+4s > 0$. Generalized Fibonacci sequence $(U_n(k,s))$ is defined by $U_0(k,s) = 0, U_1(k,s) = 1$, and $U_{n+1}(k,s) = kU_n(k,s) + sU_{n-1}(k,s)$ for $n \geqslant 1$ and generalized Lucas sequence $(V_n(k,s))$ is defined by $V_0(k,s) = 2, V_1(k,s) = k$, and $V_{n+1}(k,s) = kV_n(k,s) + sV_{n-1}(k,s)$ for $n \geqslant 1$. For $k = s = 1$, the sequences $(U_n)$ and $(V_n)$ are called Fibonacci and Lucas sequences and they are denoted as $(F_n)$ and $(L_n)$, respectively. For $k = 2$ and $s = 1$, the sequences $(U_n)$ and $(V_n)$ are called Pell and Pell-Lucas sequences and they are denoted as $(P_n)$ and $(Q_n)$, respectively. It is well known that $U_n(k,s) = (\alpha^n - \beta^n)/(\alpha - \beta)$ and $V_n(k,s) = \alpha^n + \beta^n$, where $\alpha = \left(k + \sqrt{k^2+4s}\right)/2$ and $\beta = \left(k - \sqrt{k^2+4s}\right)/2$. The above identities are known as Binet's formulae. Clearly, $\alpha + \beta = k$, $\alpha - \beta = \sqrt{k^2+4s}$, and $\alpha\beta = -s$. Moreover, generalized Fibonacci and Lucas numbers for negative subscript are defined as $U_{-n} = -(-s)^{-n}U_n$ and $V_n = (-s)^{-n}\, V_n$ for $n \in \mathbb{N}$.

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For more information about generalized Fibonacci and Lucas sequences, one can consult [8].

The general second-order equation

$$ax^2 + bxy + cy^2 = k \tag{1.1}$$

is one of the principal topics in Gauss's Disquisitions arithmeticae. Pell's equation is any Diophantine equation of the form $x^2 - dy^2 = N$, where $d$ is a given nonsquare integer and integer solutions are sought for $x$ and $y$. The name of Pell's equation arose from Leonard Euler's mistakenly attributing its study to John Pell. The equation (1.1) can be solved completely using solutions to the Pell Equation. In [3] and [5], the authors show that the equation

$$x^2 - kxy + y^2 = -1 \tag{1.2}$$

has no positive integer solutions $x$ and $y$ when $k > 3$. When $k = 3$, they show that all positive integer solutions to equation (1.2) are given by $(x, y) = (F_{2n+1}, F_{2n-1})$ with $n \geq 0$. In [3], Keskin demonstrates that the equation

$$x^2 - kxy + y^2 = 1 \tag{1.3}$$

has an infinite number of positive integer solutions $x$ and $y$ when $k \geq 3$. If $k > 3$, then it is well known that all positive integer solutions to the equation (1.3) are given by $(x, y) = (U_n(k, -1), U_{n-1}(k, -1))$ with $n \geq 2$. If $k = 3$, then all positive integer solutions to the equation (1.3) are given by $(x, y) = (F_{2n+2}, F_{2n})$ with $n \geq 1$. In addition, when $k > 3$ and $k^2 - 4$ is a square-free integer, in [2], the authors showed that all positive integer solutions to the equation

$$x^2 - kxy + y^2 = -(k^2 - 4) \tag{1.4}$$

are given by $(x, y) = (V_n(k, -1), V_{n-1}(k, -1))$ with $n \geq 1$. If $k = 3$, then all positive integer solutions to the equation (1.4) are given by $(x, y) = (L_{2n+2}, L_{2n})$ with $n \geq 0$. Moreover, they showed that the equation

$$x^2 - kxy + y^2 = k^2 - 4 \tag{1.5}$$

has no integer solutions $x$ and $y$ when $k > 3$ and $k^2 - 4$ is a square-free integer. If $k = 3$, then all positive integer solutions to the equation (1.5) are given by $(x, y) = (L_{2n+1}, L_{2n-1})$ with $n \geq 0$.

In this paper, we will show that the equation $x^2 - kxy + y^2 = -(k+2)$ has a solution iff $k = 3$ or 6. Moreover, we will show that the equation $x^2 - kxy + y^2 = k - 2$ has a solution iff $k - 2$ is a perfect square. In addition to this, when $A = k \mp 2$, we will find positive integer solutions to the equations

$$x^2 - kxy + y^2 = \mp A,$$

$$x^2 - (k^2 - 4)y^2 = \mp 4A,$$

$$x^2 - kxy + y^2 = \mp (k^2 - 4)A,$$

$$x^2 - (k^2 \mp 2)xy + y^2 = k^2,$$

$$x^2 - (k^2 + 2)xy + y^2 = -k^2,$$

and

$$x^2 + 4xy - (k^2-2)^2y^2 = 4k^2.$$

Let $d$ be a positive integer which is not a perfect square and $N$ be any nonzero fixed integer. If $a^2 - db^2 = N$, we say that $(a, b)$ is a solution to the Pell equation $x^2 - dy^2 = N$. Also, if $a$ and $b$ are both positive, we say that $(a, b)$ is positive solution to the equation $x^2 - dy^2 = N$.

Let $x_1 + y_1\sqrt{d}$ be a positive solution to the equation $x^2 - dy^2 = N$. We say that $x_1 + y_1\sqrt{d}$ is the fundamental solution of the equation $x^2 - dy^2 = N$ if $x_2 + y_2\sqrt{d}$ is a solution to the equation $x^2 - dy^2 = N$, then $x_1 + y_1\sqrt{d} \leq x_2 + y_2\sqrt{d}$.

Continued fraction plays an important role in solutions of the Pell equations $x^2 - dy^2 = 1$ and $x^2 - dy^2 = -1$. Let $d$ be a positive integer that is not a perfect square. Then there is a continued fraction expansion of $\sqrt{d}$ such that $\sqrt{d} = \left[a_0, \overline{a_1, a_2, ..., a_{l-1}, 2a_0}\right]$, where $l$ is the period length and the $a_j$'s are given by the recursion formula;

$$\alpha_0 = \sqrt{d},\ a_k = \lfloor \alpha_k \rfloor \text{ and } \alpha_{k+1} = \frac{1}{\alpha_k - a_k},\ k = 0, 1, 2, 3, ...$$

Recall that $a_l = 2a_0$ and $a_{l+k} = a_k$ for $k \geq 1$. The $n^{th}$ convergent of $\sqrt{d}$ for $n \geq 0$ is given by

$$\frac{p_n}{q_n} = [a_0, a_1, ..., a_n] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{\ddots \cfrac{1}{a_{n-1} + \frac{1}{a_n}}}}.$$

