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Representation of the self-similar structure of Rauzy fractals using integer binary operations

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Abstract Given a primitive substitution, we define a binary operation "$\star$" on $\mathcal{P}$ an infinite subset of the positive integers. This binary operation is defined with the help of the numeration system associated with the substitution. The pair $(\mathcal{P}, \star)$ is a semigroup. Let $\mathcal{P}'$ be the set generated by $\mathcal{P}$ and $\{0\}$ under the binary operation "$\star$". The pair $(\mathcal{P}', \star)$ is no longer a semigroup, it is a unital, i.e. a set with a close binary operation with identity. We give conditions for $(\mathcal{P}', \star)$ to be finitely generated. When the substitution is unimodular Pisot, we use $(\mathcal{P}', \star)$ in order to represent the self-similar structure of the Rauzy fractal associated with the substitution.

Keywords Integer semi-groups · Rauzy fractals, substitution dynamical systems · numeration systems · finite automata · graph directed iterated function systems

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1 Introduction

Integer binary operations and in particular integer semigroups have been studied in different context (*cf.* [17,18,2,26]). One of the well known binary operations is the Fibonacci multiplication, introduced by Knuth ([17]). This operation is related to the dynamical system obtained by the Fibonacci substitution (*cf.* [2]). It has been generalized in different ways to the tribonacci substitution, see [26,20,21]. The author studied the usage of binary operations on the set of non-negative integers, in order to describe the self-similar structure of the k-bonacci substitutions ([26]) and on the so-called flipped tribonacci substitution ([27]).

In the present paper, for a given primitive substitution, we introduce a binary operation on an infinite subset of the positive integers and obtain a semigroup. This operation is defined using the numeration system associated with the substitution, the so called Dumont-Thomas numeration system. We extend this operation to a subset of the non-negative integers. In this extension process we lose the associativity, therefore

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we end up with a unital. However the unital obtained, has a particular algebraic structure called quasi-semigroup. In Section 2 we give the corresponding definitions. In Theorem 2.2, we give conditions for this unital to be finitely generated.

When the substitution is unimodular Pisot, we use this unital to explore the self-similar structure of the associated Rauzy fractal. In Theorem 3.2, we prove that if the unital is finitely generated then one of the sets of the natural decomposition of the Rauzy fractal is the fixed point of an iterated function system.

2 Substitutions and Automata

A substitution on a finite alphabet $\mathcal{A} = \{1, \ldots, k\}$ is a map ζ from $\mathcal{A}$ to the set of finite words on $\mathcal{A}$, i.e. $\mathcal{A}^* = \cup_{i \geq 0} \mathcal{A}^i$. This map extends to $\mathcal{A}^*$ by concatenation by $\zeta(\emptyset) = \emptyset$ and $\zeta(UV) = \zeta(U)\zeta(V)$, for all U, $V \in \mathcal{A}^*$. Let $A^{\mathbb{N}}$ (respectively $A^{\mathbb{Z}}$) denote the set of one-sided (respectively two-sided) infinite sequences in $\mathcal{A}$. The map ζ is extended to $\mathcal{A}^{\mathbb{N}}$ and $\mathcal{A}^{\mathbb{Z}}$ in the obvious way. We call $u \in \mathcal{A}^{\mathbb{N}}$ (or $u \in \mathcal{A}^{\mathbb{Z}}$) a *fixed point* of ζ if $\zeta(u) = u$ and *periodic* if there exists $l > 0$ so that it is fixed for ζ^l.

The *incidence matrix* of the substitution ζ is defined as the matrix $M = (M_{ij})$ whose entries M_{ij} is the number of occurrences of the symbol j in the word $\zeta(i)$, for $1 \leq i, j \leq k$. We say the substitution is *primitive* if its incidence matrix is primitive, i.e. all the entries of M^l are positive for some $l > 0$. We shall assume through out the article that the substitutions are primitive.

For a primitive substitution there are a finite number of periodic points. So, we can assume without loss of generality that a primitive substitution has a fixed point, say $\mathbf{u} = u_1 u_2 \ldots$ and $u_1 = 1$, i.e. the first symbol of $\zeta(1)$ is 1.

Let $W \subset \mathcal{A}^* = \cup_{i \geq 0} \mathcal{A}^i$ be a set of finite words on the alphabet $\mathcal{A} = \{1, \ldots, k\}$. An *automaton* over $\mathcal{A}$, $\mathfrak{A} = (Q, W, E, I)$ is a direct graph labelled by elements of $\mathcal{A}$. Q is the set of *states*, $I \subset Q$ is the set of *initial states*, W is the set of *labels* and $E \subset Q \times W \times Q$ is the set of labelled *edges* or *transitions*. If $(p, w, q) \in E$ we say that w is a transition between p and q.

The prefix automaton of the substitution ζ (*cf.* [24]) is the automaton $\mathfrak{A} = (Q, W, E, I)$ so that

1. $Q = \mathcal{A}$;
2. $W =$ Pref, the set of proper prefixes of the words $\zeta(p)$, for $p \in \mathcal{A}$. We shall denote by e the empty prefix;
3. (p, w, q) is in E if p, q are elements of $\mathcal{A}$, $w \in W$ and wq is a prefix of $\zeta(p)$.
4. $I = \{1\}$.

The automaton reads words from left to right.

In Figure 1, it is shown the prefix automaton of the k-bonacci substitution, i.e. $\zeta(1) = 12, \zeta(2) = 13, \ldots, \zeta(k-1) = 1k, \zeta(k) = 1$.

