respectively. The whole Finsler connection (i.e., $Dg_{\lambda\varkappa}=0$) is determined by taking account of the intrinsic behaviour of y (i.e., δy). And some physical problems such as the averaging process $<\ >$, etc. have been considered under the assumption of „osculation" $y=y(x)$. In future, some other essential „unifications" of $\gamma_{\lambda\varkappa}(x)$ and $h_{ij}(y)$ should be investigated.

REFERENCES

1. Brans, C. and Dicke, R. H. — *Mach's principle and a relativistic theory of gravitation*. Phys. Rev., *124* (1961), 925—935
2. Einstein, A. — *Die Grundlage der allgemeinen Relativitätstheorie*. Ann. der Phys., *49* (1916), 769—822
3. Gregorash, D. and Papini, G. — *Weyl-Dirac theory with torsion*. Nuovo Cimento, *55 B* (1980), 37—51
4. Ikeda, S. — *On the theory of fields in Finsler spaces*. Jour. Math. Phys., 22 (1981), 1215—1218
5. Ikeda, S. — *Some structural features induced by the space-time fluctuation in the theory of gravitational fields*. Found. of Phys., *13* (1983), 629—636
6. Kawaguchi, A. — *Beziehung zwischen einer metrischen linearen Übertragung und einer nicht-metrischen in einem allgemeinen metrischen Raume*. Akad. Wetensch. Amsterdam Proc., *40* (1937), 596—601
7. Matsumoto, M. and Shimada, S. — *On Finsler spaces with 1-form metric*. Tensor, N.S., *32* (1978), 161—169
8. Watanabe, S. — *On some properties of generalized Finsler spaces*. Jour. Ind. Acad. Math., (to be published)

Department of Mechanical Engineering
Faculty of Science and Technology
Science University of Tokyo
Noda, Chiba 278, Japan

Analele ştiinţifice
ale Universităţii „Al. I. Cuza" din Iaşi
Tomul XXX, s. I a, Matematică, 1984 — 4

Romanian - Japanese Colloquium
on Finsler Geometry, Iaşi,
Romania, 16—25 August, 1984

NON-HOLONOMIC FRAMES IN A FINSLER SPACE

BY

HIDEO IZUMI

§0. Introduction. Let M be an n-dimensional manifold with coordinate chart (U, Φ_U) and $T(M)=(T, \pi, M, V)$ a tangent vector bundle with extended coordinate (*ex-c.*) chart $(\pi^{-1}(U), \Phi'_U)$ such that, for a vector $y\in\pi^{-1}(U)$, $\pi(y)=p\in U$, $\Phi_U(p)=(x^i)\in R_n$ and by means of natural frame $(\partial/\partial x^i)_p$, we see $y=y^i(\partial/\partial x^i)_p$, then we have $\Phi'_U(y)=(x^i, y^i)_U$. Let us denote by M_T a manifold of non vanishing vectors, that is, $M_T:=T(M)-\{0\}$, which is called a *line element space*. An n-dimensional Finsler space $F_n(M_T$ $F(x, y))$ is a manifold M_T with a Finsler metric $F(x, y)$.

The purposes of the present paper are to show the existence of non-holonomic frame $(\tilde{U}, X^i_\alpha)$ in M_T with non linear connection N on using the theory of G-structure and to introduce absolute parallel connection $(\Lambda\Gamma)$ associated with non-holonomic frame and lastly to investigate a Finsler space with metric $F(x, y)=f(\xi^\alpha(x, y))$.

Notations used here are the same as those in [2].

§1. Preliminaries of Finsler space and the existence theorem. It is well known that in a Finsler space the non linear connection $N=(G^i_k)$ is determined by a Finsler metric $F(x, y)$ uniquely. Generally, using non linear connection $N=(N^i_k)$, the tangent space $T_y(M_T)$ at y is decomposed into two distributions $H_y(M_T)$ and $V_y(M_T)$ in the following way. For the basis $(\partial/\partial x^i)_y$ and $(\partial/\partial y^i)_y$, we shall introduce a set of new basis of $T_y(M_T)$:

$$(d, d') := (\partial x, \partial y)\begin{pmatrix} \delta & 0 \\ -N & \delta \end{pmatrix}; \quad d=\partial x - N\partial y, \ d'=\partial y,$$

where, for the brevity, we use the matrix notation :

$$d=(d_k), \ d'=(d_{(k)}) ; \ \partial x=(\partial/\partial x^i), \ \partial y=(\partial/\partial y^i) \text{ and } \delta=(\delta^i_k).$$