## 2 Preliminaries

In this section, we will give necessary identities, lemmas, and theorems which will be used later. The following identities concerning generalized Fibonacci and Lucas numbers are well known:

$$L_{n+1} + L_{n-1} = 5F_n, \tag{2.1}$$

$$F_{n+1} + F_{n-1} = L_n, \tag{2.2}$$

$$U_{n+1}(k^2+2, -1) - U_n(k^2+2, -1) = U_{2n+1}(k, 1), \tag{2.3}$$

$$V_n(k, -1) = U_{n+1}(k, -1) - U_{n-1}(k, -1) = kU_n(k, -1) - 2U_{n-1}(k, -1), \tag{2.4}$$

$$V_n(k, 1) = U_{n+1}(k, 1) + U_{n-1}(k, 1) = kU_n(k, 1) + 2U_{n-1}(k, 1), \tag{2.5}$$

$$V_n^2(k, -1) - (k^2-4)U_n^2(k, -1) = 4, \tag{2.6}$$

$$V_n(k, -1)V_{n-1}(k, -1) - (k^2-4)U_n(k, -1)U_{n-1}(k, -1) = 2k, \tag{2.7}$$

$$V_{2n}^2(k, 1) - (k^2+4)U_{2n}^2(k, 1) = 4, \tag{2.8}$$

$$V_{n+1}(k, -1) - V_{n-1}(k, -1) = (k^2-4)U_n, \tag{2.9}$$

$$L_{2n}^2 - 5F_{2n}^2 = 4, \tag{2.10}$$

$$F_{2n+2}^2 - 3F_{2n+2}F_{2n} + F_{2n}^2 = 1, \tag{2.11}$$

$$U_n^2(k, -1) - kU_n(k, -1)U_{n-1}(k, -1) + U_{n-1}^2(k, -1) = 1, \tag{2.12}$$

$$U_n^2(k,1) - kU_n(k,1)U_{n-1}(k,1) - U_{n-1}^2(k,1) = (-1)^{n+1}, \tag{2.13}$$

$$V_n^2(k,-1) - kV_n(k,-1)V_{n-1}(k,-1) + V_{n-1}^2(k,-1) = -(k^2-4). \tag{2.14}$$

If we know the fundamental solution of the equation $x^2 - dy^2 = 4$, then we can give all positive integer solutions to this equation. We give the following lemma which can be useful for finding the fundamental solution.

**Lemma 2.1** *Let $d$ be a positive integer which is not perfect square and $a + b\sqrt{d}$ be a positive integer solution to the equation $x^2 - dy^2 = 4$. If $a > b^2 - 2$, then $a + b\sqrt{d}$ is the fundamental solution to the equation $x^2 - dy^2 = 4$.*

*Proof.* If $b = 1$, then the proof is trivial. Assume that $b > 1$. Let $x_1 + y_1\sqrt{d}$ be a positive solution to the equation $x^2 - dy^2 = 4$ such that $1 \le y_1 < b$. Then it follows that $a^2 - db^2 = x_1^2 - dy_1^2 = 4$ and thus $d = (x_1^2 - 4)/y_1^2 = (a^2-4)/b^2$. This shows that $x_1^2 b^2 - y_1^2 a^2 = 4b^2 - 4y_1^2 = 4(b^2 - y_1^2) > 0$. Thus, we have

$$[(x_1 b + y_1 a)/2][(x_1 b - y_1 a)/2] = b^2 - y_1^2 > 1.$$

It can be seen that $x_1 b + y_1 a$ and $x_1 b - y_1 a$ are even integers. Let $k_1 = (x_1 b + y_1 a)/2$ and $k_2 = (x_1 b - y_1 a)/2$. Then we obtain $k_1 k_2 = b^2 - y_1^2$ and $a = (k_1 - k_2)/y_1$. Thus, it follows that

$$a = \frac{k_1 - k_2}{y_1} \le \frac{k_1 k_2 - 1}{y_1} = \frac{b^2 - y_1^2 - 1}{y_1} \le b^2 - y_1^2 - 1 \le b^2 - 2,$$

which is a contradiction since $a > b^2 - 2$. □

Now we give the following theorem from [4].

**Theorem 2.2** *Let $x_1 + y_1\sqrt{d}$ be the fundamental solution to the equation $x^2 - dy^2 = 4$. Then all positive integer solutions to the equation $x^2 - dy^2 = 4$ are given by*

$$x_n + y_n\sqrt{d} = \frac{(x_1 + y_1\sqrt{d})^n}{2^{n-1}} = 2\left(\frac{x_1 + y_1\sqrt{d}}{2}\right)^n$$

*with $n \ge 1$.*

**Lemma 2.3** *Let $d$ be a positive integer which is not a perfect square and fundamental solution to the equation $x^2 - dy^2 = 4$ be $(a,b)$. Then all nonnegative integer solutions to the equation $x^2 - dy^2 = 4$ are given by $(x,y) = (V_n(a,-1), bU_n(a,-1))$ with $n \ge 0$.*

*Proof.* Assume that $(a,b)$ is the fundamental solution to the equation $x^2 - dy^2 = 4$. Then by Theorem 2.2, all nonnegative integer solutions to the equation $x^2 - dy^2 = 4$ are given by

$$x_n + y_n\sqrt{d} = 2\left(\frac{a + b\sqrt{d}}{2}\right)^n$$

with $n \ge 0$. Let $\alpha = \frac{a+b\sqrt{d}}{2}$ and $\beta = \frac{a-b\sqrt{d}}{2}$. Then it is seen that $x_n + y_n\sqrt{d} = 2\alpha^n$ and $x_n - y_n\sqrt{d} = 2\beta^n$. Thus it follows that $x_n = \alpha^n + \beta^n$ and $y_n = \frac{\alpha^n - \beta^n}{\sqrt{d}}$. On the other hand, since $\alpha + \beta = a, \alpha - \beta = b\sqrt{d}$, and $\alpha\beta = 1$, it follows that $x_n = V_n(a,-1)$ and $y_n = bU_n(a,-1)$. □

From now on, we will assume that $s$ and $t$ are positive integers.

Let $d$ be a positive integer which is not a perfect square. Then the following eight lemmas can be shown by using the previously given formulae for continued fraction expansion of $\sqrt{d}$. Note that $s^2t^2+2t, t^2-2t, 4t^2-2t, s^2t^2-2t, s^2t^2+t, s^2t^2-t, s^2t^2-4t$, and $s^2t^2+4t$ are not perfect square.

