Analele ştiinţifice
ale Universităţii „Al. I. Cuza" din Iaşi
Tomul XXX, s. I a, Matematică, 1984 — 4

Romanian - Japanese Colloquium
on Finsler Geometry, Iaşi,
Romania, 16—25 August, 1984

ON THE FINSLERIAN METRICAL STRUCTURES OF THE GRAVITATIONAL FIELD

BY

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1. Introduction. Einstein's gravitational field [2] has a four-dimensional Riemannian structure (R_4), which is wholly dominated by the Riemann metric $\gamma_{\lambda\varkappa}$. On the other hand, there have been existing several kinds of „non"-Riemannian theories ($\overline{R}_4$) of gravitational field such as Brans-Dicke theory [1], Weyl-Dirac theory with torsion [3], etc. In these theories, such „non"-Riemannian quantities as conformal scalar, torsion, etc. have been introduced, besides $\gamma_{\lambda k}$, but they are not „nonlocalized" in the sense that the independent variable is the point x alone. Therefore, if a new independent variable is attached to each point, then a new „non"-Riemannian and „nonlocalized" gravitational field can be realized. Along this line, in this paper, we shall choose the internal vector y as such independent variable and propose two Finslerian metrical structures (F_4) (i.e., eqs.(2) and (7)).

2. Finslerian metrical structure—I. Within the framework of Finslerian gravitational field [4], the most essential problem is to explain concretely how the „deviation" of the Finsler metric $g_{\lambda\varkappa}$ from $\gamma_{\lambda\varkappa}$ is stipulated by y. At this stage, there appear two fields around x: One is Einstein's gravitational field, which is the external (x)-field with R_4 spanned by points $\{x\}$, and the other is the internal (y)-field with R_4 or $\overline{R}_4$ spanned by vectors $\{y\}$. Here, it should be remarked that the indices $\varkappa$, $\lambda(=1, 2, 3, 4)$ are used for the external quantity, while the indices i, j $(=1, 2, 3, 4)$ are used for the internal quantity.

Then, our problem is to „unify" the (x)- and (y)-fields and obtain, as the final result, a Finsler metric $g_{\lambda\varkappa}$ by „unifying" $\gamma_{\lambda\varkappa}$ of the (x)-field and the Riemann metric h_{ij} (or the Minkowski metric η_{ij}) of the (y)-field. This „unification" is realized by „reflecting" y^i and h_{ij} in the (x)-field. The „reflection" process is given by the following mapping relation:

$$y^{\varkappa}=e^{\varkappa}_i(x)y^i\,; \qquad h_{\lambda\varkappa}(x,\,y)=e^i_{\lambda}(x)e^j_{\varkappa}(x)h_{ij}(y). \tag{1}$$

y^i plays a role of the one-form and h_{ij} the one-form metric [7]. Therefore, the „unified" Finsler metric $g_{\lambda\varkappa}$ can be given by

$$g_{\lambda\varkappa}(x,\,y)=\gamma_{\lambda\varkappa}(x)+h_{\lambda\varkappa}(x,\,y). \tag{2}$$

It should be remarked that the order of $h_{\lambda\varkappa}$ is considered much smaller than that of $\gamma_{\lambda\varkappa}$, so that the inverse of $g_{\lambda\varkappa}$ may be given by $g^{\varkappa\lambda}=\gamma^{\varkappa\lambda}-h^{\varkappa\lambda}$, as the first order approximation. Therefore, the indices should be raised or lowered by $\gamma^{\varkappa\lambda}$ and $\gamma_{\lambda\varkappa}$ and the higher order terms $\geqq h^2$ should be neglected in practical calculations for physical problems.