The dual basis of (d, d') are given by

$$\begin{pmatrix} \theta \\ \theta' \end{pmatrix} := \begin{pmatrix} \delta & 0 \\ N & \delta \end{pmatrix}\begin{pmatrix} dx \\ dy \end{pmatrix}; \quad \theta=dx, \quad \theta'=dy+Ndx,$$

where $\theta=(\theta^i)$, $\theta'=(\theta^{(i)})$ and $dx=(dx^i)$, $dy=(dy^i)$. The distributions H_y and V_y are the linear spaces spanned by $(d_i)_y$ and $(d_{(i)})_y$ respectively or equivalently, we find

$$H_y(M_T)=\{X\in T_y(M_T)\mid \theta'(X)=0\},$$
(1.1)
$$V_y(M_T)=\{X\in T_y(M_T)\mid \theta(X)=0\}.$$

On the intersection of two ex-c. chart, we have ex-c. transformation μ': $\bar{x}^i=\mu^i(x)$, $\bar{y}^i=\mu^i_j y^j$; $\mu^i_j:=\partial\mu^i/\partial x^j$ and it is known that

$$\begin{matrix}\bar{d}\mu=d, & \bar{d}'\mu=d', \\ \bar{\theta}=\mu\theta, & \bar{\theta}'=\mu\theta',\end{matrix} \text{ or } (\bar{d},\bar{d}')=(d,d')\begin{pmatrix}\mu^{-1} & 0\\ 0 & \mu^{-1}\end{pmatrix},\begin{pmatrix}\bar{\theta}\\ \bar{\theta}'\end{pmatrix}=\begin{pmatrix}\mu & 0\\ 0 & \mu\end{pmatrix}\begin{pmatrix}\theta\\ \theta'\end{pmatrix};\ \mu=(\mu^i_j).$$

Hence we find from (1.1) that, under any ex-c. transformation μ',
(A1) *two distributions $H_y(M_T)$ and $V_y(M_T)$ are preserved.*

Because the distributions H_y, V_y and $T_p(M)$ are isomorphic to R_n, we can define horizontal (vertical) lift $l_N(l^V)$ as follows :

$$l_N : T_p(M)\to H_y(M_T)\mid l_N\,.\,(\partial x)_p:=(d)_y=(\partial x)_y-N(\partial y)_y,$$
(1.2)
$$l^V : T_p(M)\to V_y(M_T)\mid l^V\,.\,(\partial x)_p:=(d')_y=(\partial y)_y.$$

Let $L(M)=(L,\tau,M,GL(n))$ be a frame bundle of M with ex-c. chart $(\tau^{-1}(U),\Phi'_U)$: $z_n=\{z_a\}\in L$, $z_a:=z^i_a(\partial/\partial x^i)_p$; $\tau(z_n)=\pi(z_a)=p$, $\Phi'_U(z_n)=(x^i, z^i_a)$, where n-vectors z_a form a set of basis of $T_p(M)$. On the other hand, the frame bundle $L(M_T)=(\bar{L},\bar{\tau},M_T,GL(2n))$ of M_T consists of $2n$ basis vectors of $T_y(M_T)$. Let us denote P_N the image of the product lift $\bar{l}_N:=l_N\times l^V : L(M)\to L(M_T)$. On taking account of (1.2), we find that, for $u=(B,B')\in P_N$, $B=(B_a)$, $B'=(B_{(a)})$, we get $B:=l_N\,.\,z_n$, $B':=l^V\,.\,z_n$ and then $B_a=z^i_a d_i$, $B_{(a)}=z^i_a d_{(i)}$; $(B,B')=(d,d')\begin{pmatrix}z & 0\\ 0 & z\end{pmatrix}$, $z=(z^i_a)$.

