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Regular endomorphism rings and principally injective modules

Truong Cong Quynh · Dinh Duc Tai

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Abstract A submodule N of M is called *fully invariant* if $f(N) \leq N$ for all $f \in \mathrm{End}(M)$. In this paper, we consider a generalization of fully invariant submodules and its applications. In particular, regularity of endomorphism ring of finitely generated quasi projective modules are studied. Let M be a quasi-projective module and $S = \mathrm{End}(M)$. We show that S is regular if and only if every right R-module is M-gp-injective.

Keywords M-gp-injective module · regular ring · π-fully invariant submodule

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1 Introduction

Regular and injective rings play an important role in ring and module theory. Recently, some authors studied some properties of rings and modules via regularity and injectivity of modules, homomorphisms. In particular, authors considered regularity of endomorphism rings of generalizations principally injective modules ([4], [9], [10], [8], [11]).

A ring R is called *regular* (*strongly regular*) if for any $a \in R$, there exists $b \in R$ such that $a = aba$ ($a = a^2b$ or $a = ba^2$). Note that for a module M, then $\mathrm{End}(M)$ is regular if and only if $\mathrm{Im}(f)$ and $\mathrm{Ker}(f)$ are direct summands of M for all $f \in \mathrm{End}(M)$.

Truong Cong Quynh
Department for Management of Science and Technology Development ,
Ton Duc Thang University,
Ho Chi Minh City, Vietnam,
and Faculty of Mathematics and Statistics,
Ton Duc Thang University,
Ho Chi Minh City, Vietnam,
E-mail: truongcongquynh@tdt.edu.vn

Dinh Duc Tai
Faculty of Mathematics,
Vinh University,
182 Le Duan street, Vinh City,
Nghe An Province, Vietnam,
E-mail: taidd@vinhuni.edu.vn

A right R-module N is called *M-general principally injective* (briefly, *M-gp-injective*) if, for any non-zero endomorphism ε of M, there exists $n \in \mathbb{N}$ such that $\varepsilon^n \neq 0$ and every homomorphism from $\varepsilon^n(M)$ to N can be extended to a homomorphism from M to N (see [13], [15]). If N is R_R-gp-injective, then N is called GP-injective ([8], [18]). Some generalizations of them were studied by the authors in [11], [14].

Using the concepts of general principally injective modules, we will study some characterizations of regular modules and their endomorphism ring. From the characterizations, some well-known results over ring are obtained as corollaries.

In Section 2, we study regularity of endomorphism ring with π-fully invariant condition. A submodule N of M is called π-fully invariant if for each $s \in S$ with $0 \neq s(M) \leq N$, there exists $n > 0$ such that $Ss^n(M) \leq N$ with $S = \mathrm{End}(M)$. It is well-known that a ring R is regular if and only if every right module is GP-injective see [18, Theorem 9] or [5, Corollary 2.4]. For modules, we also prove a similar result. Let M be a quasi-projective module and $S = \mathrm{End}(M)$. Then S is regular if and only if every right R-module is M-gp-injective (Theorem 2.8). Next we will give some regular results of module and its endomorphism ring. Let M be a finitely generated quasi-projective module with $S = \mathrm{End}(M)$. If any maximal submodule of M is π-fully invariant, then S is a strongly regular ring if and only if S is a right weakly regular ring (Theorem 2.6). Furthermore, we prove that M is endo-strongly regular if and only if every maximal submodule of M is π-fully invariant and every simple factor module of M is M-gp-injective (Theorem 2.11).

In Section 3, we study some properties of modules M which every simple singular factor module is M-gp-injective. Let M be a finitely generated quasi-projective module with $S = \mathrm{End}(M)$ and every simple singular factor module M is M-gp-injective. By Kasch condition, we give some characterizations of semisimple modules (Theorem 3.4). In addition, we also prove that if every maximal submodule of M is π-fully invariant, then $W(S) = 0$ (Theorem 3.7) where $S = \mathrm{End}(M)$ and $W(S) = \{s \in S \mid \mathrm{Ker}(s) \text{ is essential in } M\}$. Moreover, we indicated that if M is a ZI-module, then S is reduced and S is weakly regular (Theorem 3.11).

Throughout this article all rings are associative rings with unit and all modules are right unital modules over a ring. For a submodule N of M, we use $N \leq M$ ($N < M$) to mean that N is a submodule of M (resp., proper submodule), and we write $N \leq^e M$ to indicate that N is an essential submodule of M. We write $J(R), J(M), Soc(R), Soc(M)$ for the Jacobson radical of the ring R, for the radical of the module M, the socle of R and the socle of M, respectively. If X is a subset of R, the left (resp., right) annihilator of X in R is denoted by $l_R(X)$ (resp., $r_R(X)$) or simply $l(X)$ (resp., $r(X)$) if no confusion appears, with $l_R(X) = \{r \in R \mid rx = 0 \text{ for all } x \in X\}$ and $r_R(X) = \{r \in R \mid xr = 0 \text{ for all } x \in X\}$. The (right) singular ideal of the ring R is defined as $Z(R_R) = \{x \in R \mid xI = 0 \text{ for some essential right ideal } I \text{ of } R\}$. It is well known that $Z(R_R)$ is indeed an ideal of R. A right R-module N is called M-generated if there exists an epimorphism $M^{(I)} \to N$ for some index set I. In particular, N is called M-cyclic if it is isomorphic to M/L for some submodule L of M. Hence, any M-cyclic submodule X of M can be considered as the image of an endomorphism of M. For any term not defined here the reader is referred to [2] and [17].

