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Linear Inequations and Regularity Conditions on Ordered Semigroups

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Abstract As in the results proved by Bogdanović et al. [S. Bogdanović, M. Ćirić, P. Stanimirović and T. Petković, *Linear Equations and Regularity Conditions on Semigroups*, Semigroup Forum, **69**, 2004, 63-74], all types of the regularity including regularity, (m,n)-regularity, and (p,q,r)-regularity of elements of an ordered semigroup defined by inequations, called *linear inequations* are determined. We prove that there are 16 types of the regularity of ordered semigroups defined by such inequations.

Keywords semigroup · ordered semigroup · free semigroup · linear inequation · regular · (m,n)-regular · (p,q,r)-regular

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1 Preliminaries

J. Von Nuemann [9] introduced the notion of regularity in rings and semigroups by: an element a of a ring (semigroup) S is *regular* if the equation $a = axa$ with the variable x has a solution in S. This notion has been widely studied. Since then several kinds of regularity of a semigroup such as left and right regularity, complete regularity, intra-regularity, left and right quasi-regularity, (m,n)-regularity and (p,q,r)-regularity were defined and studied by many authors (see [2], Section 1).

The problem of classification of all types of the regularity of semigroups (without order) defined by equations of the form $a = a^m x a^n$ with $m, n \geqslant 0$, $m + n \geqslant 2$ was stated by Croisot [4]. The author showed that any of these equations determines either the regularity, left, right or complete regularity (see in [3], Section 4.1). Similarly, the problem concerning classification of all types of the regularity of elements of semigroups (without order) defined by equations of the form $a = a^p x a^q y a^r$ with $p, q, r \geqslant 0$ was studied by Lajos and Szász [8].

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In [2], Bogdanović et al. presented interesting results generalized the results obtained by Croisot, Lajos and Szász, mentioned in the last paragraph. Indeed, they determined all types of the regularity of elements of a semigroup (without order) defined by equations, called *linear equations*.

Now, a semigroup $(S, \cdot)$ together with a partial order $\leqslant$ that is *compatible* with the semigroup operation, meaning that, for any a, b, c in S,

$$a \leqslant b \Rightarrow ac \leqslant bc, ca \leqslant cb,$$

is called a *partially ordered semigroup*, or simply an *ordered semigroup* (see [1] and [5]).

For A, B non-empty subsets of an ordered semigroup $(S, \cdot, \leqslant)$, we write the set product AB for the set of all elements xy in S where $x \in A$ and $y \in B$. In particular, if $B = \{y\}$, we write Ay for $A\{y\}$, and similarly for $A = \{x\}$. Moreover, we write $(A]$ for the set of all elements x in S such that $x \leqslant a$ for some a in A, i.e.,

$$(A] = \{x \in S \mid x \leqslant a \text{ for some } a \in A\}.$$

In particular, if $A = \{x\}$, we write $(x]$ for $(\{x\}]$. It was observed (see, for example in [7]) that the following conditions hold:

(1) $A \subseteq (A]$;
(2) $A \subseteq B \Rightarrow (A] \subseteq (B]$;
(3) $(A](B] \subseteq (AB]$;
(4) $((A]] = (A]$.

An ordered semigroup $(S, \cdot, \leqslant)$ is said to be *regular* [6] if the inequations $a \leqslant axa$ with the variable x, is solvable in S. This is equivalent to say that $a \in (aSa]$ for all a in S. In general, for non-negative integers m and n, S is said to be (m, n)-*regular* [10] if the inequations $a \leqslant a^m x a^n$ with the variable x, is solvable in S. This is equivalent to say that $a \in (a^m S a^n]$ for all a in S. As in [8], for non-negative integers p, q and r, S is said to be (p, q, r)-*regular* if the inequations $a \leqslant a^p x a^q y a^r$ with the variables x, y, is solvable in S. Equivalently, $a \in (a^p S a^q S a^r]$ for all a in S. The purpose of this paper is to extend the results obtained in [2] to ordered semigroups. In fact, we prove that there are 16 types of the regularity of elements of ordered semigroups defined by inequations, called *linear inequations*.