**Lemma 2.4** $\sqrt{s^2t^2+2t}=[st,\overline{s,2st}]$.

**Lemma 2.5** *Let* $t>2$. *Then* $\sqrt{t^2-2t}=[t-2,\overline{1,2(t-2)}]$.

**Lemma 2.6** *Let* $t\geq 2$. *Then* $\sqrt{4t^2-2t}=[2t-1,\overline{2,2(2t-1)}]$.

**Lemma 2.7** *Let* $s>2$. *Then* $\sqrt{s^2t^2-2t}=[st-1,\overline{1,s-2,1,2st-2}]$.

**Lemma 2.8** $\sqrt{s^2t^2+t}=\begin{cases}[st,\overline{2s,2st}] & \textit{if } t>1,\\ [s,\overline{2s}] & \textit{if } t=1.\end{cases}$

**Lemma 2.9** $\sqrt{s^2t^2-t}=\begin{cases}[s-1,\overline{1,2(s-1)}] & \textit{if } t{=}1 \textit{ and } s>1,\\ [st-1,\overline{1,2s-2,1,2(st-1)}] & \textit{if } t{>}1 \textit{ and } s>1,\\ [t-1,\overline{2,2(t-1)}] & \textit{if } t{>}1 \textit{ and } s=1.\end{cases}$

**Lemma 2.10** *The continued fraction expansion of $\sqrt{s^2t^2-4t}$ is*

$$\begin{cases}
[s-1,\overline{1,\frac{s-3}{2},2,\frac{s-3}{2},1,2(s-1)}] & \text{if } t=1 \text{ and } s>3 \text{ is odd,}\\
[s-1,\overline{1,\frac{s-4}{2},1,2(s-1)}] & \text{if } t=1 \text{ and } s>4 \text{ is even,}\\
[3,\overline{2,6}] & \text{if } t=1 \text{ and } s=4,\\
[2s-1,\overline{1,\frac{s}{2}-2,1,2(2s-1)}] & \text{if } t=2 \text{ and } s>4 \text{ is even,}\\
[2,\overline{1,4}] & \text{if } t=2 \text{ and } s=2,\\
[5,\overline{3,2,3,10}] & \text{if } t=2 \text{ and } s=3,\\
[7,\overline{2,14}] & \text{if } t=2 \text{ and } s=4,\\
[2s-1,\overline{1,\frac{s-3}{2},2,s-1,2,\frac{s-3}{2},1,2(2s-1)}] & \\
 & \text{if } t=2 \text{ and } s>3 \text{ is odd,}\\
[st-1,\overline{1,\frac{s}{2}-2,1,2(st-1)}] & \text{if } t>2 \text{ and } s>4 \text{ is even,}\\
[2t-2,\overline{1,2(2t-2)}] & \text{if } t>2 \text{ and } s=2,\\
[4t-1,\overline{2,2(4t-1)}] & \text{if } t>2 \text{ and } s=4,\\
[st-1,\overline{1,\frac{s-3}{2},2,\frac{st-2}{2},2,\frac{s-3}{2},1,2(st-1)}] & \\
 & \text{if } t>2 \text{ is even and } s>3 \text{ is odd,}\\
[st-1,\overline{1,\frac{s-3}{2},2,\frac{st-3}{2},1,2s-2,1,\frac{st-3}{2},2,\frac{s-3}{2},1,2(st-1)}] & \\
 & \text{if } t>2 \text{ is odd and } s>3 \text{ is odd,}\\
[t-3,\overline{1,\frac{t-6}{2},1,2(t-3)}] & \text{if } t>6 \text{ is even and } s=1,\\
[t-3,\overline{1,\frac{t-5}{2},2,\frac{t-5}{2},1,2(t-3)}] & \\
 & \text{if } t>5 \text{ is odd and } s=1,\\
[2,\overline{4}] & \text{if } t=5 \text{ and } s=1,\\
[3t-1,\overline{3,\frac{3t-2}{2},3,2(3t-1)}] & \text{if } t>2 \text{ is even and } s=3,\\
[3t-1,\overline{3,\frac{3t-3}{2},1,4,1,\frac{3t-3}{2},3,2(3t-1)}] & \\
 & \text{if } t>2 \text{ is odd and } s=3.
\end{cases}$$

**Lemma 2.11**

$$\sqrt{s^2t^2+4t}=\begin{cases} [st, \overline{\tfrac{s}{2}, 2st}] & \text{if } s \text{ is even,} \\ [s, \overline{\tfrac{s-1}{2}, 1, 1, \tfrac{s-1}{2}, 2s}] & \text{if } t=1 \text{ and } s>1 \text{ is odd,} \\ [st, \overline{\tfrac{s-1}{2}, 1, 1, \tfrac{st-1}{2}, 2s, \tfrac{st-1}{2}, 1, 1, \tfrac{s-1}{2}, 2st}] & \\ \qquad \text{if } s \geq 2 \text{ is odd and } t>1 \text{ is odd,} & \\ [st, \overline{\tfrac{s-1}{2}, 1, 1, \tfrac{st-2}{2}, 1, 1, \tfrac{s-1}{2}, 2st}] & \\ \qquad \text{if } s \geq 2 \text{ is odd and } t>1 \text{ is even,} & \\ [t+1, \overline{1, \tfrac{t-2}{2}, 1, 2(t+1)}] & \text{if } s=1 \text{ and } t>2 \text{ is even,} \\ [t+1, \overline{1, \tfrac{t-1}{2}, 2, \tfrac{t-1}{2}, 1, 2(t+1)}] & \text{if } s=1 \text{ and } t>1 \text{ is odd,} \\ [2, \overline{4}] & \text{if } s=1 \text{ and } t=1, \\ [3, \overline{2, 6}] & \text{if } s=1 \text{ and } t=2. \end{cases}$$

The proof of the following lemma is easy and it can be found in many books on number theory (see [7] or [9]).

**Lemma 2.12** *Let $d$ be a positive integer which is not perfect square. The equation $x^2-dy^2=-1$ has a solution if and only if the period length of the continued fraction expansion of $\sqrt{d}$ is an odd integer.*

The following lemma is given in [6].

**Lemma 2.13** *Let $d$ be an odd positive integer which is not perfect square. Then the equation $x^2-dy^2=-4$ has a solution if and only if the equation $x^2-dy^2=-1$ has a solution.*

The following four corollaries can be given from Lemma 2.8, Lemma 2.4, Lemma 2.7, Lemma 2.9, and Lemma 2.12.

**Corollary 2.14** *The equation $x^2-(s^2t^2+t)y^2=-1$ has a solution if and only if $t=1$.*

**Corollary 2.15** *The equation $x^2-(s^2t^2+2t)y^2=-1$ has no positive integer solutions.*

**Corollary 2.16** *Let $s>2$. Then the equation $x^2-(s^2t^2-2t)y^2=-1$ has no positive integer solutions.*

**Corollary 2.17** *The equation $x^2-(s^2t^2-t)y^2=-1$ has no positive integer solutions.*

The following lemma is given in [3].

