Proposition 4.3. *If $\bar{C}^i_{jk;1}=0$, the connection $(\Lambda\Gamma)$ is metrical, that is, $g_{ij;k}=0$.*

Theorem 4.5. *If $n^i_{jk}=0$ and $\bar{C}^i_{jk;1}=0$ the space is Minkowskian. If $n^0_{j0}=0$ and $\bar{C}^i_{jk;1}=0$, the space is a Berwald space. If there exists a vector σ_i such that $n^i_{jk}=\sigma_j\delta^i_k-\sigma_k\delta^i_j$ and $\bar{C}^i_{jk}=0$, the space is conformally flat.*

(will be revisted)

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SOME REMARKABLE FINSLER STRUCTURES ON VECTOR BUNDLES

BY

FRANCISC C. KLEPP

§1. Preliminaries. At the second National Seminar on Finsler Spaces (Braşov, 1982) Radu Miron [3] has introduced an efficient method for the study of Finsler geometry on vector bundles, considering the Finsler geometry of a vector bundle $\xi=(E, \Pi, M)$ to be the Finsler geometrical objects theory on the total space E. Let $\xi=(E, \Pi, M)$ be a vector bundle of class C^∞; we suppose that the space E has $n+m$ dimensions, the base M is a differentiable manifold of dimension n, Π is the projection and the local fibre $E_P=\Pi^{-1}(P)$, $P\in M$ is an m dimensional real vector spaces.

Definition 1.1. *A non-linear connection on the total space E of ξ is a differentiable distribution $N: u\in E\to N_u\in E_u$, with the property $E_u=N_u\oplus E^v_u$, where E_u is the tangent space in the point u to the manifold E, $N: u\to N_u$ is the horizontal distribution on E, and $E^v: u\to E^v_u$ is the vertical distribution on E.*

For every vector field X on E there exists the unique decomposition: $X=X^H+X^v$; $X^H_u\in N_u$, $X^v_u\in E^v_u$, $\forall u\in E$, where X^H is called the horizontal part and X^v the vertical part of X.

A Finsler connection on the total space E can be introduced as a linear connection ∇ on E, which preserve by parallelism the horizontal distribution N and the vertical distribution E^v of ξ. The connection ∇ has four local coefficients which have the very simple transformation laws to a change of canonical coordinates on E.

Any Finsler connection ∇ can be decomposed in an unique way by:

$$\nabla_X Y=\nabla^H_X Y+\nabla^v_X Y;\quad \forall X, Y\in\mathcal{X}(E).$$

R. Miron [3] has established the torsion tensor field of a Finsler connection on E, characterised by five Finsler tensor fields, and the curvature of a Finsler connection on E, characterised by six Finsler tensor fields. In particular for $E=T(M)$ the components of the Finsler connection are equal in paires and the curvature tensor field has only three components which are the Finsler tensor fields R^2, P^2, S^2 from the known case of M. Matsumoto [2].

§2. Metric Finsler Structures on Vector Bundles. A metric structure on the total space E is defined by a second order covariant, symmetric and non-degenerated tensor field G on E. The orthogonal vectors at E^v_u in the point $u\in E$, determine by G in a unique way the horizontal linear space N_u, such that : $E_u = N_u \oplus E_u^v$. The map $N : u \to N_u$ defines in a geometrical way a non-linear connection N on E.

Proposition 2.1. *Let G be a metric structure on the total space E of a vector bundle ξ, then there exists an unique nonlinear connection N on E, such that :*

$$G(X^H, Y^v) = 0 \; ; \quad \forall X, Y \in \mathscr{X}(E) \tag{2.1}$$

For a metric structure G on E and the non-linear connection N, having the property (2.1) there exists an unique pair of Finsler tensors G^H and G^v of the type $\begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}$ respectively, such that :

$$G = G^H + G^v \tag{2.2}$$

Definition 2.1. *A linear connection ∇ on E, for which the distributions N and E^v are preserved by parallelism, and $\nabla_X G = 0$, $\forall X \in \mathscr{X}(E)$, is called a metric Finsler connection.*

Theorem 2.1. *A linear connection ∇ on E is a metric Finsler connection if and only if :*

$$\nabla_X^H G^H = 0 \; ; \quad \nabla_X^H G^v = 0 \; ; \quad \nabla_X^v G^H = 0 \; ; \quad \nabla_X^v G^v = 0.$$

§3. Almost Product Finsler Structures on Vector Bundles. Let P be an almost product structure on E, given by a tensor field of type $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, with the property : $P \circ P = I$.