The following notions concerning free semigroups will be used throughout the paper. Let X be a non-empty set called an *alphabet*. The *free semigroup* and the *free monoid* over X will be denoted by X^+ and X^*, respectively. Let $u, v \in X^+$. We say that u is a *subword* of v, denoted by $u \mid v$, if $v = puq$ for some p, q in X^*. Otherwise, we write $u \nmid v$. The word u is said to be a *prefix* of v if $v = up$ for some p in X^*, and a *suffix* of v if $v = qu$ for some q in X^*. The number of appearances of the letter x in the word u will be denoted by $|x|_u$. The first letter (the prefix of length 1) of u will be denoted by $h(u)$ and called the *head* of u, and the least letter (the suffix of length 1) of u is denoted by $t(u)$ and called the *tail* of u. We denote the prefix of u of length 2 by $h^{(2)}(u)$, and the suffix of u of length 2 by $t^{(2)}(u)$.

2 Linear inequations

Let X be a countable alphabet whose elements are called *variables*. Let c be a symbol such that $c \notin X$, called a *constant*. Let L be the set of all words $u \in (X \cup \{c\})^+$ satisfying the following conditions:

(i) the constant c appears at least once in u;
(ii) at least one of the variables appears in u;
(iii) any variable appears at most once in u.

Note that any word $u \in L$ is *linear* with respect to variables. We write

$$u(c, x_1, \ldots, x_n)$$

instead of u to emphasize that $\{x_1, x_2, \ldots, x_n\}$ is the set of all variables appearing in u.

For $u(c, x_1, \ldots, x_n) \in L$, an expression of the form

$$c \leqslant u(c, x_1, \ldots, x_n)$$

is called a *linear inequation*, *regularity inequation* or an *inequation*. For an ordered semigroup $(S, \cdot, \leqslant)$ and $a \in S$, an expression of the form $a \leqslant u(a, x_1, \ldots, x_n)$ is called an *inequation* in S. This inequation is said to be *solvable* in S if there exist $a_1, a_2, \ldots, a_n$ in S such that $a \leqslant u(a, a_1, \ldots, a_n)$.

Let $u(c, x_1, \ldots, x_n)$, $v(c, y_1, \ldots, y_m) \in L$. We shall say that the inequation $c \leqslant u(c, x_1, \ldots, x_n)$ *implies* the inequation $c \leqslant v(c, y_1, \ldots, y_m)$, denoted by

$$c \leqslant u(c, x_1, \ldots, x_n) \to c \leqslant v(c, y_1, \ldots, y_m),$$

if for every ordered semigroup $(S, \cdot, \leqslant)$ and for every $a \in S$ we have

$$a \leqslant u(a, x_1, \ldots, x_n) \text{ is solvable in } S \Rightarrow a \leqslant v(a, y_1, \ldots, y_m) \text{ is solvable in } S.$$

The inequation $c \leqslant u(c, x_1, \ldots, x_n)$ and $c \leqslant v(c, y_1, \ldots, y_m)$ are *equivalent*, denoted by

$$c \leqslant u(c, x_1, \ldots, x_n) \sim c \leqslant v(c, y_1, \ldots, y_m),$$

if for every ordered semigroup $(S, \cdot, \leqslant)$ and for every $a \in S$ we have

$$a \leqslant u(a, x_1, \ldots, x_n) \text{ is solvable in } S \Leftrightarrow a \leqslant v(a, y_1, \ldots, y_m) \text{ is solvable in } S.$$

We now consider the equivalence problem for all linear inequations. The main result (Theorem 2.17) will be proved by dividing into 16 lemmas.

Lemma 2.1 *Let $u = u(c, x_1, \ldots, x_n) \in L$ be such that $|c|_u = 1$, $h(u) \neq c$ and $t(u) \neq c$. Then*

$$c \leqslant u(c, x_1, \ldots, x_n) \sim c \leqslant xcy.$$

Proof. We consider an ordered semigroup $(S, \cdot, \leqslant)$ and $a \in S$. Then

$$a \leqslant u(a, x_1, \ldots, x_n)$$

has a solution in S if and only if $a \in (S^i a S^j]$ where i and j are the lengths of the longest prefix and suffix of u not containing the letter c, respectively. That is, $a \leqslant paq$ where $p \in S^i$ and $q \in S^j$. Then $a \in (S^i a S^j]$.