**Lemma 2.18** *All positive integer solutions to the equation $x^2-3xy+y^2=-5$ are given by $(x,y)=(L_{2n+2},L_{2n})$ with $n\geq 0$.*

The following two lemmas are given in [2] and [1].

**Lemma 2.19** *Let $k\geq 3$. Then all positive integer solutions to the equation $x^2-(k^2+4)y^2=4$ are given by $(x,y)=(V_{2n}(k,1),U_{2n}(k,1))$ with $n\geq 1$.*

**Lemma 2.20** *Let $k \geq 3$. Then all positive integer solutions to the equation $x^2-(k^2-4)y^2=4$ are given by $(x,y)=(V_n(k,-1),U_n(k-1))$ with $n \geq 1$.*

The following lemma is well known.

**Lemma 2.21** *All positive integer solutions to the equation $x^2-2y^2=-1$ are given by $(x,y)=(Q_{2n-1}/2,P_{2n-1})$ with $n>0$.*

## 3 Main Theorems

From now on, otherwise stated, we will assume that $k \geq 3$.

**Theorem 3.1** *If $k \neq 3$ and $k \neq 6$, then the equation $x^2-kxy+y^2=-(k+2)$ has no positive integer solutions. If $k=3$, then all positive integer solutions to the equation $x^2-kxy+y^2=-(k+2)$ are given by $(x,y)=(L_{2n+2},L_{2n})$ with $n \geq 0$. If $k=6$, then all positive integer solutions to this equation are given by $(x,y)=(P_{2n}+P_{2n-1},P_{2n-1}-P_{2n-2})$ with $n \geq 0$.*

*Proof.* Assume that $x^2-kxy+y^2=-(k+2)$ for some positive integers $x$ and $y$. If $x=y$, then we obtain the equation $(2-k)x^2=-(k+2)$, which is impossible. Without loss of generality, we may assume that $x>y$. Completing the square, we get $(x+y)^2-(k+2)xy=-(k+2)$. Let $k+2=s^2t$ with $t$ square-free integer and $u=x+y$, $v=x-y$. Then we get $u^2-s^2t\left(\frac{u^2-v^2}{4}\right)=-s^2t$. Therefore, we have $st \mid u$. Let $u=sta$ for some integer $a$. Then we obtain $s^2t^2a^2-s^2t\left(\frac{u^2-v^2}{4}\right)=-s^2t$, i.e.,

$$(v/2)^2-[(s^2t^2-4t)/4]a^2=-1. \tag{3.1}$$

Let $k$ be an even integer. Since $x^2-kxy+y^2+k+2=0$, it can be seen that $x$ and $y$ are of the same parity. Thus $u$ and $v$ are always even integers. On the other hand, since $k$ is an even integer, $k+2=s^2t$ is an even integer. Thus $s$ or $t$ is even. Assume that $s$ is even. Then we write $s=2r$ for some integer $r$. Thus it follows that $\frac{s^2t^2-4t}{4}=r^2t^2-t$. Then by (3.1), we get $\left(\frac{v}{2}\right)^2-(r^2t^2-t)a^2=-1$, which is impossible by Corollary 2.17. Assume that $t$ is an even integer. Then we can write $t=2m$ for some positive integer $m$. Thus it follows that $\frac{s^2t^2-4t}{4}=s^2m^2-2m$. This implies that

$$\left(\frac{v}{2}\right)^2-(s^2m^2-2m)a^2=-1 \tag{3.2}$$

by (3.1). If $s=1$ and $m>2$, then we obtain the equation $(v/2)^2-(m^2-2m)a^2=-1$ by (3.2). This is impossible since the equation $X^2-(m^2-2m)Y^2=-1$ has no positive integer solutions by Lemma 2.5 and Lemma 2.12. If $s=1$ and $m=1$, then we get $(v/2)^2+a^2=-1$ by (3.2), which is impossible. If $s=1$ and $m=2$, then we have $(v/2)^2=-1$ by (3.2), which is impossible. If $s=2$ and $m>1$, then we obtain the equation $(v/2)^2-(4m^2-2m)a^2=-1$. This is impossible since the equation $X^2-(4m^2-2m)Y^2=-1$ has no positive integer solutions by Lemma 2.6 and Lemma 2.12. If $s=2$ and $m=1$, then $k=6$ and we get $(v/2)^2-2a^2=-1$. Then we have

$(v/2, a) = (Q_{2n-1}/2, P_{2n-1})$, i.e., $(v, a) = (Q_{2n-1}, P_{2n-1})$ by Lemma 2.21. Then it follows that $u = sta = 2 \cdot 2 \cdot P_{2n-1}$. Therefore, it can be easily seen that

$$x = \frac{u+v}{2} = \frac{4P_{2n-1} + Q_{2n-1}}{2} = P_{2n} + P_{2n-1}$$

and

$$y = \frac{u-v}{2} = \frac{4P_{2n-1} - Q_{2n-1}}{2} = P_{2n-1} - P_{2n-2}.$$

If $s > 2$, then by (3.2), it can be written that $\left(\frac{v}{2}\right)^2 - (s^2m^2 - 2m)a^2 = -1$, which is impossible by Corollary 2.16. Now, let $k$ be an odd integer. Then $s^2t^2 - 4t$ is an odd integer. Since the equation $v^2 - (s^2t^2 - 4t)a^2 = -4$ has a solution, the equation $v^2 - (s^2t^2 - 4t)a^2 = -1$ has a solution by Lemma 2.13. Here, the period length of the continued fraction expansion of $\sqrt{s^2t^2 - 4t}$ must be an odd integer by Lemma 2.12. Then by Lemma 2.10, it follows that $t = 5$ and $s = 1$. Therefore, we have $k = s^2t - 2 = 3$. Since $k = 3$, we obtain the equation $x^2 - 3xy + y^2 = -5$. By Lemma 2.18, all positive integer solutions to the equation $x^2 - 3xy + y^2 = -5$ are given by $(x, y) = (L_{2n+2}, L_{2n})$ with $n \geq 0$. $\square$

**Theorem 3.2** *Let $k+2$ be not a perfect square. Then all positive integer solutions to the equation $x^2 - kxy + y^2 = k+2$ are given by $(x, y) = (U_{n+1} + U_n, U_n + U_{n-1})$ with $n > 0$, where $U_n = U_n(k, -1)$.*