3. Intrinsic behaviour of the internal variable (y). In general, the internal vector y shows its own intrinsic behaviour. Geometrically, it is grasped by its own „parallelism" δy^i in the internal field, which is embedded in the „unified" field by $\delta y^\varkappa=e_i^\varkappa\delta y^i$. An example of $\delta y^\varkappa$ can be obtained from eq. (1) as follows:

$$\delta y^\varkappa=dy^\varkappa+\Lambda^\varkappa_{\lambda\mu}(x)y^\lambda dx^\mu(=0), \tag{3}$$

where $\Lambda^\varkappa_{\lambda\mu}=e_i^\varkappa\partial e^i_\lambda/\partial x^\mu$, which resembles the one-form linear connection [7]. Eq. (3) means the absolute parallelism of e, so that the metrical conditions for $h_{\lambda\varkappa}$, i.e., $\delta h_{\lambda\varkappa}=0$ can be assumed. In our case, $\delta y^\varkappa$ must be reflected in the whole spatial structure of the Finslerian gravitational field, which is represented by the metrical Finsler connection D for $g_{\lambda\varkappa}$ of eq. (2). The connection D is represented by the following absolute differential:

$$DX^\varkappa=dX^\varkappa+F^\varkappa_{\lambda\mu}(x,y)X^\lambda dx^\mu+C^\varkappa_{\lambda\mu}(x,y)X^\lambda\delta y^\mu, \tag{4}$$

where $F^\varkappa_{\lambda\mu}\equiv\Gamma^\varkappa_{\lambda\mu}-N^\nu_\mu C^\varkappa_{\lambda\nu}$, N^ν_μ being the nonlinear connection defined by $N^\nu_\mu=\Lambda^\nu_{\lambda\mu}y^\lambda$ (see eq. (3)). It should be noted that in our case, $Dy^\varkappa\neq\delta y^\mu$, and $Dg_{\lambda\varkappa}=0$ & $Dh_{\lambda\varkappa}\neq0$, and $\delta h_{\lambda\varkappa}=0$ & $\delta g_{\lambda k}\neq0$.

The relation between Dy and δy can be obtained by reconsidering the relation $Dg_{\lambda\varkappa}=0$ & $\delta g_{\lambda\varkappa}\neq0$ as follows: The connection D is a metrical connection for $g_{\lambda\varkappa}$ derived from the non-metrical one δ. Then, by use of Kawaguchi's theorem [6], the desiring relation can be obtained in the form

$$Dy^\varkappa=\delta y^\varkappa+\frac{1}{2}g^{\varkappa\nu}(\delta g_{\nu\lambda})y^\lambda, \tag{5}$$

by which the following relation can be obtained from eqs. (3) and (4):

$$F^\varkappa_{\lambda\mu}=\Lambda^\varkappa_{\lambda\mu}+\frac{1}{2}g^{\varkappa\nu}\left(\frac{dg_{\nu\lambda}}{dx^\mu}-\Lambda^\iota_{\nu\mu}g_{\iota\lambda}-\Lambda^\iota_{\lambda\mu}g_{\nu\iota}\right), \tag{6}$$

where $\dfrac{d}{dx^\mu}=\dfrac{\partial}{\partial x^\mu}-N^\nu_\mu\dfrac{\partial}{\partial y^\nu}$.

4. Finslerian metrical structure—II. Now, if in eq. (2), $h_{\lambda\varkappa}$ is assumed to be proportional to $\gamma_{\lambda\varkappa}$, then $g_{\lambda\varkappa}$ becomes conformal to $\gamma_{\lambda\varkappa}$. On the other hand, the conformally Riemannian structure $\gamma_{\lambda\varkappa}(x)\exp 2\sigma(x)$ has been well known in connection with Brans-Dicke theory [1] (cf. [5]). Bearing in mind these situations, we shall here propose the following Finslerian conformal structure:

$$g_{\lambda\varkappa}(x,y)=\gamma_{\lambda\varkappa}(x)\exp 2\sigma(x,y), \tag{7}$$

where the conformal scalar σ becomes a function of (x,y) and summarizes the „deviation" of F_4 from R_4. Starting from eq. (7), we can determine several

kinds of metrical Finsler connections. One interesting fact has been found [8]: If σ is given by $\sigma=(\gamma_{\lambda\varkappa}y^\varkappa y^\lambda)^a(a\geqq0)$, then $F^\varkappa_{\lambda\mu}$ reduces to $\{^\varkappa_{\lambda\mu}\}$.