Thus we shall define the action of $g\in GL(n)$ to $u\in P_N$ as follows :

$$u\bar{g}:=(Bg,B'g)=(B,B')\begin{pmatrix}g & 0\\ 0 & g\end{pmatrix},\ \bar{g}:=\begin{pmatrix}g & 0\\ 0 & g\end{pmatrix}\in\bar{G}:=D(GL(n))\subset GL(2n),$$

so that (A1) is satisfied. Evidently $P_N(M_T)=(P_N,\bar{\tau},M_T,GL(n))$ is a manifold with ex-c. chart $(\bar{\tau}^{-1}\pi^{-1}(U),\Phi'_U)$ and can be identified with the sub-frame bundle $B_{\bar{G}}=(P_N,\bar{\tau},M_T,\bar{G})$ of $L(M_T)$. Summarizing the above stated, we can understand that, in virtue of the existence of non linear connection N, the structure group $GL(2n)$ of the frame bundle $L(M_T)$ is reducible to its sub-group $\bar{G}=D(GL(n))$. Hence from the theory of G-structure [1], we find that, for each $y\in M_T$, there exist a neighbourhood $\tilde{U}\subset M_T$ of y and a differential map σ: $\tilde{U}\to L(M_T)$ such that $\sigma(\tilde{U})\subset B_{\bar{G}}$. More precisely for a vector $y\in\tilde{U}$ and σ, we have $\sigma(y)=(X,X')\in B_{\bar{G}}=P_N$ and hence there, exist a matrix $X=(X^i_\alpha(x,y))\in GL(n)$ such that $X_\alpha:=X^i_\alpha(d_i)_y$ and $X^i_{(\alpha)}:=X^i_\alpha(d_{(i)})_y$ are the set of basis of $T_y(M_T)$ which satisfies (A1). A pair $(\tilde{U},X^i_\alpha)$ is called a *non-holonomic frame* on $\tilde{U}\subset M_T$. Evidently $\tilde{U}\subset\pi^{-1}\pi(\tilde{U})$, we see $\Phi'_U(y)=(x^i,y^i)$ for any $y\in\tilde{U}$. On the other hand, we can define a *non-holonomic* (n.h.) *chart* $(\tilde{U},\tilde{\Phi})$ such that $\tilde{\Phi}(y)=(x^i,\xi^\alpha(x,y))$, where $\Phi_U(\pi(y))=(x^i)$, $U=\pi(\tilde{U})$ and $y=\xi^\alpha(X_\alpha)_{\tilde{U}}$. Thus we can state.

Theorem 1.1. *If there is given a non linear connection N in a line element space M_T, we have a set of non-holonomic frame and its non-holonomic coordinate system which preserve the decomposition of $T_y(M_T)$ with respect to given non linear connection N.*

Remark. This result is available to any manifold $(M_T, N^i_k(x,y))$.

As the usual way, a connection form $\omega=(\omega^i_j)$ of $T^*(M_T)$ gives the decomposition of the tangent space $T_u(P_N(M_T))$ at $u\in P_N$, that is, $T_u(P_N)=\Gamma_u(P_N)\oplus G_u(P_N)$ such that

$$\Gamma_u(P_N):=\{X\in T_u(P_N)\mid\tilde{\omega}(X)=0\},\quad G_u(P_N):=\{X\in T_u(P_N)\mid\tilde{\theta}(X)=0\}.$$

where $\tilde{\omega}=(\tilde{\omega}^a_b)$ and $\tilde{\theta}=(\tilde{\theta},\tilde{\theta}')$;

$$\tilde{\omega}:=z^{-1}(dz+\omega z),\ \tilde{\theta}:=z^{-1}dx,\ \tilde{\theta}':=z^{-1}(dy+Ndx),\ z^{-1}=(z^a_i).$$