2 On π-fully invariant submodules and its applications

A submodule N of M is called fully invariant if $f(N) \leq N$ for all $f \in \mathrm{End}(M)$. Now we consider the following generalized concept: let N be a submodule of M and $S = \mathrm{End}(M)$. N is called *π-fully invariant* submodule of M if for each $s \in S$ with $0 \neq s(M) \leq N$, there exists $n > 0$ such that $0 \neq Ss^n(M) \leq N$.

It is easy to see that a fully invariant submodule is π-fully invariant. But the converse is not true in general. Let $R = \{\begin{pmatrix} x & x_1 & x_2 & x_3 \\ 0 & x & x_4 & x_5 \\ 0 & 0 & x & x_6 \\ 0 & 0 & 0 & x \end{pmatrix} \mid x, x_i \in \mathbb{Z}_2\}$ be a ring and $I = \{\begin{pmatrix} 0 & x_1 & x_2 & x_3 \\ 0 & 0 & x_1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \mid x_i \in \mathbb{Z}_2\}$. Then I is a π-fully invariant submodule of ${}_RR$ but not a fully invariant submodule of ${}_RR$ by [19, Example 1.2].

A module M is called *abelian* if, for any $m \in M$, $a \in R$, and idempotent $e \in R$, then $mae = mea$. A ring R is called abelian if R_R is an abelian module (see [3]).

Proposition 2.1 *The following conditions are equivalent for a module M with $S = \mathrm{End}(M)$:*

1. *S is abelian.*
2. *M is abelian as a left S-module.*
3. *$\mathrm{Ker}(e)$ is a π-fully invariant submodule of M for every $e^2 = e \in S$.*
4. *$l_S(e)$ is a π-fully invariant left ideal of S for every $e^2 = e \in S$.*

Proof. (1) $\Rightarrow$ (2). For any $m \in M$, $s \in S$ and idempotent $e \in S$, we have $es = se$ (since S is abelian). It follows that $esm = sem$ and so M is abelian as a left S-module.

(2) $\Rightarrow$ (1). For any $s \in S$ and idempotent $e \in S$, we have $se(m) = es(m)$ for all $m \in M$ (M is abelian as a left S-module). It follows that $es = es$. Thus S is abelian.

(1) $\Rightarrow$ (3). Let $e^2 = e \in S$ and $s \in S$ with $es = 0$. For all $u \in S$, we have $eus = ues = 0$. It follows that $Ss(M) \leq \mathrm{Ker}(e)$. Thus $\mathrm{Ker}(e)$ is a π-fully invariant submodule of M.

(3) $\Rightarrow$ (1). Let $e^2 = e \in S$ and $s \in S$. Since $(1-e)(M) \leq \mathrm{Ker}(e)$ and $\mathrm{Ker}(e)$ is a π-fully invariant submodule of M, $s(1-e)(M) \leq \mathrm{Ker}(e)$. Then $es = ese$. On the other hand, $\mathrm{Ker}(1-e)$ is also a π-fully invariant submodule of M and $e(M) \leq \mathrm{Ker}(1-e)$, then $se(M) \leq \mathrm{Ker}(1-e)$. It follows that $se = ese$. Thus $es = se$ or S is abelian.

Similarly (1) $\Leftrightarrow$ (4). □

It is well-known a ring R is strongly regular if and only if R is an abelian regular ring by [6, Theorem 3.5]. Then we get the following corollary:

Corollary 2.2 *The following conditions are equivalent for a module M with $S = \mathrm{End}(M)$:*

1. *S is strongly regular.*
2. *S is regular and $\mathrm{Ker}(e)$ is a π-fully invariant submodule of M for every $e^2 = e \in S$.*
3. *S is regular and $l_S(e)$ is a π-fully invariant left ideal of S for every $e^2 = e \in S$.*

Proposition 2.3 *Let M be a quasi-projective module and $I \leq L \leq M$. If L is a π-fully invariant submodule of M, then L/I is a π-fully invariant submodule of M/I.*

Proof. Let $S = \mathrm{End}(M)$ and $\bar{S} = \mathrm{End}(M/I)$. Let $\bar{f} : M/I \to M/I$ be a R-homomorphism such that $\bar{f}(M/I) \leq L/I$. Since M is quasi-projective, there exists $f \in S$ such that $\bar{f}\pi = \pi f$ where $\pi : M \to M/I$ the canonical projection. Then $f(M) \leq L$. Since L is a π-fully invariant submodule of M, there exists $n > 0$ such that $Sf^n(M) \leq L$. It is easy to check $\bar{f}^n(m + I) = f^n(m) + I$ for all $m \in M$. For any $\bar{g} \in \bar{S}$, then $\bar{g}\pi = \pi g$ for some $g \in S$ by projectivity of M. We have

$$\bar{g}\bar{f}^n(m+I) = \bar{g}(f^n(m)+I) = \bar{g}\pi(f^n(m)) = \pi g(f^n(m)) = g(f^n(m)) + I \in L/I$$

Thus L/I is a π-fully invariant submodule of M/I. $\square$

Recall that a ring R is *reduced* if it has no non-zero nilpotent elements.