Proposition 3.1. *If P is an almost product structure on E, there exists an unique decomposition of P in the Finsler tensor fields :*

$$P = \overset{1}{P} + \overset{2}{P} + \overset{3}{P} + \overset{4}{P} \tag{3.1}$$

where :

$$\begin{cases} \overset{1}{P}(\omega, X) = P(X^H, \omega^H) \; ; \quad \overset{2}{P}(\omega, X) = P(\omega^H, X^v) \; ; & \forall X \in \mathscr{X}(E) \\ \overset{3}{P}(\omega, X) = P(\omega^v, X^H) \; ; \quad \overset{4}{P}(\omega, X) = P(\omega^v, X^v) & \forall \omega \in \mathscr{X}^*(E) \end{cases} \tag{3.2}$$

We can writte :

$$\begin{gathered} P(X^H) = \overset{1}{P}(X^H) + \overset{3}{P}(X^H) \; ; \quad P(X^v) = \overset{2}{P}(X^v) + \overset{4}{P}(X^v) \; ; \\ \forall X^H \in N \; ; \quad \forall X^v \in E^v \end{gathered} \tag{3.3}$$

It follows :

$$\begin{cases} (\overset{1}{P} \circ \overset{1}{P} + \overset{2}{P} \circ \overset{3}{P})(X^H) = X^H \; ; & (\overset{3}{P} \circ \overset{1}{P} + \overset{4}{P} \circ \overset{3}{P})(X^H) = 0, \\ (\overset{1}{P} \circ \overset{2}{P} + \overset{2}{P} \circ \overset{4}{P})(X^v) = 0 \; ; & (\overset{3}{P} \circ \overset{2}{P} + \overset{4}{P} \circ \overset{4}{P})(X^v) = X^v. \end{cases} \tag{3.4}$$

The Nijenhuis tensor of P is giaen by :

$$\begin{gathered} N(X, Y) = -[X, Y] + P[PX, Y] + P[X, PY] - [PX, PY] \\ \forall X, Y \in \mathscr{X}(E) \end{gathered} \tag{3.5}$$

and the integrability conditions of the almost product structure P is : $N(X, Y) = 0$; $\forall X, Y \in \mathscr{X}(E)$.

Theorem 3.1. *The Finsler tensors from the decomposition of N are the horizontal and vertical components of the tensors : $N(X^H, Y^H)$; $N(X^H, Y^v)$, $N(X^v, Y^v)$.*

Theorem 3.2. *The almost product Finsler structure P is integrable if and only if the following Finsler tensors vanish : $[N(X^H, Y^H)]^H$; $[N(X^H, Y^H)]^v$; $[N(X^H, Y^v)]^H$; $[N(X^H, Y^v)]^v$; $[N(X^v, Y^v)]^H$; $[N(X^v, Y^v)]^v$.*

Definition 3.1. *A linear connection ∇ on E, with the properties : 1°) ∇ is an almost product connection on E ; 2°. ∇ is a Finsler connection relative to the distributions N and E^v,* i.e. $(\nabla_X Y^H)^v = 0$; $(\nabla_X Y^v)^H = 0$, *is called an almost product Finsler connection.*

Theorem 3.3. *The connection :*

$$\nabla_X = \frac{1}{2}(\mathring{\nabla}_X + P \circ \mathring{\nabla} \circ P) \; ; \quad \forall X \in \mathscr{X}(E) \tag{3.6}$$

where $\mathring{\nabla}$ is an arbitrary fixed linear connection on E, is an almost product connection.