For the manifold $(M_T, N^i_k(x,y))$, we have a natural connection parameter $(N\Gamma)(N^i_k, G^i_{jk})$, $G^i_{jk}:=\dot{\partial}_j N^i_k$ and then ${}_N\omega^i_j=G^i_{jk}\,dx^k$. For a Finsler space $(M_T, F(x,y))$, it is well known that we have two types of connection $(CF)(G^i_k, {}^*\Gamma^i_{jk}, C^i_{jk})$ and $(BF)(G^i_k, G^i_{jk})$. Using connection parameters, for example of (CF), we obtain the dual basis of $\tilde{\theta}$, $\tilde{\theta}'$, $\tilde{\omega}$ as follows :

$$H_a:=z^i_a(d_i-{}^*\Gamma^k_{ij}L^j_k),\quad H_{(a)}:=z^i_a(d_{(i)}-C^k_{ij}L^j_k),$$
$$L^b_a:=z^i_a\partial/\partial z^i_b,\quad L^j_k:=z^j_a\partial/\partial z^k_a,$$

where L^b_a is the basis of left invariant vectors of $T(GL(n))$ or the element of Lie algebra of $GL(n)$. Two distributions $\Gamma_u(P_N)$ and $G_u(P_N)$ are the linear spaces spanned by H_a, $H_{(a)}$ and L^b_a respectively. The covariant differential $D=h^*\,.\,d$ on $P_N(M_T)$ or δ on M_T defined by as usual way. Under ex-c. transformation μ', the connection ω is transformed as $d\mu+\bar{\omega}\mu=\mu\omega$ so that $\overline{\delta v^i}=\mu^i_j\delta v^j$. Hence we see that, under any ex-c. transformation μ',
(A2) *parallelism is preserved.*

For the manifolds $(M_T, N^i_k(x,y))$ and $(M_T, F(x,y))$, we see

(A3) $$\theta'=\delta y=dy+\omega y.$$

§2. **Non-holonomic objects n^i_{jk} and $\bar{C}^i_{jk}$.** The dual basis of n.h. frame $(X_\alpha, X_{(\alpha)})$ are defined by $\bar{\theta}^\alpha=X^\alpha_i dx^i$, $\bar{\theta}^{(\alpha)}=X^\alpha_i(dy^i+N^i_k dx^k)$; $X^\alpha_i X^i_\beta=\delta^\alpha_\beta$. The coordinate $\xi^\alpha(x,y)$ is also given by $\xi^\alpha=X^\alpha_i(x,y)y^i$. If the 1-form $\bar{\theta}^\alpha$ is an exact differential, the n.h. frame is called to be *holonomic*. We see

$$[d\bar{\theta}^\alpha]=[dX^\alpha_i, dx^i]=X^\alpha_{i,k}[dx^k, dx^i]+X^\alpha_{i(k)}[dy^k, dx^i];$$
(2.1)
$$X^\alpha_{i,k}:=\partial_k X^\alpha_i,\ X^\alpha_{i(k)}:=\dot{\partial}_k X^\alpha_i.$$

Therefore $d\bar{\theta}^\alpha=0$ means that $X^\alpha_{i,k}=X^\alpha_{k,i}$ and $X^\alpha_{i(k)}=0$, that is, the X^α_i is a Jacobian matrix and there exist functions $\bar{x}^\alpha$ of x^i such that $\xi^\alpha(x,y)=(\partial\bar{x}^\alpha/\partial x^i)y^i$ and $X_\alpha=\partial/\partial\bar{x}^\alpha$.

We shall introduce a set of 1-form, using $\lambda^\alpha_i:=\xi^\alpha_{(i)}=X^\alpha_i+X^\alpha_{k(i)}\,y^k$,

(2.2) $$\theta^\alpha=\lambda^\alpha_i dx^i,\ \theta^{(\alpha)}=d\xi^\alpha=\lambda^\alpha_i\bar{\delta}y^i\ ;\ \bar{\delta}y^i:=dy^i+\Lambda^i_k dx^k,$$

where $\Lambda^i_k:=\lambda^i_\alpha\lambda^\alpha_{j,k}y^j$, $\lambda^\alpha_i\lambda^i_\beta=\delta^\alpha_\beta$.

For the present, we shall confine ourselves to assume

$$[X_{(\alpha)}, X_{(\beta)}]=X_{(\alpha)}X_{(\beta)}-X_{(\beta)}X_{(\alpha)}=0, \tag{A0}$$

which implies $X^{\alpha}_{i(j)}=X^{\alpha}_{j(i)}$ so that $\lambda^{\alpha}_{i}=X^{\alpha}_{i}$ and $\theta^{\alpha}=\bar{\theta}^{\alpha}$ hold.