Lemma 2.4 *Let M be a finitely generated quasi-projective module. If for any maximal submodule of M is π-fully invariant, then $S/J(S)$ is reduced.*

Proof. Assume that there is $s \notin J(S)$ such that $s^2 \in J(S)$. Since M is quasi-projective and $s \notin J(S)$, there exists $N_0 < M$ such that $s(M) + N_0 = M$. Let N be a maximal submodule of M such that $N_0 \leq N$. Then $s(M) \nleq N$ and $s^2(M) \leq N$ (since $s^2(M) \ll M$). We have $N + s(M) = M$, whence $s(N) + s^2(M) = s(M)$. Since M is finitely generated quasi-projective and by [17, 18.4], then

$$\begin{aligned} sS &= \mathrm{Hom}(M, sM) = \mathrm{Hom}(M, s(N)) + \mathrm{Hom}(M, s^2(M)) \\ &= \mathrm{Hom}(M, s(N)) + s^2 S. \end{aligned}$$

It follows that $s = u + s^2 v$ for some $u, v \in S$ with $u(M) \leq s(N)$. We consider the following diagram

$$\begin{tikzcd} & M \arrow[dl, dotted, "g"'] \arrow[d, "u"] & \\ N \arrow[r, "s"] & s(N) \arrow[r] & 0 \end{tikzcd}$$

There exists $g \in S$ such that $g(M) \leq N$ and $u = sg$ (by projectivity of M). Therefore $s = sg + s^2 v$. Since N is π-fully invariant, there exists some positive integer n such that $Sg^n(M) \leq N$. Now we have:

$$sg^{n-1}(M) = (sg + s^2 v)g^{n-1}(M) = [sg^n + s^2 vg^{n-1}](M) \leq N$$

and

$$sg^{n-2}(M) = (sg + s^2 v)g^{n-2}(M) = [sg^{n-1} + s^2 vg^{n-2}](M) \leq N.$$

Continuing in this process, we get $s(M) \leq N$, a contradiction. Thus $S/J(S)$ is reduced. $\square$

Recall that a module M is *semiprimitive* if the Jacobson radical of M is zero (see [7]).

Corollary 2.5 *Let M be a finitely generated, quasi-projective module. If M is a semiprimitive module and for any maximal submodule of M is π-fully invariant, then S is reduced.*

Proof. By Lemma 2.4 and [17, 22.2]. □

Recall that a ring R is *right weakly regular* if $I^2 = I$ for each right ideal I of R; equivalently, $a \in aRaR$ for every $a \in R$ (see [8]).

Theorem 2.6 *Let M be a finitely generated, quasi-projective module and $S = \mathrm{End}(M)$. Assume that for any maximal submodule of M is π-fully invariant, then the following conditions are equivalent:*

1. *S is a strongly regular ring.*
2. *S is a right weakly regular ring.*

Proof. (1) $\Rightarrow$ (2) is obvious.

(2) $\Rightarrow$ (1) Assume that there exists $s \in S$ such that $s(M) + \mathrm{Ker}(s) \neq M$. Let N be a maximal submodule of M containing $s(M) + \mathrm{Ker}(s)$. As S is right weakly regular, $sS = sSsS$. Then $s = \sum st_i su_i$ for some $t_i, u_i \in S$. It follows that $\mathrm{Im}(1_M - \sum t_i su_i) \leq \mathrm{Ker}(s) \leq N$. If $(\sum t_i su_i)(M) \neq N$ then $t_k su_k(M) \neq N$ for some k and hence $t_k su_k(M) + N = M$. Since M is finitely generated quasi-projective, $S = \mathrm{Hom}(M, t_k su_k(M)) + \mathrm{Hom}(M, N)$ or $S = t_k su_k S + \mathrm{Hom}(M, N)$. Therefore $1_M = t_k su_k u + t$ for some $u, t \in S$ with $t(M) \leq N$. On the other hand, $su_k ut_k(M) \leq N$ and N is π-fully invariant, there exists n such that $S(su_k ut_k)^n(M) \leq N$. We have

$$(1_M - t)^{n+1} = (t_k su_k u)^{n+1} = t_k(su_k ut_k)^n su_k u.$$

Then $(1_M - t)^{n+1}(M) \leq N$ and $(1_M - (1_M - t)^{n+1})(M) \leq t(M) \leq N$. We deduce that $M = N$, a contradiction. It follows that $(\sum t_i su_i)(M) = N$ and so $M = N$, a contradiction. This shows that $s(M) + \mathrm{Ker}(s) = M$ for all $s \in S$. Thus S is strongly regular. □

Proposition 2.7 *Let M be a quasi-projective module and I, a fully invariant submodule of M. If any simple factor module of M is M-gp-injective, then any simple factor module of M/I is M/I-gp-injective.*