A *finite path in the automaton* $\mathfrak{A}$ is a word in E, the set of transitions:

$$(p_m, a_m, q_m)(p_{m-1}, a_{m-1}, q_{m-1}) \cdots (p_0, a_0, q_0) \tag{2.1}$$

such that $q_i = p_{i-1}$ for $1 \leq i \leq m$ and $p_m \in I$; however we usually denote the paths using only the labels, i.e. $a_m \cdots a_0$. We say that the path $a_m \cdots a_0$ *terminates* at the state q_0 and we say that the path *passes through the state* q, if there is $i \in \{0, \ldots, m\}$ such that $p_i = q$, in (2.1). We generalize the concept of a path on the automaton, in

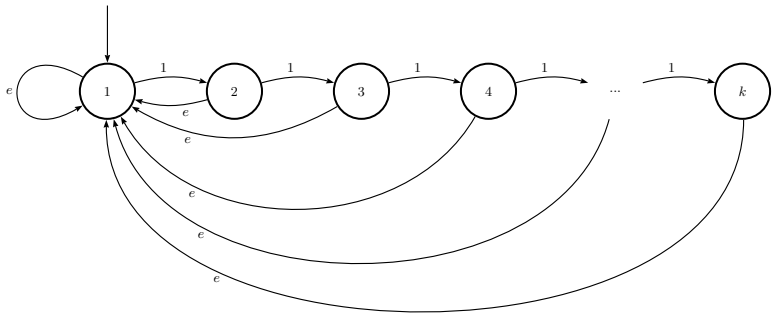

Fig. 1 The k-bonacci automaton.

the following way: we say that a path $b_n \cdots b_0$ in $\mathfrak{A}$ *starts in the state* p *and terminates in the state* q, if for some $m \geq n$ there exists a path of type (2.1), such that $b_i = a_i$, for $0 \leq i \leq n$, with $p_n = p$ and $q_0 = q$. See [11] for more on automata theory.

Theorem 2.1 ([24,9]) *Let U be a non-empty prefix of* $\mathbf{u}$*, the fixed point of the substitution ζ. Then there exists a unique path on the prefix automaton of ζ, labelled by $(a_n,a_{n-1},\ldots,a_0)$ such that $a_n \neq e$, the empty word and $U = \zeta^n(a_n)\zeta^{n-1}(a_{n-1})\cdots\zeta(a_1)a_0$. Conversely, to any such path, there corresponds a prefix of* $\mathbf{u}$*, given by the above formula.*

Due to this theorem, we can associate a numeration system for the positive integers, in the following way: Let $\epsilon : \mathbb{N} \to W^*$, given by $\epsilon(n) = a_l a_{l-1} \cdots a_1 a_0$, where $a_l \neq e$ and U the prefix of $\mathbf{u}$ of length n whose expression according Theorem 2.1 is $U = \zeta^l(a_l)\zeta^{l-1}(a_{l-1})\cdots\zeta(a_1)a_0$. Furthermore this representation is unique. This numeration system is called the *Dumont-Thomas numeration system* associated with the substitution ζ (*cf.* [9,10]). For general numeration systems see [1,5,6,10,14,15].

We consider the following subset of positive integers:

$$\mathcal{P} := \{n \in \mathbb{N} \,:\, \epsilon(n) = a_l \cdots a_0, \text{ such that } a_l \cdots a_0 \text{ is path in } \mathfrak{A} \text{ terminating at the initial state}\}.$$

We define "$\star$" an binary operation in $\mathcal{P}$, in the following way: Let $n, m \in \mathcal{P}$, such that $\epsilon(n) = a_l a_{l-1} \cdots a_1 a_0$ and $\epsilon(m) = b_r b_{r-1} \cdots b_0$, so

$$\epsilon(n \star m) := a_l a_{l-1} \cdots a_1 a_0 b_r b_{r-1} \cdots b_0.$$

This binary operation is well defined since the paths in the prefix-automaton, associated with the elements of $\mathcal{P}$ terminates at the initial state, so we can concatenate them. It is clear that the operation $\star$ is associative. Hence $(\mathcal{P}, \star)$ a semigroup.

However the binary operation $\star$ is not always commutative, e.g. in the Fibonacci case, we consider $\epsilon(2) = 1e$ and $\epsilon(3) = 1ee$, so $\epsilon(2 \star 3) = 1e1ee = \epsilon(10)$ and $\epsilon(3 \star 2) = 1ee1e = \epsilon(9)$, so $2 \star 3 = 10$ and $3 \star 2 = 9$.

If the word $\zeta(1)$ starts with the symbol 1, then in the prefix automation the transition $(1, e, 1)$ is allowed, so $e \cdots e a_l a_{l-1} \cdots a_1 a_0$ is a path in $\mathfrak{A}$, and

$$\zeta^l(a_l)\zeta^{l-1}(a_{l-1})\cdots\zeta(a_1)a_0 = \zeta^{l+i}(e)\cdots\zeta^{l+1}(e)\zeta^l(a_l)\zeta^{l-1}(a_{l-1})\cdots\zeta(a_1)a_0.$$

By convention we say that $\epsilon(0) = e$. Let $\mathcal{P}'$ be the set generated by $\mathcal{P}$ and $\{0\}$, by the operation "$\star$". Clearly $0 \star n = n$, but $n \star 0 \neq n$, since in general,

$$\zeta^l(a_l)\zeta^{l-1}(a_{l-1}) \cdots \zeta(a_1)a_0 \neq \zeta^{l+1}(a_l)\zeta^l(a_{l-1}) \cdots \zeta^2(a_1)\zeta(a_0).$$

We point out that if $n \neq 0$ and $\epsilon(n) = a_l a_{l-1} \cdots a_1 a_0$, we always consider that $a_l \neq e$. Hence the pair $(\mathcal{P}', \star)$ has a left identity. Moreover $\mathcal{P}' = \mathcal{P} \cup \{0\}$, since $0 \star n = n$ and $n \star 0 \in \mathcal{P}$. However $(\mathcal{P}', \star)$ is not a semigroup, consider $\epsilon(n \star (0 \star m)) = a_l \cdots a_1 b_r \cdots b_0$, since $0 \star m = m$ and on the other hand $\epsilon((n \star 0) \star m) = a_l \cdots a_1 e b_r \cdots b_0$, therefore $n \star (0 \star m) \neq (n \star 0) \star m$. Observe that the associativity of $n_1 \star \cdots \star n_l$ fails only in the case that $n_i = 0$ with $i \in \{2, \ldots, l\}$.

The algebraic structure of the pair $(\mathcal{P}', \star)$ is an *unital magma* or simply *unital*, a set with a closed binary operation with identity, see [7]. However it has a richer algebraic structure, it is a quasi-semigroup. We say that $(\mathcal{S}, \diamond)$ is a *quasi-semigroup* if $\diamond$ is a closed binary operation on the set $\mathcal{S}$, and there exists an element s_0 in $\mathcal{S}$ with the the property that $(\mathcal{S} \setminus \{s_0\}, \diamond)$ is a semigropup.

In [28] different integer semigroups were defined, using the Dumont-Thomas numeration systems, and studied by the author.