*Proof.* Assume that $x^2 - kxy + y^2 = k+2$ for some positive integers $x$ and $y$. Without loss of generality, we may suppose that $x \geq y$. If $x = y$, then we obtain the equation $(2-k)x^2 = k+2$, which is impossible since $k > 1$. Then we get $x > y$. Completing the square, we get $(x+y)^2 - (k+2)xy = (k+2)$. Let $k+2 = s^2t$ with $t > 1$ square-free integer and $u = x+y$, $v = x-y$. Then we have $u^2 - s^2t\left(\frac{u^2-v^2}{4}\right) = s^2t$. Thus it follows that $st \mid u$. Let $u = sta$ for some integer $a$. From here, we get $s^2t^2a^2 - s^2t\left(\frac{(sta)^2-v^2}{4}\right) = s^2t$ and therefore $v^2 - (s^2t^2 - 4t)a^2 = 4$. Using the fact that $s^2t - 2 > s^2 - 2$ for $t > 1$ and $(s^2t-2)^2 - (s^2t^2 - 4t)s^2 = 4$, we see that the fundamental solution to the equation $X^2 - (s^2t^2 - 4t)Y^2 = 4$ is $(s^2t - 2, s)$ by Lemma 2.1. Then by Lemma 2.3, we get $(v, a) = (V_n(s^2t-2, -1), sU_n(s^2t-2, -1))$ for some positive integer $n$. Thus it follows that $u = sta = s^2tU_n(s^2t-2, -1) = (k+2)U_n(k, -1)$. Since $u = x+y$ and $v = x-y$, we obtain

$$\begin{aligned} x = \frac{u+v}{2} &= \frac{(k+2)U_n(k,-1) + V_n(k,-1)}{2} \\ &= \frac{(k+2)U_n(k,-1) + kU_n(k,-1) - 2U_{n-1}(k,-1)}{2} \\ &= \frac{2(kU_n(k,-1) - U_{n-1}(k,-1)) + 2U_n(k,-1)}{2} \\ &= U_{n+1}(k,-1) + U_n(k,-1) \end{aligned}$$

and

$$\begin{aligned} y &= \frac{u-v}{2} = \frac{(k+2)U_n(k,-1) - V_n(k,-1)}{2} \\ &= \frac{(k+2)U_n(k,-1) - kU_n(k,-1) + 2U_{n-1}(k,-1)}{2} \\ &= U_n(k,-1) + U_{n-1}(k,-1) \end{aligned}$$

by (2.4).

Conversely, if $(x,y) = (U_{n+1} + U_n, U_n + U_{n-1})$ where $U_n = U_n(k,-1)$, then it can be seen that $x^2 - kxy + y^2 - k - 2 = 0$ by (2.12). This ends the proof of the theorem. □

**Theorem 3.3** *All positive integer solutions to the equation* $x^2 - (k^2-2)xy + y^2 = k^2$ *are given by* $(x,y) = (U_n(k,-1), U_{n-2}(k,-1))$ *with* $n \geq 3$.

*Proof.* Assume that $s = k^2 - 2$. Let $(x,y)$ be a positive solution to the equation $x^2 - sxy + y^2 = s + 2$. Without loss of generality, we may assume that $x \geq y$. Then $sx - 2y > 0$. Multiplying both side of the equation $x^2 - sxy + y^2 = s + 2$ by 4 and completing the square, we get $(sx-2y)^2 - (s^2-4)x^2 = 4(s+2)$, i.e., $(sx-2y)^2 - (k^2-4)k^2x^2 = 4k^2$. This shows that $sx - 2y = ka$ for some positive integer $a$. Thus we obtain $k^2a^2 - (k^2-4)k^2x^2 = 4k^2$, i.e., $a^2 - (k^2-4)x^2 = 4$. By Lemma 2.20, we get $(a,x) = (V_n(k,-1), U_n(k,-1))$ for some integer $n \geq 1$. This shows that $sx - 2y = ka = kV_n$ and thus it can be seen that

$$y = \frac{sx - kV_n}{2} = \frac{(k^2-2)U_n - kV_n}{2} = U_{n-2}.$$

When $n = 1, 2$, it follows that $y = U_{n-2} < 0$. Thus we have $n \geq 3$.

Conversely, if $x = U_n(k,-1)$ and $y = U_{n-2}(k,-1)$, then it is easily seen that $x^2 - (k^2-2)xy + y^2 = k^2$ by (2.12). □

**Theorem 3.4** *Let* $k+2$ *be not a perfect square. Then all positive integer solutions to the equation* $x^2 - (k^2-4)y^2 = 4(k+2)$ *are given by* $(x,y) = (V_n + V_{n-1}, U_n + U_{n-1})$ *with* $n > 0$, *where* $U_n = U_n(k,-1)$ *and* $V_n = V_n(k,-1)$.

*Proof.* Assume that $x^2 - (k^2-4)y^2 = 4(k+2)$ for some positive integers $x$ and $y$. Then we get

$$(x+ky)^2 - 2k(x+ky)y + (2y)^2 = 4(k+2).$$

Here, it can be seen that $2 \mid x + ky$. Then taking $u = \frac{x+ky}{2}$ and $v = y$, we obtain the equation $u^2 - kuv + v^2 = k + 2$. From Theorem 3.2, we have $(u,v) = (U_{n+1} + U_n, U_n + U_{n-1})$ for some positive integer $n$, where $U_n = U_n(k,-1)$. Thus by (2.4), it is seen that

$$x = 2u - kv = 2(U_{n+1} + U_n) - k(U_n + U_{n-1}) = V_n + V_{n-1}$$

and $y = v = U_n + U_{n-1}$, where $U_n = U_n(k,-1)$ and $V_n = V_n(k,-1)$.

Conversely, if $(x,y) = (V_n + V_{n-1}, U_n + U_{n-1})$, then it can be seen that $x^2 - (k^2-4)y^2 = 4(k+2)$ by using (2.6) and (2.7). So the proof is completed. □

**Theorem 3.5** *Let $k+2$ be a perfect square. Then all positive integer solutions of the equation $x^2-(k^2-4)y^2=4(k+2)$ are given by $(x,y)=(\sqrt{k+2}V_{n-2},U_{n-2})$ with $n>2$, where $U_n=U_n(\sqrt{k+2},-1)$ and $V_n=V_n(\sqrt{k+2},-1)$.*