Proof. Let $\bar{M} = M/I$, $\bar{S} = \mathrm{End}(\bar{M})$ and M/L be a simple R-module with $I \leq L$ (and $M/L \simeq (M/I)/(L/I)$). Since any simple factor module of M is M-gp-injective, M/L is M-gp-injective. Let $0 \neq \bar{s} \in \bar{S}$. Since M is quasi-projective, there exists $0 \neq s \in S$ such that $\pi s = \bar{s}\pi$ with $\pi : M \to M/I$ the canonical projection. There exists $n > 0$ such that $s^n \neq 0$ and any homomorphism $s^n(M) \to M/L$ can be extended to M. Assume that $\alpha : \bar{s}^n(\bar{M}) \to M/L$ is a homomphism. Let $\phi : s^n(M) \to \bar{s}^n(\bar{M})$ be a homomorphism via $\phi(s^n(m)) = \bar{s}^n(\bar{m})$ for all $m \in M$. Then there exists $h : M \to M/L$ which is extension of $\alpha\phi$. On the other hand, there exists $t \in S$ such that $\pi_L t = h$ where $\pi_L : M \to M/L$ the canonical projection (by quasi-projectivity of M). Since I is a fully invariant submodule of M, $t(I) \leq I$ and so $I \leq \mathrm{Ker}(h)$. There is a homomophism $\bar{h} : \bar{M} \to M/L$ such that $\bar{h}\pi = h$. It is easy to see that $\bar{h}$ is an extension of α. Thus M/L is $\bar{M}$-gp-injective. □

Theorem 2.8 *Let M be a quasi-projective module and $S = \mathrm{End}(M)$. The following are equivalent:*

1. *S is regular.*
2. *Every right R-module is M-gp-injective.*

Proof. (1) $\Rightarrow$ (2). Let $s \in S, s \neq 0$. Since s is regular, then $s(M)$ is a direct summand of M. Thus every homomorphism from $s(M)$ to a right R-module can be extended to M.

(2) $\Rightarrow$ (1). Assume that $s \in S$, $s \neq 0$. By (2), there exists $n > 0$ such that $s^n \neq 0$ and $M/\mathrm{Ker}(s^n)$ is M-gp-injective. Let $\phi : s^n(M) \to M/\mathrm{Ker}(s^n)$ is the canonical isomorphism. There exists a homomorphism $g : M \to M/\mathrm{Ker}(s^n)$ such that $gs^n = \phi s^n$. Moreover, since M is quasi-projective, it is $s^n(M)$-projective. Let $t \in S$ such that $t(M) \leq s^n(M)$ and $g = \phi t$. It follows that $\phi ts^n = \phi s^n$ and so $ts^n = s^n$. On the other hand, there exists $u \in S$ such that $t = s^n u$ (since M is quasi-projective). Thus $s^n = s^n us^n$. If $n = 1$, then s is a regular element. Now $n \geq 2$ and let $s_1 = s^{n-1} - s^{n-1}us^n$. Then $s_1^2 = 0$. Repeating the process above, we get s_1 is regular. It follows that s^{n-1} is also regular by [16, Lemma 2.1]. We continue this process, then s is regular. Thus S is regular. □

Corollary 2.9 ([18, Theorem 9]) *The following are equivalent for a ring R:*

1. *R is regular.*
2. *Every right R-module is GP-injective.*

It is well-known a ring R is strongly regular if and only if for any $a \in R$, $R = aR + r(a)$, where $r(a)$ is the right annihilator of $\{a\}$ in R. From this, we call a module M *endo-strongly regular* if, for any $s \in \mathrm{End}(M)$, $M = \mathrm{Im}(s) + \mathrm{Ker}(s)$ or equivalently, for any $s \in \mathrm{End}(M)$, $s(M) = s^2(M)$. We get the following result:

Proposition 2.10 *Let M be a quasi-projective module and $S = \mathrm{End}(M)$. The following conditions are equivalent:*

1. *S is strongly regular.*
2. *M is endo-strongly regular.*
3. *For any $s \in S$, $\mathrm{Im}(s) = \mathrm{Im}(e)$ for some a central idempotent $e \in S$.*
4. *S is regular and every M-cyclic submodule of M is π-fully invariant.*

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) are obvious.

(4) $\Rightarrow$ (1) Let $s \in \mathrm{End}(M) = S$. There exists $t \in S$ such that $s = sts$. By the hypothesis and $(ts)^2 = ts$, then $Sts(M) \leq ts(M)$. It follows that $sts(M) \leq ts(M)$ and so $sts = tsu$ for some $u \in S$ by quasi-projectivity of M. Thus $s = sts = tsu = t(sts)u = ts(tsu) = ts^2$. □

Theorem 2.11 *The following conditions are equivalent for a finitely generated quasi-projective module M:*

1. *M is endo-strongly regular.*
2. *Every maximal submodule of M is π-fully invariant and every simple factor module of M is M-gp-injective.*

Proof. (1) $\Rightarrow$ (2) is obvious.