Let $(\mathcal{S}, \diamond)$ be a quasi-semigroup. In order to simplify the notation we denote $(\cdots(s_1 \diamond s_2) \diamond \ldots s_{k-1}) \diamond s_k$ by $s_1 \diamond s_2 \diamond \ldots \diamond s_{k-1} \diamond s_k$. We say that a quasi-semigroup $(\mathcal{S}, \diamond)$ is *finitely generated* if there exists $\mathfrak{G} = \{s_1, \ldots, s_m\}$ with $s_i \in \mathcal{S}$, such that for all $s \in \mathcal{S}$ we have $s = s_{i_1} \diamond \cdots \diamond s_{i_l}$, for some l and $s_{i_j} \in \mathfrak{G}$, $1 \leq j \leq l$. The elements of $\mathfrak{G}$ are called *generators* of $(\mathcal{S}, \diamond)$. In a similar way we say that a semigroup is finitely generated.

Let us consider the tribonacci case, and the numbers $n_1 = 0$, $n_2 = 2$ and $n_3 = 6$ their representation are $\epsilon(n_1) = e$, $\epsilon(n_2) = 1e$, and $\epsilon(n_3) = 11e$. The corresponding paths e, $1e$ and $11e$, are all the paths in the prefix automaton that pass through the initial state only when they start and terminate. Therefore all the paths that start and terminate at the initial state are concatenation of these three paths. Hence the set $\mathcal{P}'$ is generated by these three numbers, i.e. for every $n \in \mathcal{P}'$, we have $n = n_{i_1} \star \cdots \star n_{i_l}$, for some $l \geq 1$. We point out that $(\mathcal{P}', \star)$ is finitely generated while $(\mathcal{P}, \star)$ is not, since one of the generators, n_1 is not in $\mathcal{P}$. The same situation occurs for the k-bonacci substitution for all $k \geq 2$. This was considered in [26]. This is a special case of the following theorem.

Theorem 2.2 *The quasi-semigroup $(\mathcal{P}', \star)$ is finitely generated if and only if for any state p in the prefix automaton, different of the initial state, the paths in automaton from p to itself have to pass through the initial state.*

Proof. Let us suppose that there is a state p in the prefix automaton $\mathfrak{A}$, different of the initial state, i.e. $p \neq 1$; such that there is at least one path in $\mathfrak{A}$ from p to itself that does not pass through the initial state. Let $c_r \ldots c_0$ be such path. By primitivity there exist a path $a_l \ldots a_0$ in $\mathfrak{A}$ from the initial state to p and $b_m \ldots b_0$ a path in $\mathfrak{A}$ from p to the state 1. So we have the following paths in $\mathfrak{A}$: $a_l \ldots a_0 c_r \ldots c_0 b_m \ldots b_0$, $a_l \ldots a_0 c_r \ldots c_0 c_r \ldots c_0 b_m \ldots b_0$ and so on. They start at the state 1 and terminate at the state 1, but they do not pass through the state 1 in any intermediate step. Since

$$\epsilon(a_l \ldots a_0 \underbrace{(c_r \ldots c_0) \cdots (c_r \ldots c_0)}_{s} b_m \ldots b_0)$$

are elements of $\mathcal{P}$, for $s \geq 1$ and the path $c_r \dots c_0$ does not correspond to an element in $\mathcal{P}'$. We conclude $\mathcal{P}'$ is not finitely generated.

Let us suppose that the paths in the prefix automaton from any state, different from the initial, to itself pass through the initial state. Since there is no path from any state p, with $p \neq 1$, to itself that does not pass through the state 1, there is a finite number of paths that pass only through the initial state when they start and terminate. Let $a_{j_s} \dots a_{j_0}$, for $1 \leq j \leq r$, be these paths. And let $n_1, \dots, n_r$, the numbers in $\mathcal{P}'$, with $\epsilon(n_j) = a_{j_s} \dots a_{j_0}$. So, we have that any path in the prefix automaton that starts and terminates at the initial state, is a concatenation of these paths. Therefore the numbers $n_1, \dots, n_r$ generate $(\mathcal{P}', \star)$. □

3 Rauzy Fractals

A substitution is *Pisot* if the Perron-Frobenius eigenvalue of the the incidence matrix is a Pisot number and the characteristic polynomial is irreducible. i.e. all non-Perron-Frobenius eigenvalues are nonzero and less than one in modulus. A Pisot substitution is primitive [8]. A substitution is *unimodular Pisot* if it is Pisot and the absolute value of the determinant of the incidence matrix is 1.

Let ζ be a unimodular Pisot substitution and M its incidence matrix. We order the eigenvalues of M as follows:

$$\lambda, \beta_1, \dots, \beta_{r-1}, \beta_r, \overline{\beta_r}, \beta_{r+1}, \overline{\beta_{r+1}}, \dots, \beta_{r+s}, \overline{\beta_{r+s}}$$

where λ is the Perron-Frobenius eigenvalue of M, the first r are distinct real eigenvalues and the last $2s$ are pairs of complex conjugate eigenvalues, $r + 2(s+1) = k$. We identify $\mathbb{R}^{r-1} \times \mathbb{C}^{s+1}$ with $\mathbb{R}^{k-1}$. Let B be the matrix associated with the contractive space of M, i.e.

$$B = \operatorname{diag}(\beta_1, \dots, \beta_{r-1}, \beta_r, \overline{\beta_{r+1}}, \dots, \beta_{r+s}, \overline{\beta_{r+s}}).$$

Let $\Delta : \mathcal{A}^* \to \mathbb{R}^{k-1}$ be a map with the properties $\Delta(UV) = \Delta(U) + \Delta(V)$ and $\Delta(\zeta(U)) = B\Delta(U)$, for all $U, V \in \mathcal{A}^*$. There exists a map with these properties and is called a *valuation* of ζ (*cf.* [16,29]).

Let

$$\mathfrak{R} := \overline{\left\{ \sum_{j=0}^{n} B^j \Delta(a_j) \ : \ a_n \cdots a_0 \text{ is a path in the automaton } \mathfrak{A} \right\}},$$

where $\overline{A}$ is the closure of A in the standard topology of $\mathbb{R}^{k-1}$. The set $\mathfrak{R}$ is known as the *Rauzy fractal associated with the substitution* ζ. The set $\mathfrak{R}$ has positive Lebesgue mesure and its boundary null Lebesgue measure (*cf.* [29]).