Conversely, assume that $a \leqslant xay$ has a solution in S. Then $a \leqslant paq$ for some p, q in S. We have $a \leqslant p^n a q^n$ for every $n \in \mathbb{N}$ (let $\mathbb{N}$ denote the set of all positive integers). Let $s = \max\{i, j\}$. Then $a \leqslant p^s a q^s \in S^i a S^j$, and hence $a \leqslant u(a, x_1, \ldots, x_n)$ has a solution in S. □

Similarly, we have the following two lemmas.

Lemma 2.2 *Let $u = u(c, x_1, \ldots, x_n) \in L$ be such that $|c|_u = 1$, $t(u) = c$. Then*

$$c \leqslant u(c, x_1, \ldots, x_n) \sim c \leqslant xc.$$

Lemma 2.3 *Let $u = u(c, x_1, \ldots, x_n) \in L$ be such that $|c|_u = 1$, $h(u) = c$. Then*

$$c \leqslant u(c, x_1, \ldots, x_n) \sim c \leqslant cx.$$

Lemma 2.4 *Let $u = u(c, x_1, \ldots, x_n) \in L$ be such that $|c|_u \geqslant 2$, $c^2 \nmid u$, $h(u) \neq c$ and $t(u) \neq c$. Then*

$$c \leqslant u(c, x_1, \ldots, x_n) \sim c \leqslant xcycz.$$

Proof. We consider an ordered semigroup $(S, \cdot, \leqslant)$ and $a \in S$. Then $a \leqslant u(a, x_1, \ldots, x_n)$ has a solution in S if and only if

$$a \in (S^{i_1} a S^{i_2} \cdots a S^{i_m}], \tag{2.1}$$

where $m \geqslant 3$ and $i_1, i_2, \ldots, i_m$ are respectively the lengths of the largest subwords of u not containing c. That is, $a \leqslant paqar$ where $p \in S^{i_1}, q \in S^{i_2} \cdots S^{i_{m-1}}, r \in S^{i_m}$.

Conversely, assume that $a \leqslant xayaz$ is solvable in S. That is, $a \leqslant paqar$ for some p, q, r in S. By induction on m, we will prove that (2.1) holds for any $m \geqslant 3$ and $i_1, \ldots, i_m \in \mathbb{N}$.

Assume that $m = 3$. By $a \leqslant paqar$, it follows that $a \leqslant p^n a (qar)^n$ for all $n \in \mathbb{N}$. Then, for $i = \max\{i_1, i_2, i_3\}$, we have

$$\begin{aligned} a &\leqslant p^{2i+1} a (qar)^{2i+1} \\ &= p^{2i+1} a (qar)^i qar (qar)^i \\ &\in S^{i_1} a S^{i_2} a S^{i_3}. \end{aligned}$$

That is, $a \in (S^{i_1} a S^{i_2} a S^{i_3}]$.

Suppose that (2.1) holds for some $m \geqslant 3$, i.e., $a \leqslant p_1 a p_2 a \cdots p_{m-1} a p_m$ for some $p_1 \in S^{i_1}, \ldots, p_m \in S^{i_m}$. Let $i_{m+1} \in \mathbb{N}$. Then, for $i = \max\{i_m, i_{m+1}\}$,

$$\begin{aligned} a &\leqslant p_1 a p_2 a \cdots p_{m-1} a p_m \\ &\leqslant p_1 a p_2 a \cdots p_{m-1} p^{2i+1} a (qar)^{2i+1} p_m \\ &= p_1 a p_2 a \cdots p_{m-1} p^{2i+1} a (qar)^i qar (qar)^i p_m \\ &\in S^{i_1} a S^{i_2} \cdots S^{i_{m-1}} a S^{i_m} a S^{i_{m+1}}. \end{aligned}$$

That is,

$$a \in (S^{i_1} a S^{i_2} \cdots S^{i_{m-1}} a S^{i_m} a S^{i_{m+1}}].$$

□

The following three lemmas can be proved similarly.