(2) $\Rightarrow$ (1) Assume first that $0 \neq s \in S$ with $s^2 = 0$. Let N be maximal submodule of M containing $\mathrm{Ker}(s)$ (since M is finitely generated). Therefore M/N is a simple module and so by the hypothesis it is M-gp-injective. Define $f : s(M) \to M/N$ via $f(s(m)) = m + N$. It is easy to see that f is a homomorphism. We consider the following diagram

$$\begin{tikzcd} & M/N & \\ 0 \arrow[r] & s(M) \arrow[u, "f"] \arrow[r] & M \arrow[ul, dotted, "g"'] \end{tikzcd}$$

Since M/N is M-gp-injective, there exists $g : M \to M/N$ such that $g(s(m)) = f(s(m))$ for all $m \in M$. On the other hand, $s(M) \leq N$ and N is π-fully invariant, $Ss(M) \leq N$. Moreover, there exists $\varphi \in S$ such that $p\varphi = g$ where $p : M \to M/N$ the canonical projection (since M is quasi-projective). It follows that $\varphi s(M) \leq N$ or $p\varphi s(M) = 0$ and so $gs(M) = 0$. For any $m \in M$, we have

$$m + N = f(s(m)) = g(s(m)) = 0, \text{ or } \ m \in N.$$

That means $N = M$, a contradiction. This shows that for all $s \in S$, $s^2 = 0$ if and only if $s = 0$. Thus

$$s^m \neq 0 \text{ for all } m \in \mathbb{N} \text{ and } s \in S, s \neq 0 \quad (*).$$

Let $s \in S, s \neq 0$. Assume that there exists $m \in \mathbb{N}$ such that $M = \mathrm{Ker}(s^m) + s(M)$. Then $s^m(M) = s^{m+1}(M)$ and so $s^m = s^{m+1}t$ for some $t \in S$ (by quasi-projectivity of M). By (*), we have $r_S(s^m) = r_S(s)$ and so $1_M - st \in r_S(s^m) = r_S(s)$ or $s = s^2t$.

Otherwise, let $A = \sum_{k\in\mathbb{N}} [\mathrm{Ker}(s^k) + s(M)]$. Since M is finitely generated, $A \neq M$. Let N be a maximal submodule of M containing A (since M is finitely generated). Therefore M/N is a simple module and N is π-fully invariant. There is $n > 0$ such that $Ss^n(M) \leq N$ and $s^n \neq 0$ (by (*)). On the other hand, by the hypothesis M/N is M-gp-injective, there exists $m > 0$ such that $s^{nm} \neq 0$ and every homomorphism from $s^{nm}(M)$ to M/N can be extended to a homomorphism from M to M/N. Define $f : s^{nm}(M) \to M/N$ via $f(s^{nm}(x)) = x + N$. It is easy to see that f is a homomorphism. We consider the following diagram

$$\begin{tikzcd} & M/N & \\ 0 \arrow[r] & s^{nm}(M) \arrow[u, "f"] \arrow[r] & M \arrow[ul, dotted, "g"'] \end{tikzcd}$$

There exists $g : M \to M/N$ such that $g(s^{nm}(x)) = f(s^{nm}(x))$ for all $x \in M$. On the other hand, there exists $\varphi \in S$ such that $p\varphi = g$ where $p : M \to M/N$ the canonical projection (since M is quasi-projective). We have

$$\varphi s^{nm}(M) = \varphi s^{n(m-1)} s^n(M) \leq N.$$

It follows that $p\varphi s^{nm}(M) = 0$ and so $gs^{nm}(M) = 0$. Therefore $N = M$, a contradiction. $\square$

A ring R is said to be right *quasi-duo* if every maximal right ideal is an ideal (see [10]).

Corollary 2.12 ([10]) *The following conditions are equivalent.*

1. *R is strongly regular.*
2. *R is a right quasi-duo ring whose simple right modules are GP-injective.*

Lemma 2.13 *Let ${}_SM$ be an abelian module, $S = \mathrm{End}(M)$ and $s \in S$. If N is a maximal submodule of M such that $\mathrm{Ker}(s) + \mathrm{Im}(s) \leq N$, then N is essential in M.*

Proof. If N is not essential in M, then N is a direct summand of M (by maximality of N). Let $0 \neq e^2 = e \in S$ and $N = \mathrm{Ker}(e)$. Since $\mathrm{Im}(s) \leq N$, $es = 0$. It follows that $se = 0$. Therefore $e(M) \leq \mathrm{Ker}(s) \leq N = \mathrm{Ker}(e)$ and so $e = 0$, a contradiction. Thus N is essential in M. $\square$

Corollary 2.14 *Let M be a finitely generated quasi-projective module. If ${}_SM$ is an abelian module, then the following are equivalent:*

1. *M is endo-strongly regular.*
2. *Every maximal submodule of M is π-fully invariant and every singular simple factor module of M is M-gp-injective.*

Proof. By Lemma 2.13 and proving of Theorem 2.11. $\square$

3 On simple singular factor modules

For convenience, we always assume that M is a finitely generated quasi-projective right R-module and $S = \mathrm{End}(M)$ for all the following results in this section.

Let M be a module. We denote $W(S) = \{s \in S | \ \mathrm{Ker}(s) \leq^e M\}$. Then $W(S)$ is a two-sided ideal of S (see [17]).