For $i \in \{1, \dots, k\}$, let

$$\mathfrak{R}_i := \overline{\left\{ \sum_{j=0}^{n} B^j \Delta(a_j) \ : \ a_n \cdots a_0 \text{ path in } \mathfrak{A} \text{ terminating at the state } i \right\}}.$$

So $\{\mathfrak{R}_1, \dots, \mathfrak{R}_k\}$ is a partition of $\mathfrak{R}$ into k subsets. It is called *the natural decomposition of* $\mathfrak{R}$. The boundary $\mathfrak{R}_i$ has null Lebesgue measure (*cf.* [29]). There is a long standing

conjecture, the so called *Pisot conjecture*, that sets R_i have disjoint interiors. For references on the Rauzy fractal and conditions under which the Pisot conjecture is true, among other references see [23,3,8,16,22,29,25,30].

The natural decomposition of the Rauzy fractals satisfies the following equations

$$\mathfrak{R}_j = \bigcup_{i\,:\,M_{ij}>0} f_{i,w,j}(\mathfrak{R}_i), \tag{3.1}$$

where i,j are states of the automaton $\mathfrak{A}$ and (i,w,j) is a transition in $\mathfrak{A}$; the maps $f_{i,w,j}(z) := Bz + \Delta(w)$ and $z \in \mathfrak{R}_i$. This is a *graph-directed iterated function system or GIFS*. Moreover $\{\mathfrak{R}_1, \ldots, \mathfrak{R}_k\}$ is the only compact solution of the equation (3.1), since the maps $f_{i,w,j}$ are contractions and the matrix M is primitive. When a GIFS has only a unique compact solution, we say that the solution is the fixed point of the GIFS, or the attractor of the GIFS. For more details see [29] and for general properties of GIFS, see [19] or [13].

The equation (3.1) can also be written as

$$\mathfrak{R}_j = \bigcup_{(i,w,j)\in E} f_{i,w,j}(\mathfrak{R}_i), \tag{3.2}$$

where E is the set of transitions in the automaton $\mathfrak{A}$.

We remark that if $a_l \cdots a_0$ is a path in the prefix automaton $\mathfrak{A}$, that terminates at the state r, more precisely: $(p_l,a_l,q_l)(p_{l-1},a_{l-1},q_{l-1})\cdots(p_0,a_0,q_0)$ is a path in $\mathfrak{A}$, with $a_l \neq e$, $q_i = p_{i-1}$, for $1 \leq i \leq l$, $p_l = 1$ and $q_0 = r$, then we have

$$f_{p_0,a_0,q_0} \circ f_{p_1,a_1,q_1} \circ \cdots \circ f_{p_{l-1},a_{l-1},q_{l-1}} \circ f_{p_l,a_l,q_l}(\mathfrak{R}_1) \subset \mathfrak{R}_r.$$

In order to simplify the notation, we shall denote $f_{a_0} \circ f_{a_1} \circ \cdots \circ f_{a_l}$ the map $f_{p_0,q_0,a_0} \circ f_{p_1,q_1,a_1} \circ \cdots \circ f_{p_l,q_l,a_l}$. So we define $\mathfrak{R}(n) := F_n(\mathfrak{R}_1)$, where n is a positive integer with $\epsilon(n) = a_l \cdots a_0$, as before and $F_n = f_{a_0} \circ f_{a_1} \circ \cdots \circ f_{a_l} : \mathfrak{R}_1 \to \mathfrak{R}_r$. It follows from the definitions:

$$\mathfrak{R}(n \star m) = F_m \circ F_n(\mathfrak{R}_1).$$

For this reason we say $(\mathcal{P}, \star)$ represents the self-similar structure of the set $\mathfrak{R}$, and its natural decomposition. Theorems 3.1 and 3.2 reinforce this concept.

Theorem 3.1 *The union of the sets $\mathfrak{R}(n)$, over all $n \in \mathcal{P}$ is the set $\mathfrak{R}_1$, i.e.*

$$\mathfrak{R}_1 = \bigcup_{n\in\mathcal{P}} \mathfrak{R}(n).$$

Proof. By previous remarks, clearly $\mathfrak{R}(n) \subset \mathfrak{R}_1$, for all $n \in \mathcal{P}$.

Let $x \in \mathfrak{R}_1$ of the form $\sum_{j=0}^{l} B^j(\Delta(a_j))$, for some path $a_l \ldots a_0$ in $\mathfrak{A}$ that starts and terminates at the state 1, and $a_l \neq e$. Let n be the corresponding element in $\mathcal{P}$, i.e. $\epsilon(n) = a_l \ldots a_0$.

For any $z \in \mathfrak{R}_1$, we have

$$\begin{aligned} f_{a_0} \circ \cdots \circ f_{a_n}(z) &= B(\cdots B(B(z + \Delta(a_l)) + \Delta(a_{l-1})) + \cdots + \Delta(a_1)) + \Delta(a_0) \\ &= B^{l+1}z + B^l(\Delta(a_l)) + \cdots + B(\Delta(a_1)) + \Delta(a_0) \\ &= B^{l+1}z + x. \end{aligned}$$

Let us remark that 0 is an element of $\mathfrak{R}_1$, since $(1, e, 1)$ is a transition in the prefix automaton and $\Delta(e) = 0$. Taking $z = 0$, then $f_{a_0} \circ \cdots \circ f_{a_n}(z) = x$, so $x \in \mathfrak{R}(n)$.

If $x \in \mathcal{R}_1$, is not of the previous form, then it is the limit of points $x_{l_m} = \sum_{j=0}^{l_m} B^j(\Delta(a_j))$, where $a_{l_m} \dots a_0$ are paths in $\mathfrak{A}$ that start and terminate at the state 1. In this case, we can denote $x = \sum_{j\geq 0} B^j(\Delta(a_j))$. Let us fix a large $l = l_i$, let $\epsilon(n) = a_l \dots a_0$, with $a_l \neq e$. By the previous argument $x_l \in \mathfrak{R}(n)$ and $f_{a_0} \circ \cdots \circ f_{a_l}(\mathfrak{R}_1) = \mathfrak{R}(n)$, so

$$\mathfrak{R}(n) = B^{l+1}(\mathfrak{R}_1) + x_l.$$

On the other hand, the point $\sum_{j\geq 0} B^j(\Delta(a_{j+l}))$ is in $\mathfrak{R}_1$, since for the infinite increasing sequence $\{l_m\}_{m\geq 1}$, the paths $a_{l_m} \dots a_{l_1+1}$ are in $\mathfrak{A}$ and they start and terminate at the initial state. Hence $|\sum_{j>l} B^j(\Delta(a_j))| \leq \text{diameter}(B^{l+1}(\mathfrak{R}_1))$. Therefore we can conclude $x \in \mathfrak{R}(n)$. □

An *iterated function system or IFS* on $X \subset \mathbb{R}^n$ is $\{g_1, \dots, g_k\}$, where $g_j : X \to X$ are contractions, for $1 \leq j \leq k$. There exits a unique compact set $K \subset X$ that satisfies:

$$K = g_1(K) \cup \cdots \cup g_k(K).$$

We say that K is the fixed point or the attractor of the IFS (*cf.* [12]).