Lemma 2.5 *Let $u = u(c, x_1, \ldots, x_n) \in L$ be such that $|c|_u \geqslant 2$, $c^2 \nmid u$, $h(u) \neq c$ and $t(u) = c$. Then*

$$c \leqslant u(c, x_1, \ldots, x_n) \sim c \leqslant xcyc.$$

Lemma 2.6 *Let $u = u(c, x_1, \ldots, x_n) \in L$ be such that $|c|_u \geqslant 2$, $c^2 \nmid u$, $h(u) = c$ and $t(u) \neq c$. Then*

$$c \leqslant u(c, x_1, \ldots, x_n) \sim c \leqslant cxcy.$$

Lemma 2.7 *Let $u = u(c, x_1, \ldots, x_n) \in L$ be such that $|c|_u \geqslant 2$, $c^2 \nmid u$, $h(u) = c$ and $t(u) = c$. Then*

$$c \leqslant u(c, x_1, \ldots, x_n) \sim c \leqslant cxc.$$

Lemma 2.8 *Let $k \in \mathbb{N}$ be such that $k \geqslant 2$. Let $u = u(c, x_1, \ldots, x_n) \in L$ be such that $c^k \mid u$, $c^{k+1} \nmid u$, $h(u) \neq c$ and $t(u) \neq c$. Then*

$$c \leqslant u(c, x_1, \ldots, x_n) \sim c \leqslant xc^k y.$$

Proof. It is clear that $c \leqslant u(c, x_1, \ldots, x_n)$ implies $c \leqslant xc^k y$.

Conversely, we consider an ordered semigroup $(S, \cdot, \leqslant)$ and $a \in S$ such that the inequation $a \leqslant xa^k y$ has a solution in S, i.e., $a \leqslant pa^k q$ for some p, q in S. It suffices to show that for every $m \geqslant 2$ and $i_1, \ldots, i_m \in \mathbb{N}$ the following holds

$$a \in (S^{i_1} a^k S^{i_2} a^k \cdots S^{i_{m-1}} a^k S^{i_m}]. \tag{2.2}$$

This will be proceed by induction on m.

Case $m = 2$. By $a \leqslant pa^k q$, it follows that $a \leqslant p^n a(a^{k-1}q)^n$ for every $n \in \mathbb{N}$. Thus, for $i = \max\{i_1, i_2\}$, we have

$$\begin{aligned} a &\leqslant p^{i+1} a(a^{k-1}q)^{i+1} \\ &= p^{i+1} a^k q(a^{k-1}q)^i \\ &\in S^{i_1} a^k S^{i_2}. \end{aligned}$$

That is, $a \in (S^{i_1} a^k S^{i_2}]$.

Suppose that (2.2) holds for some $m \geqslant 2$. Then

$$a \leqslant p_1 a^k p_2 a^k \cdots p_{m-1} a^k p_m$$

for some $p_1 \in S^{i_1}, \ldots, p_m \in S^{i_m}$. Let $i_{m+1} \in \mathbb{N}$. Set $p_0 = p_1 a^k p_2 a^k \cdots p_{m-1}$. We have

$$p_0 \in S^{i_1} a^k S^{i_2} \cdots a^k S^{i_{m-1}} \text{ and } a \leqslant p_0 a^k p_m = (p_0 a^{k-1}) a p_m.$$

Then

$$a \leqslant (p_0 a^{k-1})^n a p_m^n \text{ for every } n \in \mathbb{N}.$$

Thus, for $i = \max\{i_m, i_{m+1}\}$, we have

$$\begin{aligned} a &\leqslant (p_0a^{k-1})^{i+3}ap_m^{i+3} \\ &= (p_0a^{k-1})^2(p_0a^{k-1})^i(p_0a^{k-1})ap_m^{i+3} \\ &= (p_0a^{k-1})^2(p_0a^{k-1})^ip_0a^kp_m^{i+3} \\ &\in (S^{i_1}a^kS^{i_2}a^k\cdots S^{i_{m-1}})^2S^{i_m}a^kS^{i_{m+1}} \\ &\subseteq S^{i_1}a^kS^{i_2}a^k\cdots S^{i_{m-1}}a^kS^{i_m}a^kS^{i_{m+1}}. \end{aligned}$$

That is,

$$a \in (S^{i_1}a^kS^{i_2}a^k\cdots S^{i_{m-1}}a^kS^{i_m}a^kS^{i_{m+1}}].$$

□

Lemma 2.9 *Let $k \in \mathbb{N}, k \geqslant 2$. Let $u = u(c, x_1, \ldots, x_n) \in L$ be such that $c^k \mid u$, $c^{k+1} \nmid u$, $h(u) \neq c$, $t(u) = c$ and $t^{(2)}(u) \neq c^2$. Then*

$$c \leqslant u(c, x_1, \ldots, x_n) \sim c \leqslant xc^kyc.$$

Proof. It is clear that $c \leqslant u(c, x_1, \ldots, x_n)$ implies $c \leqslant xc^kyc$.