Using (2.1), (2.2) and (A0), we have

$$[d\theta^{\alpha}]=-\lambda^{\alpha}_{i}(\tfrac{1}{2}n^{i}_{jk}[dx^{j}, dx^{k}]+\bar{C}^{i}_{jk}[dx^{j}, \bar{\delta}y^{k}]),$$

$$n^{i}_{jk}:=\lambda^{i}_{\alpha}\bar{d}_{k}\lambda^{\alpha}_{j}-j\,|\,k,\ \bar{C}^{i}_{jk}:=\lambda^{i}_{\alpha}\lambda^{\alpha}_{j(k)},\ \bar{d}_{k}:=\partial_{k}-\Lambda^{h}_{k}\dot{\partial}_{h},$$

where $j\,|\,k$ means the interchange of indices j, k of the foregoing terms.

Proposition 2.1. *The necessary and sufficient conditions for a n.h. frame $(\bar{U}, X^{i}_{\alpha}(x, y))$ of M_{T} to be holonomic are that the tensors n^{i}_{jk} and $\bar{C}^{i}_{jk}$ vanish.*

§3. Absolute parallel connection associated with a.n.h. frame. On the domain $\bar{U}$ of M_{T}, we shall define 1-form $\theta^{i}_{j}:=\lambda^{i}_{\alpha}d\lambda^{\alpha}_{j}$.

Proposition 3.1. *The connection form $\theta=(\theta^{i}_{j})$ is absolute parallel, that is, ${}_{\Lambda}\Omega:=[d\theta]+[\theta, \theta]=0$.*

Proof. Since $d\lambda=\lambda\theta$ and $d\lambda^{-1}=-\theta\lambda^{-1}$ $\lambda=(\lambda^{\alpha}_{i})$, $\lambda^{-1}=(\lambda^{i}_{\alpha})$, we find $[d\theta]=[d\lambda^{-1}, d\lambda]=-[\theta, \theta]$.

If we denote $\theta^{i}_{j}=\Lambda^{i}_{jk}dx^{k}+\bar{C}^{i}_{jk}\bar{\delta}y^{k}$, we find $\Lambda^{i}_{jk}:=\lambda^{i}_{\alpha}\bar{d}_{k}\lambda^{\alpha}_{j}$. Hence $(\Lambda\Gamma)$ $(\Lambda^{i}_{k}, \Lambda^{i}_{jk}, \bar{C}^{i}_{jk})$ is a connection parameter on $\bar{U}$ of M_{T} such that

$$v^{i}_{;k}:=\bar{d}_{k}v^{i}+\Lambda^{i}_{hk}v^{h};\ v^{i}_{;(k)}:=v^{i}{}_{(k)}+\bar{C}^{i}_{hk}v^{h},\quad \text{for } v^{i}(x, y). \tag{3.1}$$

For the curvature tensor ${}_{\Lambda}\Omega=({}_{\Lambda}R, {}_{\Lambda}P, {}_{\Lambda}S)$, we have

$${}_{\Lambda}R^{i}_{hjk}:=\bar{d}_{k}\Lambda^{i}_{hj}+\Lambda^{m}_{hj}\Lambda^{i}_{mk}+\bar{C}^{i}_{hm}\bar{d}_{k}\Lambda^{m}_{j}-j\,|\,k=0,$$

$${}_{\Lambda}H^{i}_{hjk}:=\bar{d}_{k}\Lambda^{i}_{hj}+\Lambda^{m}_{hj}\Lambda^{i}_{mk}-j\,|\,k=0,\quad {}_{\Lambda}H^{i}_{jk}:={}_{\Lambda}H^{i}_{0jk}=\bar{d}_{k}\Lambda^{i}_{j}-j\,|\,k=0, \tag{3.2}$$

$${}_{\Lambda}P^{i}_{hjk}:=\Lambda^{i}_{hj(k)}-\bar{C}^{i}_{hk;j}=0,\quad S^{i}_{hjk}:=\bar{C}^{i}_{hj(k)}+\bar{C}^{m}_{hj}\bar{C}^{i}_{mk}-j\,|\,k=0.$$