Theorem 3.2 *If $(\mathcal{P}', \star)$ is finitely generated then the set $\mathfrak{R}_1$ is the fixed point of an IFS.*

Before giving the proof of Theorem 3.2, we present some examples:

Example 1. Let us consider, the well known example of the tribonacci substitution. Here the GIFS (3.1) is given by:

$$\mathfrak{R}_1 = f_{1,e,1}(\mathfrak{R}_1) \bigcup f_{2,e,1}(\mathfrak{R}_2) \bigcup f_{3,e,1}(\mathfrak{R}_3),\ \mathfrak{R}_2 = f_{1,1,2}(\mathfrak{R}_1),\ \mathfrak{R}_3 = f_{2,1,3}(\mathfrak{R}_2);$$

where $f_{i,e,j}(z) = \beta z$ and $f_{i,1,j}(z) = \beta z + 1$, where β is one of the complex roots of the polynomial $x^3 - x^2 - x - 1$.

As we remarked in Section 2, the set $\mathcal{P}'$ is generated by n_1, n_2 and n_3, where $\epsilon(n_1) = e$, $\epsilon(n_2) = 1e$ and $\epsilon(n_3) = 11e$. So the set $\mathfrak{R}_1$ is the fixed point of the IFS: $\{F_{n_1}, F_{n_2}, F_{n_3}\}$. Where $F_{n_1}(z) = \beta z$, $F_{n_2}(z) = \beta^2 z + \beta$ and $F_{n_3}(z) = \beta^3 z + \beta^2 + \beta$. Due to the particular fact $f_{1,e,1} = f_{2,e,1} = f_{3,e,1}$, we have that $\mathfrak{R}$ is also is the fixed point of the IFS: $\{G_1, G_2, G_3\}$, where $G_1(z) = \beta z$, $G_2(z) = \beta^2 z + 1$ and $G_3(z) = \beta^3 z + \beta + 1$. As it can be noticed this property is a very special fact of the tribonacci substitution, in general k-bonacci. This unital $(\mathcal{P}', \star)$ is related to the semigroup introduced and studied in [26].

Example 2. Let ζ be the "flipped tribonacci" substitution: $\zeta(1) = 12$, $\zeta(2) = 31$, $\zeta(3) = 1$. Here the GIFS (3.1) is given by:

$$\mathfrak{R}_1 = f_{1,e,1}(\mathfrak{R}_1) \bigcup f_{2,3,1}(\mathfrak{R}_2) \bigcup f_{3,e,1}(\mathfrak{R}_3),$$

$$\mathfrak{R}_2 = f_{1,1,2}(\mathfrak{R}_1),$$

$$\mathfrak{R}_3 = f_{2,0,3}(\mathfrak{R}_2).$$

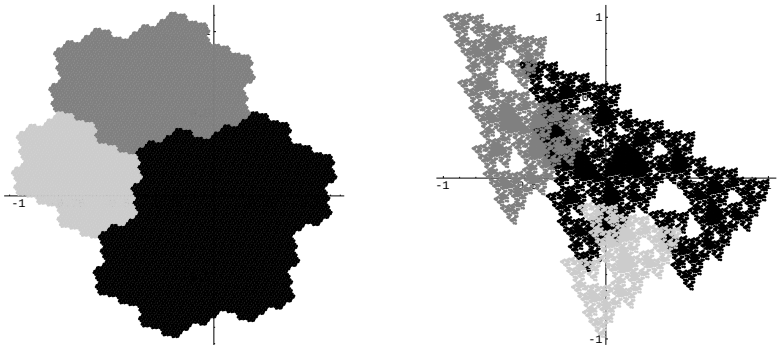

Fig. 2 Rauzy fractals of Examples 1 (left) and 2 (right) and their natural decomposition

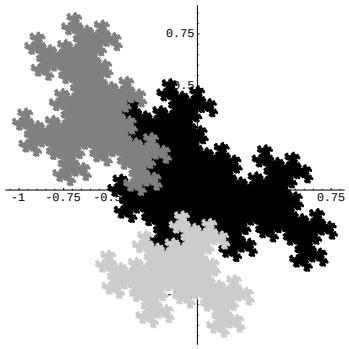

Fig. 3 The Rauzy fractal of Example 3 and its natural decomposition

Let β be one of the complex roots of the polynomial $x^3 - x^2 - x - 1$.

Since the valuation is $\Delta(e) = 0$ and $\Delta(1) = 1$, we have $\Delta(\zeta(3)) = \beta\Delta(3)$, so $\Delta(3) = \beta^{-1}$. Therefore the maps that define the GIFS are: $f_{i,e,j}(z) = \beta z$, $f_{1,1,2}(z) = \beta z + 1$ and $f_{2,3,1}(z) = \beta z + \beta^{-1}$.

The unital $(\mathcal{P}', \star)$ is generated by n_1, n_2, n_3 the non-negative numbers whose representation in the numeration system defined by Theorem 2.1 are $\epsilon(n_1) = e$, $\epsilon(n_2) = 13$ and $\epsilon(n_3) = 1ee$. So the set $\mathfrak{R}_1$ is the fixed point of the IFS $\{F_{n_1}, F_{n_2}, F_{n_3}\}$. This unital $(\mathcal{P}', \star)$ is related to the semigroup introduced and studied in [27].