Theorem 3.4. *If ∇ is an almost product connection on E, then the set of all almost product connections on E is given by :*

$$\bar{\nabla}_X = \nabla_X + \Omega A_X \; ; \quad \forall X \in \mathscr{X}(E), \quad \forall A \in \tau_2^1(E) \tag{3.7}$$

where $\Omega : \tau_1^1(E) \to \tau_1^1(E)$ is the Obata operator of P, given by :

$$\Omega A_X = \frac{1}{2}(A_X + P \circ A_X \circ P) \tag{3.8}$$

§4. Almost Complex Finsler Structures on Vector Bundles. Let F be an almost complex structure on E, given by a tensor field of type $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ with the property :

$$F \circ F = -I \tag{4.1}$$

Proposition 4.1. *If F is an almost complex structure on E, then there exists an unique decomposition of F in the Finsler tensor fields :*

$$F = \overset{1}{F} + \overset{2}{F} + \overset{3}{F} + \overset{4}{F} \tag{4.2}$$

where :

$$\begin{cases} \overset{1}{F}(\omega, X) = F(\omega^H, X^H) \; ; \quad \overset{2}{F}(\omega, X) = F(\omega^H, X^v) \; ; & \forall X \in \mathscr{X}(E) \\ \overset{3}{F}(\omega, X) = F(\omega^v, X^H) \; ; \quad \overset{4}{F}(\omega, X) = F(\omega^v, X^v) \; ; & \forall \omega \in \mathscr{X}^*(E) \end{cases} \tag{4.3}$$

We can writte :

$$\begin{gathered} F(X^H) = \overset{1}{F}(X^H) + \overset{3}{F}(X^H) \; ; \quad F(X^v) = \overset{2}{F}(X^v) + \overset{4}{F}(X^v) \; ; \\ \forall X^H \in N \; ; \quad \forall X^v \in E^v \end{gathered} \tag{4.2}$$

We obtain:

$$(4.5)\qquad \begin{cases} (\overset{1}{F}\circ\overset{1}{F}+\overset{2}{F}\circ\overset{3}{F})(X^H)=-X^H\,; & (\overset{3}{F}\circ\overset{1}{F}+\overset{4}{F}\circ\overset{3}{F})(X^H)=0 \\ (\overset{1}{F}\circ\overset{2}{F}+\overset{2}{F}\circ\overset{4}{F})(X^v)=0\,; & (\overset{3}{F}\circ\overset{2}{F}+\overset{4}{F}\circ\overset{4}{F})(X^v)=-X^v \end{cases}$$

In the adopted basis for the distributions N and E^v the relations (3.5) are written as:

$$(4.6)\qquad \begin{cases} \overset{1}{F}{}^i_j\overset{1}{F}{}^j_k+\overset{2}{F}{}^i_a\overset{3}{F}{}^a_k=-\delta^i_k\,, & \overset{3}{F}{}^a_j\overset{2}{F}{}^j_k+\overset{4}{F}{}^a_b\overset{3}{F}{}^b_k=0, \\ \overset{1}{F}{}^i_j\overset{2}{F}{}^j_b+\overset{2}{F}{}^i_a\overset{4}{F}{}^a_b=0, & \overset{3}{F}{}^a_j\overset{2}{F}{}^j_b+\overset{4}{F}{}^a_c\overset{4}{F}{}^c_b=-\delta^a_b\,. \end{cases}$$

The Nijenhuis tensor of F is given by:

$$(4.7)\qquad N^*(X, Y)=[X, Y]+F[FX, Y]+F[X, FY]-[FX, FY].$$

Theorem 4.1. *The Finsler tensors from the decomposition of N^* are the horizontal and vertical components of the tensors $N^*(X^H, Y^H)$, $N^*(X^H, Y^v)$; $N^*(X^v, Y^v)$.*