In order to consider more physically the case of eq. (7), in the following we shall regard y as the space-time fluctuation and assume y to be given by a function of x. Then, the F_4-structure is reduced to conformally Riemannian or non-Riemannian, according as the torsion$=0$ or $\neq0$. In this case, $g_{\lambda\varkappa}$, $F^\varkappa_{\lambda\mu}$ and $R^\varkappa_{\nu\lambda\mu}$ (third curvature) are reduced to, respectively,

$$\begin{aligned}\bar{g}_{\lambda\varkappa}(x)&=\gamma_{\lambda\varkappa}(x)\exp 2\sigma(x,y(x)),\\ \bar{F}^\varkappa_{\lambda\mu}(x)&=\{^\varkappa_{\lambda\mu}\}+(\delta^\varkappa_\lambda\sigma_\mu+\delta^\varkappa_\mu\sigma_\lambda-\bar{g}^{\varkappa\nu}\bar{g}_{\lambda\mu}\sigma_\nu)+\Theta^\varkappa_{\lambda\mu}(\bar{T}),\\ \bar{R}^\varkappa_{\nu\lambda\mu}(x)&=K^\varkappa_{\nu\lambda\mu}(\{\ \})+M^\varkappa_{\nu\lambda\mu}(\bar{T}),\end{aligned} \tag{8}$$

where the torsion $\bar{T}^\varkappa_{\lambda\mu}(\equiv\bar{F}^\varkappa_{\lambda\mu}-\bar{F}^\varkappa_{\mu\lambda})$ has been introduced. (In eq. (8), $K^\varkappa_{\nu\lambda\mu}$ denotes the Riemannian curvature, $M^\varkappa_{\nu\lambda\mu}$ is defined as the rest and $\Theta^\varkappa_{\lambda\mu}$ represents the contortion tensor; $\sigma_\lambda=\partial\sigma/\partial x^\lambda$). Concerning eq. (8), if the torsion $\bar{T}^\varkappa_{\lambda\mu}$ has such a special form as (cf. [5])

$$\bar{T}^\varkappa_{\lambda\mu}(x)=\delta^\varkappa_\lambda\sigma_\mu-\delta^\varkappa_\mu\sigma_\lambda, \tag{9}$$

then the torsional effects do not appear at the stage of curvature. Therefore, Einsteinian field equation becomes conformally invariant. This kind of conformal scalar is now considered within the framework of gauge covariant vacuum equation (cf. [3, 5]).

Finally, we shall proceed to the averaging process with respect to y. In general, this process is formally written as, e.g.,

$$<g_{\lambda\varkappa}>=\int g_{\lambda\varkappa}(x,y)f(y)\,(dy)^4, \tag{10}$$

where $f(y)$ means a distribution function of $\{y\}$. But it is very difficult to apply $<\ >$ to Finsler quantites in their general formulas, so that we cannot help assuming again that y is a function of x. Then, the averaged values of eq. (8) can be given by, in general,

$$\begin{aligned}&<\bar{g}_{\lambda\varkappa}>=\gamma_{\lambda\varkappa},\quad <\bar{F}^\varkappa_{\lambda\mu}>=\{^\varkappa_{\lambda\mu}\}+<\Theta^\varkappa_{\lambda\mu}(\bar{T})>,\\ &<\bar{R}^\varkappa_{\nu\lambda\mu}>=K^\varkappa_{\nu\lambda\mu}(\{\ \})+<M^\varkappa_{\nu\lambda\mu}(\bar{T})>,\end{aligned} \tag{11}$$

where we have assumed the reasonable conditions $<\exp\sigma>=1$ and $<\sigma_\lambda>=0$. Eq. (11) shows that the resulting macro-field does not necessarily become Riemannian owing to the contributions of $\bar{T}^\varkappa_{\lambda\mu}$. But, if eq. (9) is assumed, then the macro-field becomes purely Riemannian.