*Proof.* Assume that $x^2-(k^2-4)y^2=4(k+2)$ for some positive integers $x$ and $y$. Then we get

$$(x+ky)^2-2k(x+ky)y+(2y)^2=4(k+2).$$

It can be seen that $2 \mid x+ky$. Let $u=\frac{x+ky}{2}$ and $v=y$. Let $s^2=k+2$ for some integer $s$. Then we obtain $u^2-(s^2-2)uv+v^2=s^2$. By Theorem 3.3, it follows that $u=U_n(s,-1)$ and $v=U_{n-2}(s,-1)$ with $n\geq 3$. Since $s^2=k+2$, we get

$$\begin{aligned} x=2u-ky &= 2U_n-kU_{n-2} \\ &= 2\sqrt{k+2}U_{n-1}-2U_{n-2}-kU_{n-2} \\ &= \sqrt{k+2}(U_{n-1}+\sqrt{k+2}U_{n-2}-U_{n-3}-\sqrt{k+2}U_{n-2}) \\ &= \sqrt{k+2}(U_{n-1}-U_{n-3})=\sqrt{k+2}V_{n-2} \end{aligned}$$

and $y=v=U_{n-2}$ by (2.4), where $U_n=U_n(s,-1)=U_n(\sqrt{k+2},-1)$ and $V_n=V_n(\sqrt{k+2},-1)$. Conversely, if $(x,y)=(\sqrt{k+2}V_{n-2},U_{n-2})$ with $n>2$, then it can be seen that $x^2-(k^2-4)y^2=4(k+2)$. □

**Theorem 3.6** *The equation $x^2-(k^2-4)y^2=-4(k+2)$ has a positive integer solution only when $k=3$ or $6$. If $k=3$, then all positive integer solutions to this equation are given by $(x,y)=(5F_{2n},L_{2n})$ with $n>0$. If $k=6$, then all positive integer solutions to this equation are given by $(x,y)=(8P_{2n-2},P_{2n-1}-P_{2n-2})$ with $n>1$.*

*Proof.* Assume that $x^2-(k^2-4)y^2=-4(k+2)$ for some positive integers $x$ and $y$. It can be seen that $2 \mid x+ky$. Let $u=\frac{x+ky}{2}$ and $v=y$. Then we get $u^2-kuv+v^2=-(k+2)$. From Theorem 3.1, we obtain $k=3$ or $6$. Firstly, let $k=3$. Then we have $(u,v)=(L_{2n+2},L_{2n})$ for some nonnegative integer $n$ by Theorem 3.1. Thus by the identity (2.4), it is seen that

$$x=2u-kv=2L_{2n+2}-3L_{2n}=5F_{2n}$$

and $y=v=L_{2n}$. If $n=0$, then we get $x=0$. Thus it follows that $n>0$. Now, let $k=6$. Then we get $(u,v)=(P_{2n}+P_{2n-1},P_{2n-1}-P_{2n-2})$ for some nonnegative integer $n$ by Theorem 3.1. Thus by the identity (2.4), it is seen that

$$x=2u-kv=2(P_{2n}+P_{2n-1})-6(P_{2n-1}-P_{2n-2})=8P_{2n-2}$$

and $y=v=P_{2n-1}-P_{2n-2}$. Conversely, if $k=3$ and $(x,y)=(5F_{2n},L_{2n})$ with $n>0$ or $k=6$ and $(x,y)=(8P_{2n-2},P_{2n-1}-P_{2n-2})$ with $n>1$, then it can be seen that $x^2-(k^2-4)y^2=-4(k+2)$. □

We omit the proof of the following theorem.

**Theorem 3.7** *The equation $x^2-(k^2-4)y^2=4(k-2)$ has a solution if and only if $k-2$ is a perfect square. Moreover, if $k-2$ is a perfect square, then all positive integer solutions to this equation are given by $(x,y)=(\sqrt{k-2}V_{2n-2},U_{2n-2})$ with $n>1$, where $U_n=U_n(\sqrt{k-2},1)$ and $V_n=V_n(\sqrt{k-2},1)$.*

**Theorem 3.8** *All positive integer solutions to the equation* $x^2 - kxy + y^2 = -(k-2)$ *are given by* $(x, y) = (U_{n+1} - U_n, U_n - U_{n-1})$ *with* $n \geq 0$*, where* $U_n = U_n(k, -1)$.

*Proof.* Assume that $x^2 - kxy + y^2 = -(k-2)$ for some positive integers $x$ and $y$. Without loss of generality, we may assume that $x \geq y$. Then $x - y \geq 0$. Completing the square, we get $(x-y)^2 - (k-2)xy = -(k-2)$. Let $k - 2 = s^2 t$ with $t$ square-free integer. Taking $u = x + y$ and $v = x - y$, we get $v^2 - s^2 t\left(\frac{u^2 - v^2}{4}\right) = -s^2 t$. Here, it follows that $st \mid v$. Let $v = sta$ for some integer $a$. Then we get $s^2 t^2 a^2 - s^2 t\left(\frac{u^2 - (sta)^2}{4}\right) = -s^2 t$ and therefore $u^2 - (s^2 t^2 + 4t)a^2 = 4$. Since $(s^2 t + 2)^2 - (s^2 t^2 + 4t)s^2 = 4$ and $s^2 t + 2 > s^2 - 2$ for $t > 0$, it follows that $(s^2 t + 2, s)$ is the fundamental solution to the equation $X^2 - (s^2 t^2 + 4t)Y^2 = 4$ by Lemma 2.1. Thus $(u, a) = (V_n(s^2 t + 2, -1), sU_n(s^2 t + 2, -1))$ for some $n \geq 0$ by Lemma 2.3. This shows that $v = sta = s^2 t U_n(s^2 t + 2, -1) = (k-2)U_n(k, -1)$. Since $u = x + y$ and $v = x - y$, it can be shown that

$$x = \frac{u+v}{2} = \frac{V_n(k,-1) + (k-2)U_n(k,-1)}{2} = U_{n+1}(k,-1) - U_n(k,-1)$$

and

$$y = \frac{u-v}{2} = \frac{V_n(k,-1) - (k-2)U_n(k,-1)}{2} = U_n(k,-1) - U_{n-1}(k,-1).$$

Conversely, if $(x, y) = (U_{n+1} - U_n, U_n - U_{n-1})$, then it can be easily seen that $x^2 - kxy + y^2 = k - 2$ by (2.12). $\square$

**Corollary 3.9** *Let* $k \geq 1$. *All positive integer solutions of the equation* $x^2 - (k^2+2)xy + y^2 = -k^2$ *are given by* $(x, y) = (U_{2n+1}, U_{2n-1})$ *with* $n \geq 0$*, where* $U_n = U_n(k, 1)$.

*Proof.* Assume that $x^2 - (k^2 + 2)xy + y^2 = -k^2$ for some positive integers $x$ and $y$. Let $s = k^2 + 2$. Then it follows that $x^2 - sxy + y^2 = -s + 2$. Since $s = k^2 + 2 > 2$, we get $(x, y) = (U_{n+1} - U_n, U_n - U_{n-1})$ for some nonnegative integer $n$ by Theorem 3.8, where $U_n = U_n(s, -1)$. Since $s = k^2 + 2$, we obtain

$$x = U_{n+1}(k^2 + 2, -1) - U_n(k^2 + 2, -1) = U_{2n+1}(k, 1)$$

and

$$y = U_n(k^2 + 2, -1) - U_{n-1}(k^2 + 2, -1) = U_{2n-1}(k, 1),$$

by the identity (2.3). Conversely, if $(x, y) = (U_{2n+1}, U_{2n-1})$, then it can be seen that $x^2 - (k^2 + 2)xy + y^2 = -k^2$. $\square$

**Theorem 3.10** *All positive integer solutions of the equation* $x^2 - (k^2 - 4)y^2 = -4(k-2)$ *are given by* $(x, y) = (V_n - V_{n-1}, U_n - U_{n-1})$ *with* $n > 0$*, where* $U_n = U_n(k, -1)$ *and* $V_n = V_n(k, -1)$.