Lemma 3.1 *Assume that every simple singular factor module of M is M-gp-injective. Then for any $s \in S$, there exist $n > 0$ and $K \leq M$ such that $M = [\mathrm{Ker}(s^k) + SsM] \oplus K$.*

Proof. Firstly we suppose that s is nilpotent, then there exists a positive integer m such that $s^m \neq 0$ and $s^{m+1} = 0$. Call $K \leq M$ with $L = [SsM + \mathrm{Ker}(s^m)] \oplus K$ essential in M. If $L \neq M$, there exists a maximal submodule N of M containing L. Hence $N \leq^e M$ and M/N is M-gp-injective by hypothesis. We have $(s^m)^2 = 0$, so any R-homomorphism of $s^m(M)$ into M/N extends to M. It is easy to see that the map $f : s^m(M) \to M/N$ via $f(s^m(x)) = x + N$ is a homomorphism. We consider the following diagram

$$\begin{tikzcd}
& M/N & \\
0 \arrow[r] & s^m(M) \arrow[u, "f"] \arrow[r] & M \arrow[ul, dotted, "g"']
\end{tikzcd}$$

There exists $g : M \to M/N$ such that $g(s^m(x)) = f(s^m(x))$ for all $x \in M$. On the other hand, there exists $\varphi \in S$ such that $p\varphi = g$ where $p : M \to M/N$ the canonical projection (since M is quasi-projective). Since $\varphi(s^n(M)) \leq N$, $p\varphi s^m(M) = 0$ and so $gs^m(M) = 0$. For all $x \in M$, we have

$$x + N = f(s^m(x)) = g(s^m(x)) = 0, \text{ or } \ x \in N.$$

That means $N = M$, a contradiction.

It remains to show that the case when s is not nilpotent (and so $s \neq 0$). Let $A = \sum_{k\in\mathbb{N}} [\mathrm{Ker}(s^k) + SsM]$. Call $K \leq M$ with $A \oplus K \leq^e M$. Assume that $A \neq M$. Repeating the proving of Theorem 2.11, we get a decomposition $M = A \oplus K$. Since M is finitely generated and $\mathrm{Ker}(s^k) + SsM \leq \mathrm{Ker}(s^{k+1}) + SsM$ for all $k \in \mathbb{N}$, there exists $n > 0$ such that $M = [\mathrm{Ker}(s^n) + SsM] \oplus K$. $\square$

Lemma 3.2 *If every simple singular factor module of M is M-gp-injective, then* $W(S) \cap J(S) = 0$

Proof. Assume that $W(S) \cap J(S) \neq 0$. Let $0 \neq z \in W(S) \cap J(S)$. Then $\mathrm{Ker}(z) \leq^e M$. We first show that there exists $c \in S$ such that $0 \neq zc(M) \leq \mathrm{Ker}(z)$. In fact, by Lemma 3.1, there exists $n > 0$ such that $M = \mathrm{Ker}(z^n) + SzM$. It follows that $z^nM = z^nSsM$. Since M is finitely generated, quasi-projective, there exists $v \in SzS \leq J(S)$ such that $z^n = z^nv$ or $z^n(1_M - v) = 0$. Note that $0 \neq v \in J(S)$, we get $z^n = 0$. From this, we can choose the integer $k \geq 2$ such that $z^{k-1} \neq 0$ and $z^k = 0$. It follows that $0 \neq z^{k-1}(M) \leq \mathrm{Ker}(z)$. Thus there exists $c \in S$ such that $0 \neq zc(M) \leq \mathrm{Ker}(z)$. We have $(zcz)^2 = zczzcz = zc(z^2c)z = 0$. By Lemma 3.1, there exists $K \leq M$ such that $R = [SzczM + \mathrm{Ker}(zcz)] \oplus K$ which implies $K = 0$ (since $\mathrm{Ker}(zcz) \leq^e M$) and so $M = SzczM + \mathrm{Ker}(zcz)$. It follows that $zcz(M) = zcz(SzczM)$. Since M is finitely generated quasi-projective, there exists $a \in SzczS \leq J(S)$ such that $zcz = zcza$. Then $zcz(1-a) = 0$ and so $zcz = 0$ (since $a \in J(S)$). Therefore $(zc)^2 = 0$. We continue this process, we have $zc = 0$, a contradiction. $\square$

Lemma 3.3 *A simple submodule N of M is either a direct summand of M or* $\mathrm{Hom}(M,N)N = 0$.

Proof. Let N be a simple submodule of M. Assume that $\mathrm{Hom}(M,N)N \neq 0$. It follows that $\sum\{f(N)|\ f \in \mathrm{Hom}(M,N)\} \neq 0$. There exists a homomorphism $f : M \to N$ such that $f(N) \neq 0$. Since N is simple, $f(N) = N$, whence $N = f(N) = f(M)$ and so $M = N + \mathrm{Ker}(f)$. On the other hand, we have $N \cap \mathrm{Ker}(f) = \mathrm{Ker}(f|_N)$ and $f|_N : N \to N$ is an isomorphism (since N is simple). Then $N \cap \mathrm{Ker}(f) = 0$ and so $M = N \oplus \mathrm{Ker}(f)$. $\square$

Recall that a module M is *Kasch* if it contains a copy of every simple module in $\sigma[M]$, where $\sigma[M]$ denotes the full subcategory of Mod-R whose objects are submodules of M-generated modules (see [1]).