In future, we should consider the relation between the conformal scalar σ which makes the field equation conformally invariant and the conformal scalar σ which satisfies the conditions $<\exp\sigma>=1$ and $<\sigma_\lambda>=0$. As is understood from the above, however, the averaging process $<\ >$ itself cannot yet be „geometrized". This is a very difficult open problem.

5. Conclusions. Thus, two kinds of Finslerian metrical structures, i.e., eqs. (2) and (7) have been proposed, where the „deviations" are prescribed by the „addition" process $(\gamma_{\lambda\varkappa}+)$ and the „multiplication" process $(\gamma_{\lambda\varkappa}\times)$

respectively. The whole Finsler connection (i.e., $Dg_{\lambda\varkappa}=0$) is determined by taking account of the intrinsic behaviour of y (i.e., δy). And some physical problems such as the averaging process $<\ >$, etc. have been considered under the assumption of „osculation" $y=y(x)$. In future, some other essential „unifications" of $\gamma_{\lambda\varkappa}(x)$ and $h_{ij}(y)$ should be investigated.

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NON-HOLONOMIC FRAMES IN A FINSLER SPACE

BY

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§0. Introduction. Let M be an n-dimensional manifold with coordinate chart (U, Φ_U) and $T(M)=(T, \pi, M, V)$ a tangent vector bundle with extended coordinate (*ex-c.*) chart $(\pi^{-1}(U), \Phi'_U)$ such that, for a vector $y\in\pi^{-1}(U)$, $\pi(y)=p\in U$, $\Phi_U(p)=(x^i)\in R_n$ and by means of natural frame $(\partial/\partial x^i)_p$, we see $y=y^i(\partial/\partial x^i)_p$, then we have $\Phi'_U(y)=(x^i, y^i)_U$. Let us denote by M_T a manifold of non vanishing vectors, that is, $M_T:=T(M)-\{0\}$, which is called a *line element space*. An n-dimensional Finsler space $F_n(M_T$ $F(x, y))$ is a manifold M_T with a Finsler metric $F(x, y)$.

The purposes of the present paper are to show the existence of non-holonomic frame $(\tilde{U}, X^i_\alpha)$ in M_T with non linear connection N on using the theory of G-structure and to introduce absolute parallel connection $(\Lambda\Gamma)$ associated with non-holonomic frame and lastly to investigate a Finsler space with metric $F(x, y)=f(\xi^\alpha(x, y))$.

Notations used here are the same as those in [2].

§1. Preliminaries of Finsler space and the existence theorem. It is well known that in a Finsler space the non linear connection $N=(G^i_k)$ is determined by a Finsler metric $F(x, y)$ uniquely. Generally, using non linear connection $N=(N^i_k)$, the tangent space $T_y(M_T)$ at y is decomposed into two distributions $H_y(M_T)$ and $V_y(M_T)$ in the following way. For the basis $(\partial/\partial x^i)_y$ and $(\partial/\partial y^i)_y$, we shall introduce a set of new basis of $T_y(M_T)$:

$$(d, d') := (\partial x, \partial y)\begin{pmatrix} \delta & 0 \\ -N & \delta \end{pmatrix}; \quad d=\partial x - N\partial y, \ d'=\partial y,$$

where, for the brevity, we use the matrix notation :

$$d=(d_k), \ d'=(d_{(k)}) ; \ \partial x=(\partial/\partial x^i), \ \partial y=(\partial/\partial y^i) \text{ and } \delta=(\delta^i_k).$$

The dual basis of (d, d') are given by

$$\begin{pmatrix} \theta \\ \theta' \end{pmatrix} := \begin{pmatrix} \delta & 0 \\ N & \delta \end{pmatrix}\begin{pmatrix} dx \\ dy \end{pmatrix}; \quad \theta=dx, \quad \theta'=dy+Ndx,$$

where $\theta=(\theta^i)$, $\theta'=(\theta^{(i)})$ and $dx=(dx^i)$, $dy=(dy^i)$. The distributions H_y and V_y are the linear spaces spanned by $(d_i)_y$ and $(d_{(i)})_y$ respectively or equivalently, we find