*Proof.* Assume that $x^2 - (k^2 - 4)y^2 = -4(k-2)$ for some positive integers $x$ and $y$. Similarly, it can be seen that $2 \mid x + ky$. Let $u = \frac{x+ky}{2}$ and $v = y$. Then it follows that $u^2 - kuv + v^2 = -k + 2$. From Theorem 3.8, we have $(u, v) = (U_{n+1} - U_n, U_n - U_{n-1})$ for some nonnegative integer $n$, where $U_n = U_n(k, -1)$. Thus by (2.4), it is seen that

$$x = 2u - kv = 2(U_{n+1} - U_n) - k(U_n - U_{n-1}) = V_n - V_{n-1}$$

and $y = v = U_n - U_{n-1}$ with $n \geq 0$, where $U_n = U_n(k, -1)$ and $V_n = V_n(k, -1)$. Moreover, if $n = 0$, then $x = 2 - k$. Since $k > 2$, this is impossible. Thus we have $n > 0$.

Conversely, if $(x, y) = (V_n - V_{n-1}, U_n - U_{n-1})$, then it can be seen that $x^2 - (k^2 - 4)y^2 = -4(k - 2)$ by (2.6) and (2.8). □

**Theorem 3.11** *Let $k \geq 1$. All positive integer solutions of the equation $x^2 - (k^2 + 2)xy + y^2 = k^2$ are given by $(x, y) = (U_{2n}(k, 1), U_{2n-2}(k, 1))$ with $n > 1$.*

*Proof.* Let $s = k^2 + 2$ and $(x, y)$ be a positive solution of the equation $x^2 - sxy + y^2 = s - 2$. Without loss of generality, we may assume that $x \geq y$. Then it can be seen that $sx - 2y > 0$. Multiplying both sides of the equation $x^2 - sxy + y^2 = s - 2$ by 4 and completing the square, we get $(sx - 2y)^2 - (s^2 - 4)x^2 = 4(s - 2)$ and therefore $(sx - 2y)^2 - (k^2 + 4)k^2x^2 = 4k^2$. This shows that $sx - 2y = ka$ for some positive integer $a$. Thus we obtain $k^2a^2 - (k^2 + 4)k^2x^2 = 4k^2$, i.e., $a^2 - (k^2 + 4)x^2 = 4$. Then $(a, x) = (V_{2n}(k, 1), U_{2n}(k, 1))$ for some positive integer $n$ by Lemma 2.19. Since $sx - 2y = ka = kV_{2n}$, we get

$$\begin{aligned} y &= \frac{sx - kV_{2n}}{2} = \frac{sU_{2n} - kV_{2n}}{2} = \frac{(k^2 + 2)U_{2n} - k(kU_{2n} + 2U_{2n-1})}{2} \\ &= \frac{2U_{2n} - 2kU_{2n-1}}{2} = -kU_{2n-1} + U_{2n} = U_{2n-2} \end{aligned}$$

by (2.5).

Conversely, if $x = U_{2n}(k, 1)$ and $y = U_{2n-2}(k, 1)$, then it is easily seen that $x^2 - (k^2 + 2)xy + y^2 - k^2 = 0$ by (2.13). □

**Theorem 3.12** *The equation $x^2 - kxy + y^2 = k - 2$ has a solution if and only if $k - 2$ is a perfect square.*

*Proof.* Assume that $x^2 - kxy + y^2 = k - 2$ for some positive integers $x$ and $y$. It can be seen that $x \neq y$. Without loss of generality, we may assume that $x > y$. Completing the square, we get $(x - y)^2 - (k - 2)xy = k - 2$. Let $k - 2 = s^2t$ with $t$ square-free integer. Taking $u = x + y$ and $v = x - y$, we see that $v^2 - s^2t\left(\frac{u^2 - v^2}{4}\right) = s^2t$. Thus we obtain $st \mid v$ and therefore $v = sta$ for some integer $a$. Consequently, we obtain

$$ta^2 - \left(\frac{u^2 - s^2t^2a^2}{4}\right) = 1.$$

This shows that

$$u^2 - (s^2t^2 + 4t)a^2 = -4. \tag{3.3}$$

Let $k$ be an even integer. Since $x^2 - kxy + y^2 = k - 2$, it can be seen that $x$ and $y$ have the same parity. Thus $u$ and $v$ are always even integers. As a result of (3.3), we obtain

$$\left(\frac{u}{2}\right)^2 - \left(\frac{s^2t^2 + 4t}{4}\right)a^2 = -1. \tag{3.4}$$

On the other hand, since $k$ is an even integer, $k - 2 = s^2t$ is even. Thus $s$ or $t$ is even. Assume that $s$ is even. Then we get $s = 2r$ for some integer $r$. Thus it follows that $\frac{s^2t^2+4t}{4} = r^2t^2+t$. Therefore by (3.4), we obtain $\left(\frac{u}{2}\right)^2 - (r^2t^2+t)a^2 = -1$. Then by Corollary 2.14, it follows that $t = 1$. Thus we have $k-2 = s^2$, which implies that $k-2$ is a perfect square. Assume that $t$ is even. Then $t = 2m$ for some integer $m$. Thus it follows that $\frac{s^2t^2+4t}{4} = s^2m^2+2m$. Therefore by (3.4), we get $\left(\frac{u}{2}\right)^2 - (s^2m^2+2m)a^2 = -1$. But this is impossible by Corollary 2.15. Now, let $k$ be odd. Since $k-2 = s^2t$ and $k$ is odd, it follows that $s$ and $t$ are odd. Then $s^2t^2+4t$ is odd. Since the equation $u^2-(s^2t^2+4t)a^2 = -4$ has a solution, the equation $u^2-(s^2t^2+4t)a^2 = -1$ has a solution by Lemma 2.13. Moreover, by Lemma 2.12, the period length of the continued fraction expansion of $\sqrt{s^2t^2+4t}$ must be odd. As a result, by Lemma 2.11, it follows that $t = 1$. Then we get $k-2 = s^2$. That is, $k-2$ is a perfect square. Conversely, if $k-2$ is a perfect square, then the equation $x^2-kxy+y^2 = k-2$ has a solution by Theorem 3.11. □