Moreover we have

Proposition 3.2. *It holds that $y^{i}_{;k}=0$ and $\Lambda^{i}_{k(j)}=\Lambda^{i}_{jk}$.*

The direct calculations give us

$$\begin{aligned}
&\text{a) } \bar{d}_{k}\bar{d}_{j}v^{i}-j\,|\,k=0, && \text{b) } (\bar{d}_{j}v^{i})_{(k)}-\bar{d}_{j}v^{i}{}_{(k)}=-\Lambda^{h}_{kj}v^{i}_{(h)},\\
&\text{c) } v^{i}_{;j;k}-j\,|\,k=-n^{h}_{jk}v^{i}_{;h}, && \text{d) } v^{i}_{;(j);(k)}-j\,|\,k=0,\\
&\text{e) } v^{i}_{;j(k)}-v^{i}_{(k);j}=\Lambda^{i}_{hj(k)}v^{h}, && \text{f) } v^{i}_{;j;k}-v^{i}_{;(k);j}=-\bar{C}^{h}_{jk}v^{i}_{;h},\\
&\text{g) } v^{i}_{;(j)(k)}-v^{i}_{(k);(j)}=\bar{C}^{i}_{hj(k)}v^{h}+\bar{C}^{h}_{jk}v^{i}_{(h)}.
\end{aligned} \tag{3.3}$$

Proposition 3.3. *Bianchi's identities $[\bar{\delta}\Omega]=0$ are derived from the identities $[dd\theta^{\alpha}]=0$.*

Proposition 3.4. *The identities $[dd\theta^{\alpha}]=0$ give*

$$\text{a) } n^{i}_{jk;l}+n^{h}_{jk}n^{i}_{lh}+j\,|\,k\,|\,l=0,\quad \text{b) } n^{i}_{jk(l)}=\bar{C}^{i}_{lj;k}-\bar{C}^{i}_{lk;j}, \tag{3.4}$$

where $j\,|\,k\,|\,l$ means the cyclic permutation of indices j, k, l.

In virtue of well known 'Lagrangean Method', we can choose a function σ so that $e^{\sigma}\theta^{\alpha}$ is holonomic (called a *conformal frame*).

Theorem 3.5. *The necessary and sufficient conditions for a n.h. frame $(\bar{U}, X^{i}_{\alpha})$ to be a conformal frame are that there exists a vector $\sigma_{i}(x)$ such that $n^{i}_{jk}=\sigma_{j}\delta^{i}_{k}-\sigma_{k}\delta^{i}_{j}$ and $\bar{C}^{i}_{jk}=0$.*

§4. A Finsler space with metric $F(x, y)=f(\xi^{\alpha}(x, y))$. We know that in the manifold $(M_{T}, N^{i}_{k}(x, y))$ we have two connections $(N\Gamma)$ and $(\Lambda\Gamma)$, the difference of which is an arbitrary 1-form and in a Finsler space with metric $F(x, y)$ we have the connections (CF), (BF) and $(\Lambda\Gamma)$. However, we can show

Theorem 4.1. *In a Finsler space with metric $F(x, y)=f(\xi^{\alpha}(x, y))$, we determine (CF) and (BF) by $(\Lambda\Gamma)$ uniquely.*

The proof of Theorem 4.1 follows from the following calculations.

$$F(x, y):f(\xi^{\alpha}(x, y)),\qquad \lambda^{\alpha}_{i}:=\xi^{\alpha}_{(i)},$$