Conversely, we consider an ordered semigroup $(S, \cdot, \leqslant)$ and $a \in S$ such that the inequation $a \leqslant xa^kya$ has a solution in S, i.e., $a \leqslant pa^kqa$ for some p, q in S. To prove that the inequation $a \leqslant u(a, x_1, \ldots, x_n)$ is satisfied in S, it suffices to show that for every $m \geqslant 2$ and $i_1, \ldots, i_m \in \mathbb{N}$ the following holds

$$a \in (S^{i_1}a^kS^{i_2}a^k\cdots S^{i_{m-1}}a^kS^{i_m}a].$$

As in the proof of Lemma 2.8 we have

$$a \in (S^{i_1}a^kS^{i_2}a^k\cdots S^{i_{m-1}}a^kS^{i_m}].$$

Thus

$$\begin{aligned} a &\leqslant pa^kqa \\ &= paa^{k-1}qa \\ &\in p(S^{i_1}a^kS^{i_2}a^k\cdots S^{i_{m-1}}a^kS^{i_m}]a^{k-1}qa \\ &\subseteq (p](S^{i_1}a^kS^{i_2}a^k\cdots S^{i_{m-1}}a^kS^{i_m}](a^{k-1}qa] \\ &\subseteq (pS^{i_1}a^kS^{i_2}a^k\cdots S^{i_{m-1}}a^kS^{i_m}a^{k-1}qa] \\ &\subseteq (S^{i_1}a^kS^{i_2}a^k\cdots S^{i_{m-1}}a^kS^{i_m}a]. \end{aligned}$$

Hence

$$a \in ((S^{i_1}a^kS^{i_2}a^k\cdots S^{i_{m-1}}a^kS^{i_m}a]] = (S^{i_1}a^kS^{i_2}a^k\cdots S^{i_{m-1}}a^kS^{i_m}a].$$

□

The following two lemmas can be proved similarly.

Lemma 2.10 *Let $k \in \mathbb{N}, k \geqslant 2$. Let $u = u(c, x_1, \ldots, x_n) \in L$ be such that $c^k \mid u$, $c^{k+1} \nmid u$, $h(u) = c$, $h^{(2)}(u) \neq c^2$ and $t(u) \neq c$. Then*

$$c \leqslant u(c, x_1, \ldots, x_n) \sim c \leqslant cxc^k y.$$

Lemma 2.11 *Let $k \in \mathbb{N}, k \geqslant 2$. Let $u = u(c, x_1, \ldots, x_n) \in L$ be such that $c^k \mid u$, $c^{k+1} \nmid u$, $h(u) = c$, $h(u) \neq c^2$, $t(u) = c$ and $t^{(2)}(u) \neq c^2$. Then*

$$c \leqslant u(c, x_1, \ldots, x_n) \sim c \leqslant cxc^k yc.$$

Lemma 2.12 *Let $u = u(c, x_1, \ldots, x_n) \in L$ be such that $h(u) \neq c$ and $t^{(2)}(u) = c^2$. Then*

$$c \leqslant u(c, x_1, \ldots, x_n) \sim c \leqslant xc^2.$$

Proof. Clearly, $c \leqslant u(c, x_1, \ldots, x_n)$ implies $c \leqslant xc^2$.