Theorem 3.4 *The following conditions are equivalent for module M:*

1. *M is semisimple.*
2. *M is a Kasch module and every simple singular factor module of M is M-gp-injective.*
3. *M is a Kasch module and $W(S) = 0$.*
4. *M is a Kasch module and S is semiprime.*

Proof. $(1) \Rightarrow (2), (3), (4)$ are obvious.

$(2) \Rightarrow (1)$. We need show that M has no maximal essential submodule. In fact, assume that N is a maximal essential submodule of M. Since M is a Kasch module, there exists a monomorphism $\varphi : M/N \longrightarrow M$. Define $s : M \longrightarrow M$ via $s(m) = \varphi(m + N)$. Then $0 \neq s \in S$ and $N = \mathrm{Ker}(s)$. But N is essential in M, and so $s \in W(S)$. Note that $s(M) \simeq M/\mathrm{Ker}(s)$ is simple. By Lemma 3.3, either $s(M)$ a direct summand of M or $\mathrm{Hom}(M, s(M))s(M) = 0$.

Assume that $s(M)$ is a direct summand of M. Then $s(M)$ is M-projective. It follows that $M/\mathrm{Ker}(s)$ is also M-projective. Since $\mathrm{Ker}(s)$ is essential in M, $\mathrm{Ker}(s) = M$, a contradiction. Thus we have $\mathrm{Hom}(M, s(M))s(M) = 0$. Moreover, $sSsS(M) \leq \mathrm{Hom}(M, s(M))s(M) = 0$ and so $(sS)^2 = 0$ or $s \in J(S)$. It follows that $s \in W(S) \cap J(S) = 0$ by Lemma 3.2, a contradiction.

$(4) \Rightarrow (3)$. Assume that $W(S) \neq 0$. Let $0 \neq s \in S$ and $K \leq M$ such that $s(M) \oplus K \leq^e M$. If $s(M) \oplus K = M$, then $s(M)$ is M-projective. But $\mathrm{Ker}(s) \leq^e$ and so $\mathrm{Ker}(s) = M$, a contradiction. Thus $s(M) \oplus K \neq M$. Let N be a maximal submodule of M containing $s(M) \oplus K$. Then $N = \mathrm{Ker}(u)$ and $u \in W(S)$ for some $0 \neq u \in S$. Similarly as proving of $(2) \Rightarrow (1)$, we get $(uS)^2 = 0$ and so $u = 0$ (since S is semiprime), this is a contradiction. Thus $W(S) = 0$.

$(3) \Rightarrow (1)$. Let N be a maximal essential submodule of M. Then by proving of $(2) \Rightarrow (1)$, we get $N = \mathrm{Ker}(s)$ for some $s \in W(S) = 0$. Thus $N = \mathrm{Ker}(s) = M$, a contradiction. Thus M is semisimple. □

Corollary 3.5 ([4, Theorem 3.9]) *The following conditions are equivalent:*

1. *R is a semisimple Artinian ring.*
2. *R is right Kasch and every simple singular module is GP-injective.*
3. *R is right Kasch and $Z(S_S) = 0$.*
4. *R is right Kasch and S is semiprime.*

Proposition 3.6 *If every simple singular factor module of M is M-gp-injective, then no proper fully invariant submodule of M is isomorphic to M.*

Proof. Call $s : M \to M$ a monomorphism. Assume that $H = s(M)$ is a proper fully invariant submodule of M. By Lemma 3.1, there exist $n > 0$ and $K \leq M$ such that $M = [\mathrm{Ker}(s^n) + SsM] \oplus K$. Since s is a monomorphism, then $\mathrm{Ker}(s^n) = 0$. It follows that $M = Ss(M) \oplus K$ and so $M = H \oplus K = s(M) \oplus K$ (since H is fully invariant submodule of M). Thus $s(M) = e(M)$ for some $e^2 = e \in S$. But M is quasi-projective, $s = es$ and $e = st$ for some $t \in S$. It follows that $s = sts$, whence $(1_M - ts)(M) \leq \mathrm{Ker}(s) = 0$ or $1_M - ts = 0$. Therefore $M = ts(M) \leq t(H) \leq H$, and so $M = H$, a contradiction. □

Theorem 3.7 *If every maximal submodule of M is π-fully invariant and every simple singular factor module of M is M-gp-injective, then $W(S) = 0$.*

Proof. Assume that $W(S) \neq 0$. We claim that there exists $s \in W(S), s \neq 0$ and $s^2 = 0$. Otherwise, for all $u \in W(S)$, u is not nilpotent. Let $A = \sum_{k \in \mathbb{N}} [\mathrm{Ker}(u^k) + u(M)]$ and then $A \leq^e M$. Assume that $A \neq M$. Repeating the proving of Theorem 2.11, we get $M = A$. Since M is finitely generated and $\mathrm{Ker}(u^k) + uM \leq \mathrm{Ker}(u^{k+1}) + uM$ for all