Theorem 4.2. *The almost complex Finsler structure F is integrable if and only if the following Finsler tensors vanish:*

$$[N^*(X^H, Y^H)]^H \quad [N^*(X^H, Y^H)]^V, \quad [N^*(X^H, Y^v)]^H,$$
$$[N^*(X^H, Y^v)]^V, \quad [N^*(X^v, Y^v)]^H, \quad [N^*(X^v, Y^v)]^V$$

Definition 4.1. *A linear connection ∇ on E, with the properties: 1°. ∇ is an almost complex connection; 2°. ∇ is a Finsler connection relative to the distributions N and E^v, i.e. $[\nabla_X Y^H]^V=0$, $[\nabla_X Y^v]^H=0$ is called an almost complex Finsler connnection.*

Theorem 4.3. *The connection*

$$(4.8)\qquad \nabla_X=\frac{1}{2}(\mathring{\nabla}_X-F\circ\mathring{\nabla}_X\circ F), \quad \forall X\in\mathcal{X}(E)$$

where $\mathring{\nabla}$ is an arbitrary fixed linear connection on E, is an almost complex connection.

Theorem 4.4. *If ∇ is an almost complex connection on E, then the set of all almost complex connections on E is given by:*

$$(4.9)\qquad \overline{\nabla}_X=\nabla_X+\Omega A_X, \quad \forall X\in\mathcal{X}(E)\,;\ \forall A\in\tau^1_2(E)$$

where $\Omega: \tau^1_1(E)\to\tau^1_1(E)$ is the Obata operator of F, given by:

$$\Omega A_X=\frac{1}{2}(A_X-F\circ A_X\circ F).$$

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INDUCED FINSLER CONNECTIONS OF A HYPERSURFACE

BY

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I published my first paper on Finsler geometry in 1960, so 24 years have passed since then. During the first decade I have mainly studied Finsler connections. The preface of my monograph [1] published in 1970 began with the following paragraph:

It was undesirable for the author to give the title „Differential geometry of Finsler spaces" to the present book, because the Finsler *geometry* was not yet developed satisfactorily in this book. We have been able to study, at the present stage, only the theory of general connections appearing in the theory of Finsler spaces.

Since 1970 I have gradually tended to considerations of Finsler spaces with some characteristic properties and their metrics. In particular the notion of constant curvature has excited my interest recently [2], and I was naturally attracted into the theory of hypersurfaces, because a sphere is a hypersurface of an euclidean space and a typical example of spaces of constant curvature.

Now, to our greatest regret, we can not see the father of differential geometry of Romania, M. Haimovici. Following E. Cartan, he at once developed the theory of Finslerian subspaces, not only derived the Gauss-Codazzi equations, but also showed important theorems on the existence of totally geodesic subspaces.

Let $F^n=(M^n, L(x, y), F\Gamma)$ be a Finsler space equipped with a Finsler metric $L(x, y)$ and a Finsler connection $F\Gamma$. Then a hypersurface M^{n-1} of M^n is regarded as a Finsler space with the induced metric $\underline{L}(u, v)$ which is given from $L(x, y)$ by treating only the supporting element y tangential to M^{n-1}:

$$M^{n-1}: \ x^i=x^i(u^\alpha), \quad B^i_\alpha(u)=\partial x^i/\partial u^\alpha, \quad y^i=B^i_\alpha(u)v^\alpha,$$
$$\underline{L}(u, v)=L(x(u), B^i_\alpha(u)v^\alpha).$$

We have naturally two Finsler connections on the Finslerian hypersurface $F^{n-1}=(M^{n-1}, \underline{L}(u, v))$; one is the *intrinsic* connection $F\underline{\Gamma}$, derived from $\underline{L}(u, v)$ in the way (2) which is similar to (2), and another is *induced* connection $IF\Gamma$, derived from $F\Gamma$ by the so-called projection.

$$\begin{tikzcd} & \underline{L}(x, v) \arrow[r, "(2)"] & F\underline{\Gamma} \\ L(x, y) \arrow[ur, "(1)"] \arrow[dr, "(2)"'] & & \\ & F\Gamma \arrow[r, "(3)"] & IF\Gamma \end{tikzcd}$$