$$\begin{aligned}
&l_{i}:=F_{(i)}=f_{(\alpha)}\lambda^{\alpha}_{i},\ y_{i}=FF_{(i)}=M_{\alpha}\lambda^{\alpha}_{i},\ M_{\alpha}:=ff_{(\alpha)},\ f_{(\alpha)}:\partial f/\partial\xi^{\alpha}.\\
&g_{jk}=m_{jk}+\bar{C}_{jk},\ m_{jk}:=M_{\alpha\beta}\lambda^{\alpha}_{j}\lambda^{\beta}_{k},\ M_{\alpha\beta}:=M_{\alpha(\beta)},\ \bar{C}_{jk}:=\bar{C}^{0}_{jk}.\\
&2C_{ijk}=M_{ijk}+m_{ih}\bar{C}^{h}_{jk}+m_{jh}\bar{C}^{h}_{ik}+m_{kh}\bar{C}^{h}_{ij}+N_{ijk},\\
&M_{ijk}:M_{\alpha\beta(\gamma)}\lambda^{\alpha}_{i}\lambda^{\beta}_{j}\lambda^{\gamma}_{k},\ N_{ijk}:=M_{\alpha}\lambda^{\alpha}_{i\,j(k)},\\
&C^{i}_{jk}=\bar{C}^{i}_{jk}+g^{ih}(g_{hj;(k)}+g_{hk;(j)}-g_{jk;(h)})/2.
\end{aligned} \tag{4.1}$$

Proposition 4.2. *It holds that*

$$\begin{aligned}
&F_{;k}=0,\ y^{i}{}_{;k}=0,\ y_{i;k}=0,\ g_{ij;k}=\bar{C}_{ij;k},\ m_{ij;k}=0,\ m_{jk}=y_{j;(k)},\\
&g_{ij;(k)}=m_{ij;(k)}+\bar{C}_{ij;(k)},\ m_{ij;(k)}=M_{ijk},\ y^{i}{}_{;(k)}=\delta^{i}_{k}.
\end{aligned} \tag{4.2}$$

For the connections (CF) and (BF), we have

$$\begin{aligned}
&G^{i}_{k}=\Lambda^{i}_{k}+E^{i}_{k},\quad \Lambda^{i}_{jk}=\Lambda^{i}_{k(j)},\quad E^{i}_{jk}=E^{i}_{k(j)},\\
&G^{i}_{jk}=\Lambda^{i}_{jk}+E^{i}_{jk},\quad P^{i}_{jk}=E^{i}_{jk}-D^{i}_{jk},\quad \Lambda^{i}_{jk}=\lambda^{i}_{\alpha}\bar{d}_{k}\lambda^{\alpha}_{j},\\
&{}^{*}\Gamma^{i}_{jk}=\Lambda^{i}_{jk}+D^{i}_{jk},\quad n^{i}_{jk}=\Lambda^{i}_{jk}-\Lambda^{i}_{kj}=-E^{i}_{jk}+E^{i}_{kj}=-D^{i}_{jk}+D^{i}_{kj},
\end{aligned} \tag{4.3}$$

where we denote

$$\begin{aligned}
&2E^{i}_{k}:=n^{a}_{bc}(g^{ib}\delta^{c}_{k}y_{a}+g^{ib}g_{ak}y^{c}+\delta^{i}_{a}\delta^{b}_{k}y^{c}-2C^{ib}_{k}y_{a}y^{c})+g^{ih}\bar{C}_{hk;0},\\
&2g_{hi}D^{h}_{jk}:=n^{h}_{ik}g_{jh}+n^{h}_{ij}g_{hk}-n^{h}_{jk}g_{hi}+\bar{C}_{ij;k}+\bar{C}_{ik;j}-\bar{C}_{jk;i}\\
&\qquad +2(E^{h}_{k}C_{ijh}+E^{h}_{j}C_{ikh}-E^{h}_{i}C_{jkh}).
\end{aligned} \tag{4.4}$$

If we put $\Phi^{i}:=g^{ih}n^{0}_{h0}$, we get from (4.3) and (4.4)

$$\begin{aligned}
&G^{i}_{k}=\Lambda^{i}_{k}+(\Phi^{i}_{(k)}-n^{i}_{0k})/2,\quad E^{i}_{k}=(\Lambda^{i}_{k}-n^{i}_{0k})/2,\quad \Phi^{i}_{k}:=\Phi^{i}_{(k)},\\
&E^{i}_{jk}=(\Phi^{i}_{jk}+\bar{C}^{i}_{jk;0}-n^{i}_{jk})/2,\quad \bar{C}^{i}_{jk;0}=n^{i}_{jk}-n^{i}_{0k(j)},\quad \Phi^{i}_{jk}:=\Phi^{i}_{j(k)},\\
&G^{i}_{jkl}=(\Phi^{i}_{jkl}+\bar{C}^{i}_{jk;0(l)}+\bar{C}^{i}_{jl;k}+\bar{C}^{i}_{kl;j})/2,\quad \Phi^{i}_{jkl}:=\Phi^{i}_{jk(l)}.
\end{aligned}$$