**Theorem 3.13** *Let $k > 3$ and $k^2-4$ be a square-free integer. Then the equation $x^2-kxy+y^2 = -(k-2)(k^2-4)$ has no integer solutions.*

*Proof.* Assume that $x^2-kxy+y^2 = -(k-2)(k^2-4)$ for some positive integers $x$ and $y$. Then multiplying both sides of this equation by 4 and completing the square, we get

$$(2x-ky)^2-(k^2-4)y^2 = -4(k-2)(k^2-4)$$

and

$$(2y-kx)^2-(k^2-4)x^2 = -4(k-2)(k^2-4).$$

On the other hand, without loss of generality, we may suppose that $x \geq y$. If $x = y$, then we obtain the equation $(2-k)x^2 = -(k-2)(k^2-4)$. This is impossible since $k^2-4$ is a square-free integer. Then we get $x > y$ and therefore $kx-2y > 0$. Since $k^2-4$ is a square-free integer, it can be seen that $k^2-4 \mid kx-2y$ and $k^2-4 \mid 2x-ky$. Let $u = \frac{kx-2y}{k^2-4}$ and $v = \frac{2x-ky}{k^2-4}$. Then we get $u^2-kuv+v^2 = k-2$. From Theorem 3.12, it follows that $k-2$ is a perfect square, which is impossible since $k^2-4$ is square-free. □

**Theorem 3.14** *Let $k^2-4$ be a square-free integer. Then all positive integer solutions of the equation $x^2-kxy+y^2 = -(k+2)(k^2-4)$ are given by $(x,y) = (V_n+V_{n-1}, V_{n-1}+V_{n-2})$ with $n > 0$, where $V_n = V_n(k,-1)$.*

*Proof.* Assume that $x^2-kxy+y^2 = -(k+2)(k^2-4)$ for some positive integers $x$ and $y$. Then without loss of generality, we may suppose that $x \geq y$. Then we get $kx-2y > 0$. Thus multiplying both sides of the equation $x^2-kxy+y^2 = -(k+2)(k^2-4)$ by 4 and completing the square, we get

$$(kx-2y)^2-(k^2-4)x^2 = -4(k+2)(k^2-4). \tag{3.5}$$

Since $k^2-4$ is a square-free integer, it is seen that $k^2-4 \mid kx-2y$. Then we have $kx-2y = (k^2-4)a$ for some integer $a$. By using (3.5), we obtain $(k^2-4)^2a^2-(k^2-4)x^2 = -4(k^2-4)(k+2)$, i.e., $x^2-(k^2-4)a^2 = 4(k+2)$. From Theorem 3.4, we get

$(x, a) = (V_n + V_{n-1}, U_n + U_{n-1})$ for some positive integer $n$, where $U_n = U_n(k, -1)$ and $V_n = V_n(k, -1)$. Thus by using (2.9), it can be seen that

$$\begin{aligned} y &= \frac{kx - (k^2-4)a}{2} = \frac{k(V_n + V_{n-1}) - (k^2-4)(U_n + U_{n-1})}{2} \\ &= \frac{kV_n + kV_{n-1} - (V_{n+1} - V_{n-1}) - (V_n - V_{n-2})}{2} \\ &= \frac{kV_n + kV_{n-1} - kV_n + V_{n-1} + V_{n-1} - kV_{n-1} + V_{n-2} + V_{n-2}}{2} \\ &= V_{n-1} + V_{n-2}. \end{aligned}$$

Conversely, if $(x, y) = (V_n + V_{n-1}, V_{n-1} + V_{n-2})$, then it can be seen that $x^2 - kxy + y^2 = -(k+2)(k^2-4)$ by (2.9) and (2.14). Then the proof follows. □

**Theorem 3.15** *Let $k^2 - 4$ be a square-free integer. Then the equation $x^2 - kxy + y^2 = (k+2)(k^2-4)$ has a positive integer solution only when $k = 3$ and in this case all positive integer solutions to this equation are given by $(x, y) = (5F_{2n}, 5F_{2n-2})$ with $n > 1$.*

*Proof.* Assume that $x^2 - kxy + y^2 = (k+2)(k^2-4)$ for some positive integers $x$ and $y$. Then without loss of generality, we may suppose that $x \geq y$. If $x = y$, then we obtain the equation $(2-k)x^2 = (k+2)(k^2-4)$. This is impossible. Then we get $x > y$ and therefore $kx - 2y > 0$. Multiplying both sides of the equation $x^2 - kxy + y^2 = (k+2)(k^2-4)$ by 4 and completing the square, we get

$$(kx - 2y)^2 - (k^2-4)x^2 = 4(k+2)(k^2-4). \tag{3.6}$$

Since $k^2 - 4$ is a square-free integer, it can be seen that $k^2 - 4 \mid kx - 2y$. Then we have $kx - 2y = (k^2-4)a$ for some positive integer $a$. By using (3.6), we obtain $(k^2-4)^2a^2 - (k^2-4)x^2 = 4(k^2-4)(k+2)$, i.e., $x^2 - (k^2-4)a^2 = -4(k+2)$. From Theorem 3.6, we get $k = 3$ or 6. Since $k^2 - 4$ is a square-free integer, we have $k = 3$. Then from Theorem 3.6, we obtain $(x, a) = (5F_{2n}, L_{2n})$ for some positive integer $n$. Thus by using (2.2), it can be seen that

$$\begin{aligned} y &= \frac{kx - (k^2-4)a}{2} = \frac{15F_{2n} - 5L_{2n}}{2} = 5\frac{3F_{2n} - F_{2n+1} - F_{2n-1}}{2} \\ &= 5\frac{3F_{2n} - F_{2n} - F_{2n-1} - F_{2n-1}}{2} = 5\frac{2(F_{2n} - F_{2n-1})}{2} \\ &= 5F_{2n-2}. \end{aligned}$$

If $n = 1$, then we get $y = 5F_{2n-2} = F_0 = 0$. But this is impossible since $y$ is a positive integer. Thus we have $n > 1$. Conversely, if $k = 3$ and $(x, y) = (5F_{2n}, 5F_{2n-2})$ with $n > 1$, then it can be seen that $x^2 - kxy + y^2 = (k+2)(k^2-4)$ by (2.11). So the proof is completed. □

We omit the proof of the following theorem.

**Theorem 3.16** *Let $k^2 - 4$ be a square-free integer. Then all positive integer solutions to the equation $x^2 - kxy + y^2 = (k-2)(k^2-4)$ are given by $(x, y) = (V_n - V_{n-1}, V_{n-1} - V_{n-2})$ with $n > 1$, where $V_n = V_n(k, -1)$.*

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