Conversely, we consider an ordered semigroup $(S, \cdot, \leqslant)$, and let $a \in S$ be such that $a \leqslant xa^2$ has a solution in S, i.e., $a \leqslant pa^2$ for some p in S. To show that the inequation $a \leqslant u(a, x_1, \ldots, x_n)$ is solvable in S, it suffices to show that

$$a \in (S^{i_1} a^{k_1} S^{i_2} a^{k_2} \cdots S^{i_m} a^{k_m}]$$

for every $m \in \mathbb{N}$ and $i_1, \ldots, i_m, k_1, \ldots, k_m \in \mathbb{N}$. By $a \leqslant pa^2$, it follows that $a \leqslant p^t a^{t+1}$ for all $t \in \mathbb{N}$. And, for $i = i_1 + i_2 + \cdots i_m + k_1 + \cdots + k_m$, we have

$$a \leqslant p^i a^{i+1} \in S^{i_1} a^{k_1} S^{i_2} a^{k_2} \cdots S^{i_m} a^{k_m}.$$

That is, $a \in (S^{i_1} a^{k_1} S^{i_2} a^{k_2} \cdots S^{i_m} a^{k_m}]$. □

Similarly, we have the following two lemmas:

Lemma 2.13 *Let $u = u(c, x_1, \ldots, x_n) \in L$ be such that $h^{(2)}(u) = c^2$ and $t(u) \neq c$. Then*

$$c \leqslant u(c, x_1, \ldots, x_n) \sim c \leqslant c^2 x.$$

Lemma 2.14 *Let $u = u(c, x_1, \ldots, x_n) \in L$ be such that $h^{(2)}(u) = c^2$ and $t^{(2)}(u) = c^2$. Then*

$$c \leqslant u(c, x_1, \ldots, x_n) \sim c \leqslant c^2 xc^2.$$

Lemma 2.15 *Let $u = u(c, x_1, \ldots, x_n) \in L$ be such that $h(u) = c$, $h^{(2)}(u) \neq c^2$ and $t^{(2)}(u) = c^2$. Then*

$$c \leqslant u(c, x_1, \ldots, x_n) \sim c \leqslant cxc^2.$$

Proof. It is easy to see that $c \leqslant u(c, x_1, \ldots, x_n)$ implies $c \leqslant cxc^2$.

Conversely, we consider an ordered semigroup $(S, \cdot, \leqslant)$ and $a \in S$ such that $a \leqslant axa^2$ is solvable in S, i.e., $a \leqslant apa^2$ for some p in S. It suffices to show that

$$a \in (aS^{i_1} a^{k_1} S^{i_2} a^{k_2} \cdots S^{i_m} a^{k_m}]$$

for every $m \in \mathbb{N}$ and $i_1, \ldots, i_m, k_1, \ldots, k_m \in \mathbb{N}$. As in the proof of Lemma 2.12 we have

$$a \in (S^{i_1} a^{k_1} S^{i_2} a^{k_2} \cdots S^{i_m} a^{k_m}].$$

Thus

$$\begin{aligned} a &\leqslant (apa)a \\ &\in apa(S^{i_1}a^{k_1}S^{i_2}a^{k_2}\cdots S^{i_m}a^{k_m}] \\ &\subseteq (apa](S^{i_1}a^{k_1}S^{i_2}a^{k_2}\cdots S^{i_m}a^{k_m}] \\ &\subseteq (apaS^{i_1}a^{k_1}S^{i_2}a^{k_2}\cdots S^{i_m}a^{k_m}] \\ &\subseteq (aS^{i_1}a^{k_1}S^{i_2}a^{k_2}\cdots S^{i_m}a^{k_m}]. \end{aligned}$$

Hence

$$a \in ((aS^{i_1}a^{k_1}S^{i_2}a^{k_2}\cdots S^{i_m}a^{k_m}]] = (aS^{i_1}a^{k_1}S^{i_2}a^{k_2}\cdots S^{i_m}a^{k_m}].$$

□

The following lemma can be proved in the same manner as Lemma 2.15.

Lemma 2.16 *Let $u = u(c, x_1, \ldots, x_n) \in L$ be such that $h^{(2)}(u) = c^2$, $t(u) = c$ and $t^{(2)}(u) \neq c^2$. Then*

$$c \leqslant u(c, x_1, \ldots, x_n) \sim c \leqslant c^2xc.$$

We now state the main result of the paper:

Theorem 2.17 *Let $u(c, x_1, \ldots, x_n) \in L$. Then the inequation $c \leqslant u(c, x_1, \ldots, x_n)$ is equivalent to one of the following inequations:*