Example 3. Let ζ be the substitution: $\zeta(1) = 12$, $\zeta(2) = 3$, $\zeta(3) = 1$. The corresponding semigroup $\mathcal{P}'$ is generated by $\epsilon(n_1) = e$, $\epsilon(n_2) = 1ee$. The natural decomposition $\{\mathfrak{R}_1, \mathfrak{R}_2, \mathfrak{R}_3\}$ of its Rauzy fractal is the fixed point of the GIFS:

$$\mathfrak{R}_1 = f_{1,e,1}(\mathfrak{R}_1) \bigcup f_{3,e,1}(\mathfrak{R}_3), \mathfrak{R}_2 = f_{1,1,2}(\mathfrak{R}_1), \mathfrak{R}_3 = f_{2,e,3}(\mathfrak{R}_2),$$

where $f_{i,e,j}(z) = \beta z$ and $f_{1,1,2}(z) = \beta z + 1$, where β is one of the complex roots of the polynomial $x^3 - x^2 - 1$.

We can simplify the notation by introducing the maps $g_0 := f_{i,e,j}$ and $g_1 := f_{1,1,2}$ Here $\mathfrak{R}_1$ is the fixed point of the IFS $\{F_{n_1}, F_{n_2}\}$, where $F_{n_1} := g_0$ and $F_{n_2} := g_0 \circ g_0 \circ g_1$. The set $\mathfrak{R}_2$ is also the fixed point of the IFS $\{g_1 \circ F_{n_1}, g_2 \circ F_{n_2}\}$, similarly $\mathfrak{R}_3$.

Proof of Theorem 3.2. Let $\mathcal{I} := \{i \ : \ M_{i1} > 0\}$ and $i_1, \ldots, i_r$ be its elements. Since the fixed point of the substitution ζ starts with the symbol 1, then $1 \in \mathcal{I}$. Let us suppose that $i_1 = 1$.

Let $C_{i_1} := \{(1, w, 1) \in E\}$, where E is the set of transitions of the prefix automaton $\mathfrak{A}$. For $i_s \in \mathcal{I}$, let C_{i_s} be the set of paths $(p_m, a_m, q_m) \cdots (p_0, a_0, q_0)$ in the automaton $\mathfrak{A}$ of type (2.1) such that $p_m = 1$, $q_0 = 1$, $p_0 = i_s$ and $q_i \neq 1$ for $1 \leq i \leq m$, i.e. C_{i_s} is the set of paths in the automaton $\mathfrak{A}$ such that they start and terminate in the initial state and pass through the state i_s, just before terminating.

Since $\mathcal{P}$ is finitely generated, according to Theorem 2.2, the sets C_{i_s} are well defined and not empty.

By GIFS (3.1) or (3.2):

$$\begin{aligned}\mathfrak{R}_1 &= \bigcup_{i \,:\, M_{i1>0}} f_{i,w,1}(\mathfrak{R}_i) \\ &= \bigcup_{(1,w,1)\in C_{i_1}} f_{1,w,1}(\mathfrak{R}_1) \cup \bigcup_{s=2}^{r} \left(\bigcup_{(i_s,w,1)\in E} f_{i_s,w,1}(\mathfrak{R}_{i_s}) \right).\end{aligned}$$

If we use (3.2) to one of the terms of this identity, we get

$$\bigcup_{(i_s,w,1)\in E} f_{i_s,w,1}(\mathfrak{R}_{i_s}) = \bigcup_{(j_s,w,i_s),(i_s,w,1)\in E} f_{i_s,w,1}(f_{j_s,w,i_s}(\mathfrak{R}_{j_s}))$$

By Theorem 2.2, all the paths in $\mathfrak{A}$ from any state to itself have to pass through the initial state. So, iterating the previous identity, we eventually get to the state 1. Therefore

$$\bigcup_{(i_s,w,1)\in E} f_{i_s,w,1}(\mathfrak{R}_{i_s}) = \bigcup_{(p_l,a_l,q_l)\cdots(p_0,a_0,q_0)\in C_{i_s}} f_{p_0,a_0,q_0} \circ f_{p_1,a_1,q_1} \circ \cdots \circ f_{p_{l-1},a_{l-1},q_{l-1}} \circ f_{p_l,a_l,q_l}(\mathfrak{R}_1).$$

Hence

$$\mathfrak{R}_1 = \bigcup_{s=1}^{r} \bigcup_{(p_l,a_l,q_l)\cdots(p_0,a_0,q_0)\in C_{i_s}} f_{p_0,a_0,q_0} \circ f_{p_1,a_1,q_1} \circ \cdots \circ f_{p_{l-1},a_{l-1},q_{l-1}} \circ f_{p_l,a_l,q_l}(\mathfrak{R}_1). \quad (3.3)$$

Let n_j be the non-negative integers such that $\epsilon(n_j) = a_l \ldots a_0$, for $(p_l, a_l, q_l) \cdots (p_0, a_0, q_0) \in C_{i_s}$, for $1 \leq s \leq r$. We can say $1 \leq j \leq N$, for some N. So each n_j is associated with a path in $\mathfrak{A}$ that starts and terminates at the initial state, and it does not pass through the initial state at any intermediate stage. On the other hand all those paths are elements of some C_{i_s}. Therefore the $n_1, \ldots, n_N$ are the generators of $(\mathcal{P}', \star)$.

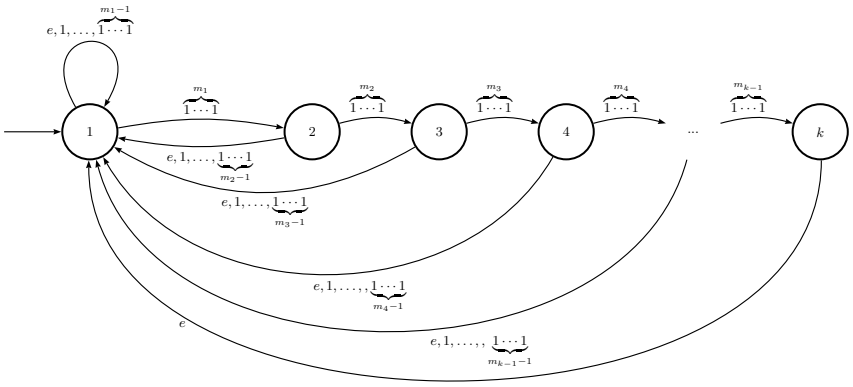

Fig. 4 The automaton of the substitution of type (3.4).