$k \in \mathbb{N}$, there exists $n > 0$ such that $M = \mathrm{Ker}(u^n) + uM$. It follows that $u^n M = u^{n+1}M$. Since M is quasi-projective, there exists $v \in S$ such that $u^n = u^{n+1}v$. We have

$$\begin{aligned} u^n = u^{n+1}v = u.u^n v \quad &= uu^{n+1}v^2 \\ = u^2u^nv^2 \quad &= u^2u^{n+1}v^3 \\ = u^3u^nv^3 \quad &= u^3u^{n+1}v^4 \\ = \cdots \quad &= \cdots \\ = u^{2n}u^nv^{2n} & \end{aligned}$$

Call $w = u^n$. Then $w = w^2t$ for some $t \in S$. This gives that $w^2 = w^2tw$ and so $(w - wtw)(M) \leq \mathrm{Ker}(w) \cap w(M)$. Assume that $w(1_M - tw)(M) = 0$, then $w = wtw$. It follows that $w(M)$ is a direct summand of M and so it is M-projective. Note that $\mathrm{Ker}(w) \leq^e M$. This implies that $w = u^n = 0$, a contradiction. So $w(1_M - tw)(M) \neq 0$. Call $z = 1_M - tw$. Then $wz \neq 0$ and $w^2z = 0$. It follows that $(wzw)^2 = 0$ and so $wzw = 0$ by our assumption. Therefore $(wz)^2 = 0$, which implies that $wz = 0$ again by our assumption, a contradiction.

Thus, there exists $s \in W(S), s \neq 0$ and $s^2 = 0$. By proving of Theorem 2.11, we get a contradiction. Thus $W(S) = 0$. □

Corollary 3.8 *If every maximal right ideal of R is π-fully invariant and every simple singular R-module is GP-injective, then $Z(R_R) = 0$.*

Corollary 3.9 ([4, Theorem 3.2]) *If every maximal right ideal of R is an ideal and every simple singular R-module is GP-injective, then $Z(R_R) = 0$.*

Let $S = \mathrm{End}(M)$. A module M is *ZI-module* if $sm = 0$ implies that $sSm = 0$ for any $m \in M$ and $s \in S$ (see [12]).

Lemma 3.10 *Let M be a ZI-module. Then*

1. *Every idempotent of S is central.*
2. $\mathrm{Ker}(s)$ *is a fully invariant submodule of M for any $s \in S$.*

Proof. (1) For all $e^2 = e \in S$ and $s \in S$. For any $m \in M$, we have $e(1-e)m = 0$, which implies $es(1-e)m = 0$. Thus $es(1-e) = 0$. Similarly, $(1-e)se = 0$. Thus $se = es$.

(2) Let $s \in S$. For all $m \in \mathrm{Ker}(s)$, $s(m) = 0$ and so $sSm = 0$. Therefore $sfm = 0$ or $f(m) \in \mathrm{Ker}(s)$ for all $f \in S$. Thus $f(\mathrm{Ker}(s)) \leq \mathrm{Ker}(s)$ for all $f \in S$. □

Theorem 3.11 *If M is a ZI-module and every simple singular factor module of M is M-gp-injective then S is reduced and S is weakly regular.*

Proof. We first show that $SsM + \mathrm{Ker}(s) = M$ for all $s \in S$. In fact, by Lemma 3.1, there exists $K \leq M$ such that $M = [\mathrm{Ker}(s) + SsM] \oplus K$. There exists $e^2 = e \in S$ such that $\mathrm{Ker}(s) + SsM = \mathrm{Ker}(e)$ and $K = eM$. It follows that $es = 0$ and so $se = 0$, which implies that $K = eM \leq \mathrm{Ker}(s) \leq (\mathrm{Ker}(s) + SsM) \cap K = 0$. Therefore $K = 0$ and so $M = \mathrm{Ker}(s) + SsM$.

We next show that S is reduced. Indeed, assume that $0 \neq s \in S$ with $s^2 = 0$. By the above proof, we have $M = \mathrm{Ker}(s) + SsM$. Moreover, as M is a ZI-module, $SsM \leq \mathrm{Ker}(s)$ and so $M = \mathrm{Ker}(s)$. It follows that $s = 0$. This shows that S is reduced.

We now claim that S is right weakly regular. In fact, for all $s \in S$, $SsM + \mathrm{Ker}(s) = M$. Then $s(M) = sSsM$. Since M is finitely generated quasi-projective, $s = ss_1sh_1 + ... + ss_ksh_k$ for some $s_i, h_i \in S$. Therefore $s \in sSsS$ or S is right weakly regular. As S is reduced, S is also left weakly regular. $\square$

Corollary 3.12 *Let M be a ZI-module. The following are equivalent:*

1. *M is endo-strongly regular.*
2. *Every maximal submodule of M is π-fully invariant and every singular simple factor module of M is M-gp-injective.*

Corollary 3.13 ([8, Theorem 4]) *Let R be a ZI-ring. If every simple singular module is GP-injective, then R is reduced and R is weakly regular.*

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