(1) $c \leqslant xcy$;
(2) $c \leqslant xc$;
(3) $c \leqslant cx$;
(4) $c \leqslant xcycz$;
(5) $c \leqslant xcyc$;
(6) $c \leqslant cxcy$;
(7) $c \leqslant cxc$;
(8) $c \leqslant xc^ky$, *for some* $k \in \mathbb{N}$;
(9) $c \leqslant xc^kyc$, *for some* $k \in \mathbb{N}$;
(10) $c \leqslant cxc^ky$, *for some* $k \in \mathbb{N}$;
(11) $c \leqslant cxc^kyc$, *for some* $k \in \mathbb{N}$;
(12) $c \leqslant xc^2$;
(13) $c \leqslant c^2x$;
(14) $c \leqslant c^2xc^2$;
(15) $c \leqslant cxc^2$;
(16) $c \leqslant c^2xc$.

Proof. The proof of the assertion follows by Lemmas 2.1 - 2.16. □

3 Regularity Conditions

Given an inequation $c \leqslant u(c, x_1, \ldots, x_n)$, the *regularity condition* determined by this inequation means an expression of the form

$$\langle c \leqslant u(c, x_1, \ldots, x_n) \rangle .$$

And, we say that an ordered semigroup $(S, \cdot, \leqslant)$ *satisfies the regularity condition* $\langle c \leqslant u(c, x_1, \ldots, x_n) \rangle$ if the inequation $a \leqslant u(a, x_1, \ldots, x_n)$ is solvable in S for all a in S.

Now, for any two regularity conditions

$$\langle c \leqslant u(c, x_1, \ldots, x_n) \rangle , \langle c \leqslant v(c, y_1, \ldots, y_m) \rangle,$$

we say that $\langle c \leqslant u(c, x_1, \ldots, x_n) \rangle$ *implies* $\langle c \leqslant v(c, y_1, \ldots, y_m) \rangle$, denoted by

$$\langle c \leqslant u(c, x_1, \ldots, x_n) \rangle \to \langle c \leqslant v(c, y_1, \ldots, y_m) \rangle ,$$

if for all ordered semigroups $(S, \cdot, \leqslant)$,

$$S \text{ satisfies } \langle c \leqslant u(c, x_1, \ldots, x_n) \rangle \Rightarrow S \text{ satisfies } \langle c \leqslant v(c, y_1, \ldots, y_m) \rangle .$$

And, these two regularity conditions are *equivalent*, denoted by

$$\langle c \leqslant u(c, x_1, \ldots, x_n) \rangle \sim \langle c \leqslant v(c, y_1, \ldots, y_m) \rangle ,$$

if for all ordered semigroups $(S, \cdot, \leqslant)$,

$$S \text{ satisfies } \langle c \leqslant u(c, x_1, \ldots, x_n) \rangle \Leftrightarrow S \text{ satisfies } \langle c \leqslant v(c, y_1, \ldots, y_m) \rangle .$$

For an arbitrary $u(c, x_1, \ldots, x_n) \in L$, using Theorem 2.17, we obtain that the regularity condition $\langle c \leqslant u(c, x_1, \ldots, x_n) \rangle$ is equivalent to one of the following conditions:

(1) $\langle c \leqslant xcy \rangle$;
(2) $\langle c \leqslant xc \rangle$;
(3) $\langle c \leqslant cx \rangle$;
(4) $\langle c \leqslant xcycz \rangle$;
(5) $\langle c \leqslant xcyc \rangle$;
(6) $\langle c \leqslant cxcy \rangle$;
(7) $\langle c \leqslant cxc \rangle$;
(8) $\left\langle c \leqslant xc^k y \right\rangle$, for some $k \in \mathbb{N}$;
(9) $\left\langle c \leqslant xc^k yc \right\rangle$, for some $k \in \mathbb{N}$;
(10) $\left\langle c \leqslant cxc^k y \right\rangle$, for some $k \in \mathbb{N}$;
(11) $\left\langle c \leqslant cxc^k yc \right\rangle$, for some $k \in \mathbb{N}$;
(12) $\left\langle c \leqslant xc^2 \right\rangle$;
(13) $\left\langle c \leqslant c^2 x \right\rangle$;
(14) $\left\langle c \leqslant c^2 xc^2 \right\rangle$;
(15) $\left\langle c \leqslant cxc^2 \right\rangle$;
(16) $\left\langle c \leqslant c^2 xc \right\rangle$.

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