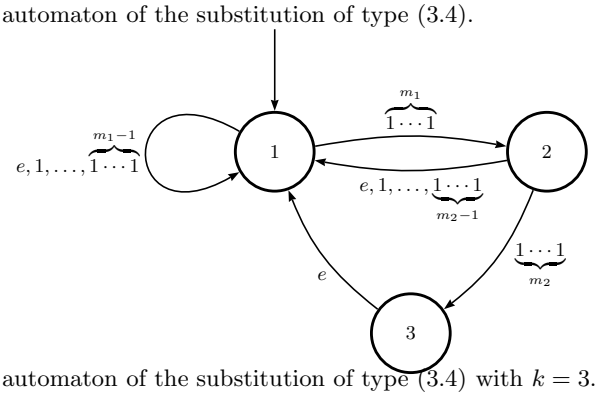

Fig. 5 The automaton of the substitution of type (3.4) with $k = 3$.

Let $F(n_j)$ be the maps defined before. By (3.3):

$$\mathfrak{R}_1 = F_{n_1}(\mathfrak{R}_1) \cup \cdots \cup F_{n_N}(\mathfrak{R}_1).$$

Since the $f_{i,w,j}$ are contractions then F_{n_j} are also contractions. Therefore $\mathfrak{R}_1$ is the fixed point of the IFS $\{F_{n_1}, \ldots, F_{n_N}\}$. □

Corollary 3.3 *If the set $\mathfrak{R}_1$ is a fixed point of an IFS, whose maps are associated with paths in pre-fix automaton then $(\mathcal{P}', \star)$ is finitely generated.*

Proof. Follows from the proof of Theorem 3.2 □

Example 4. Let us consider the family of substitutions:

$$1 \to \underbrace{1\cdots 1}_{m_1} 2,\ 2 \to \underbrace{1\cdots 1}_{m_2} 3,\ \ldots,\ (k-1) \to \underbrace{1\cdots 1}_{m_{k-1}} k,\ k \to 1, \tag{3.4}$$

with $m_1 \geq m_2 \geq \cdots m_{k-1} \geq 1$. This is a generalization of the k-bonacci substitutions. Their prefix-automaton can be seen in Figure 4, when $k = 3$ the corresponding automaton is shown in Figure 5. In these figures different transitions between two states are represented by same "arrow", with different labels. This family of substitutions has been studied by different authors, for instance see [30].

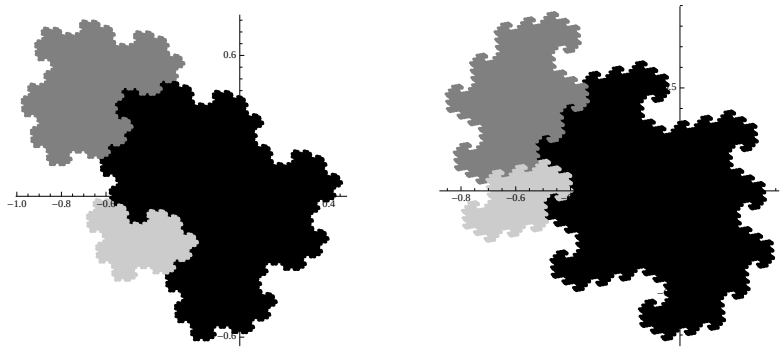

Fig. 6 Rauzy fractals of Example 3 and their natural decomposition, with $k = 3$, $m_1 = 2$, $m_2 = 1$ (left) and $m_1 = m_2 = 2$ (right).

For simplicity, we suppose $k = 3$. Let β be one of the complex eigenvalues of the incidence matrix of the substitution. The maps of the GIFS are: $f_{ijw} = \beta z + \Delta(w)$, where $\Delta(e) = 0$, and $\Delta(1) = 1$. Since w is of the form e or $\underbrace{1 \cdots 1}_{l}$, $\Delta(w) = 0$ or $\Delta(w) = l$. In order to simplify the notation we introduce the maps $g_{i,l,j}(z) = f_{i,w,j}(z)$, where $\Delta(w) = l$.

Let $\{\mathfrak{R}_1, \mathfrak{R}_2, \mathfrak{R}_3\}$ be the compact solution of the GIFS:

$$\mathfrak{R}_1 = \left(\bigcup_{l=0}^{m_1-1} g_{1,l,1}(\mathfrak{R}_1) \right) \cup \left(\bigcup_{l=0}^{m_2-1} g_{2,l,1}(\mathfrak{R}_2) \right) \cup g_{3,0,1}(\mathfrak{R}_3),$$

$$\mathfrak{R}_2 = g_{1,m_1,2}(\mathfrak{R}_1),\ \mathfrak{R}_3 = g_{2,m_2,3}(\mathfrak{R}_3).$$

Let s_i be the positive integers, such that $\epsilon(s_i) = \overbrace{1 \cdots 1}^{i}$, for $1 \leq m_1 - 1$. We define

$$n_i := \begin{cases} s_i, \text{ if } 1 \leq i \leq m_1 - 1; \\ m_1 \star s_i, \text{ if } m_1 \leq i \leq m_1 + m_2 - 1; \\ m_1 \star m_2 \star 0, \text{ if } i = m_1 + m_2. \end{cases}$$

These $m_1 + m_2$ numbers are the generators $(\mathcal{P}', \star)$, mentioned in Theorem 2.2. According to Theorem 3.2, the set $\mathfrak{R}_1$ is the fixed point of the IFS, $\{F_{n_i}\}_{i=1}^{m_1+m_2}$, where

$$F_{n_i} := \begin{cases} f_{1,\epsilon(s_i),1}, \text{ if } 1 \leq i \leq m_1 - 1; \\ f_{2,\epsilon(s_i),1} \circ f_{1,\epsilon(m_1),2}, \text{ if } m_1 \leq i \leq m_1 + m_2 - 1; \\ f_{3,e,1} \circ f_{2,\epsilon(m_2),3} \circ f_{1,\epsilon(m_1),2}, \text{ if } i = m_1 + m_2. \end{cases}$$

This fact allow us to have to have an easy description of the self-similar structure of the other sets of the natural decomposition, since $\mathfrak{R}_2 = f_{1,\epsilon(m_1),2}(\mathfrak{R}_1)$, and $\mathfrak{R}_3 = f_{2,\epsilon(m_2),3} \circ f_{1,\epsilon(m_1),2}(\mathfrak{R}_1)$.

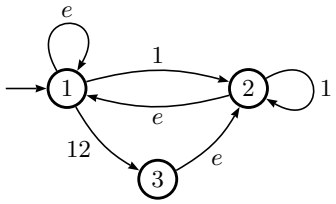

Fig. 7 The prefix automaton of the Kolakoski substitution.