Proposition 4.3. *If $\bar{C}^i_{jk:1}=0$, the connection $(\Lambda\Gamma)$ is metrical, that is, $g_{ij:k}=0$.*

Theorem 4.5. *If $n^i_{jk}=0$ and $\bar{C}^i_{jk:1}=0$ the space is Minkowskian. If $n^0_{j0}=0$ and $\bar{C}^i_{jk:1}=0$, the space is a Berwald space. If there exists a vector σ_i such that $n^i_{jk}=\sigma_j\delta^i_k-\sigma_k\delta^i_j$ and $\bar{C}^i_{jk}=0$, the space is conformally flat.*

(will be revisted)

REFERENCES

1. F u j i m o t o, A. — *Theory of G-structure*. Publ. of Study group of Geometry, No 1 (1972.
2. I z u m i, H. — *Non-holonomic frames in a Finsler space with 1-form metric*, Tensor, N.S., 40 (1983), 189—192

Department of Mathematics
National Defense Academy
Yokosuka 239, Japan

Analele ştiinţifice
ale Universităţii „Al. I. Cuza" din Iaşi
Tomul XXX, s. I a, Matematică, 1984 — 4

Romanian-Japanese Colloquium
on Finsler Geometry, Iaşi,
Romania, 16—25 August, 1984

SOME REMARKABLE FINSLER STRUCTURES ON VECTOR BUNDLES

BY

FRANCISC C. KLEPP

§1. Preliminaries. At the second National Seminar on Finsler Spaces (Braşov, 1982) Radu M i r o n [3] has introduced an efficient method for the study of Finsler geometry on vector bundles, considering the Finsler geometry of a vector bundle $\xi=(E, \Pi, M)$ to be the Finsler geometrical objects theory on the total space E. Let $\xi=(E, \Pi, M)$ be a vector bundle of class C^∞; we suppose that the space E has $n+m$ dimensions, the base M is a differentiable manifold of dimension n, Π is the projection and the local fibre $E_P=\Pi^{-1}(P)$, $P\in M$ is an m dimensional real vector spaces.

Definition 1.1. *A non-linear connection on the total space E of ξ is a differentiable distribution $N: u\in E\to N_u\in E_u$, with the property $E_u=N_u\oplus E^v_u$, where E_u is the tangent space in the point u to the manifold E, $N: u\to N_u$ is the horizontal distribution on E, and $E^v: u\to E^v_u$ is the vertical distribution on E.*

For every vector field X on E there exists the unique decomposition: $X=X^H+X^v$; $X^H_u\in N_u$, $X^v_u\in E^v_u$, $\forall u\in E$, where X^H is called the horizontal part and X^v the vertical part of X.

A Finsler connection on the total space E can be introduced as a linear connection ∇ on E, which preserve by parallelism the horizontal distribution N and the vertical distribution E^v of ξ. The connection ∇ has four local coefficients which have the very simple transformation laws to a change of canonical coordinates on E.

Any Finsler connection ∇ can be decomposed in an unique way by:

$$\nabla_X Y=\nabla^H_X Y+\nabla^v_X Y;\quad \forall X, Y\in\mathscr{X}(E).$$

R. M i r o n [3] has established the torsion tensor field of a Finsler connection on E, characterised by five Finsler tensor fields, and the curvature of a Finsler connection on E, characterised by six Finsler tensor fields. In particular for $E=T(M)$ the components of the Finsler connection are equal in paires and the curvature tensor field has only three components which are the Finsler tensor fields R^2, P^2, S^2 from the known case of M. M a t s u m o t o [2].

§2. Metric Finsler Structures on Vector Bundles. A metric structure on the total space E is defined by a second order covariant, symmetric and non-degenerated tensor field G on E. The orthogonal vectors at E^v_u in the point $u\in E$, determine by G in a unique way the horizontal linear space N_u,