Example 5. The substitution $\zeta(1) = 123, \zeta(2) = 12, \zeta(3) = 2$, is known as the *Kolakoski substitution* and it has been studied in [4]. Its prefix automaton is shown in Figure 7. This substitution is Pisot unimodular, however $(\mathcal{P}', \star)$ is not finitely generated, since there is a transition from the state 2 to itself. According to Corollary 3.3 the elements of the natural decompositions are not fixed points of IFS, whose maps are associated with paths in the prefix automaton.

References

1. Allouche, J.-P.; Shallit, J. – *Automatic Sequences. Theory, applications, generalizations.*, Cambridge University Press, Cambridge, 2003, xvi+571 pp.
2. Arnoux, P. – *Some remarks about Fibonacci multiplication*, Appl. Math. Lett., **2** (1989), no. 4, 319–320.
3. Arnoux, P.; Itô, S. – *Pisot substitutions and Rauzy fractals*, Journes Montoises d'Informatique Thorique (Marne-la-Valle, 2000), Bull. Belg. Math. Soc. Simon Stevin, **8** (2001), no. 2, 181–207.
4. Baake, M.; Sing, B. – *Kolakoski-*$(3, 1)$ *is a (deformed) model set*, Canad. Math. Bull. **47** (2004), no. 2, 168–190.
5. Berthé, V.; Rigo, M. – *Abstract numeration systems and tilings*, in *Proc. 32nd Symposium, Mathematical Foundations of Computer Science*, Lecture Notes in Computer Science, **3618**, 131–143. Springer-Berlin, 2005.
6. Berthé, V.; Rigo, M. (ed.) – *Preliminaries. Combinatorics, automata and number theory*, Encyclopedia of Mathematics and its Applications **135**, Cambridge University Press, Cambridge, 2010, 1–33.
7. Bourbaki, N. – *Éléments de mathématique. Algbre. Chapitres 1 à 3.* (French), Hermann, Paris 1970 xiii+635 pp.
8. Canterini, V.; Siegel, A. – *Geometric representation of substitutions of Pisot type*, Trans. Amer. Math. Soc., **353** (2001), no. 12, 5121–5144.
9. Dumont, J.-M.; Thomas, A. – *Systèmes de numération et fonctions fractales relatifs aux substitutions* (French) [[Numbering systems and fractal functions relative to substitutions]], Theoret. Comput. Sci. **65** (1989), no. 2, 153–169.
10. Dumont, J.-M.; Thomas, A. – *Digital sum moments and substitutions*, Acta Arith. **64** (1993), no. 3, 205–225.
11. Eilenberg, S. – *Automata, languages and machines* Vol. A. Pure and Applied Mathematics, Vol. 58., Academic Press [A subsidiary of Harcourt Brace Jovanovich, Publishers], New York, 1974, xvi+451 pp.
12. Falconer, K. – *Fractal geometry. Mathematical foundations and applications*, John Willey & Sons, Ltd. Chichester, 1990, xxii+288 pp.
13. Falconer, K. – *Techniques in fractal geometry*, John Willey & Sons, Ltd., Chichester, 1997, xviii+256 pp.
14. Frougny, C. – *Representations of numbers and finite automata*, Math. Systems Theory **25** (1992), no. 1, 37–60 .
15. Grabner, P.J.; Liardet, P.; Tichy, R.F. – *Odometers and systems of numeration*, Acta Arith. **70** (1995), no. 2, 103–123.
16. Holton, C.; Zamboni, L.Q. – *Geometric realization of substitutions*, Bull. Soc. Math. France **126** (1998), no. 2, 149–179.
17. Knuth, D.E. – *Fibonacci multiplication*, Appl. Math. Lett., **1** (1988), no. 1, 57–60.

18. Fraenkel, A.S.; Porta, H.; Stolarsky, K.B. – *Some arithmetic semigroups*, Analytic Number Theory (Allerton Park, IL, 1989): Proceedings of a Conference in Honor of Paul T. Bateman, edited by B.C. Berndt et al., Progr. Math., 85, Birkhauser Boston, Boston, MA, 1990, 255-264.
19. Mauldin, R.D.; Williams, S.C. – *Hausdorff dimension in graph directed constructions*, Trans. Amer. Math. Soc., **309** (1988), no. 2, 811–829.
20. Messaoudi, A. – *Généralisation de la multiplication de Fibonacci* (French) [[Generalization of Fibonacci multiplication]], Math. Slovaca, **50** (2000), no. 2, 135–148.
21. Messaoudi, A. – *Tribonacci multiplication*, Appl. Math. Lett., **15** (2002), no. 8, 981–985.
22. Pytheas Fogg, N. – Substitutions in Dynamics, Arithmetics and Combinatorics, (edited by V. Berthé, S. Ferenczi, C. Mauduit, et A. Siegel), Lecture Notes in Mathematics **1794**, Springer, Berlin, 2002.
23. Rauzy, G. – *Nombres algébriques et substitutions* (French) [[Algebraic numbers and substitutions]], Bull. Soc. Math. France **110** (1982), no. 2, 147–178 .
24. Rauzy, G. – *Sequences defined by iterated morphisms*, in: R.M. Capocelli (ed.), Sequences, Springer, New York, 1990, 275–286.
25. Sing, B. – *Pisot Substitutions and Beyond*, PhD-thesis, Universität Bielefeld, 2007. Available at http://nbn-resolving.de/urn/resolver.pl?urn=urn:nbn:de:hbz:361-11555.
26. Sirvent, V.F. – *A semigroup associated with k-bonacci numbers with dynamic interpretation*, The Fibonacci Quarterly, **35** (1997), no. 4, 335–340.
27. Sirvent, V.F. – *Semigroups and the self-similar structure of the flipped Tribonacci substitution*, Appl. Math. Lett., **12** (1999), no. 1, 25–29.
28. Sirvent, V.F. – *Integer semi-groups associated with Dumont-Thomas numeration systems*, ISRN Combinatorics., 2014, Article Id. 275260, 5p. Available at https://www.hindawi.com/journals/isrn/2014/275260/.
29. Sirvent, V.F.; Wang, Y. – *Self-affine tiling via substitution dynamical systems and Rauzy fractals*, Pacific J. Math., **206** (2002), no. 2, 465–485.
30. Thuswaldner, J.M. – *Unimodular Pisot substitutions and their associated tiles*, J. Th. Nombres Bordeaux **18** (2006), no. 2, 487–536.