We obtain:

$$(4.5)\quad \begin{cases} (\overset{1}{F}\circ\overset{1}{F}+\overset{2}{F}\circ\overset{3}{F})(X^H)=-X^H\,; & (\overset{3}{F}\circ\overset{1}{F}+\overset{4}{F}\circ\overset{3}{F})(X^H)=0 \\ (\overset{1}{F}\circ\overset{2}{F}+\overset{2}{F}\circ\overset{4}{F})(X^v)=0\,; & (\overset{3}{F}\circ\overset{2}{F}+\overset{4}{F}\circ\overset{4}{F})(X^v)=-X^v \end{cases}$$

In the adopted basis for the distributions N and E^v the relations (3.5) are written as:

$$(4.6)\quad \begin{cases} \overset{1}{F}{}^i_j\overset{1}{F}{}^j_k+\overset{2}{F}{}^i_a\overset{3}{F}{}^a_k=-\delta^i_k\,, & \overset{3}{F}{}^a_j\overset{2}{F}{}^j_k+\overset{4}{F}{}^a_b\overset{3}{F}{}^b_k=0, \\ \overset{1}{F}{}^i_j\overset{2}{F}{}^j_b+\overset{2}{F}{}^i_a\overset{4}{F}{}^a_b=0, & \overset{3}{F}{}^a_j\overset{2}{F}{}^j_b+\overset{4}{F}{}^a_c\overset{4}{F}{}^c_b=-\delta^a_b\,. \end{cases}$$

The Nijenhuis tensor of F is given by:

$$(4.7)\qquad N^*(X, Y)=[X, Y]+F[FX, Y]+F[X, FY]-[FX, FY].$$

Theorem 4.1. *The Finsler tensors from the decomposition of N^* are the horizontal and vertical components of the tensors $N^*(X^H, Y^H)$, $N^*(X^H, Y^v)$; $N^*(X^v, Y^v)$.*

Theorem 4.2. *The almost complex Finsler structure F is integrable if and only if the following Finsler tensors vanish:*

$$[N^*(X^H, Y^H)]^H \quad [N^*(X^H, Y^H)]^V, \quad [N^*(X^H, Y^v)]^H,$$
$$[N^*(X^H, Y^v)]^V, \quad [N^*(X^v, Y^v)]^H, \quad [N^*(X^v, Y^v)]^V$$

Definition 4.1. *A linear connection ∇ on E, with the properties: 1°. ∇ is an almost complex connection; 2°. ∇ is a Finsler connection relative to the distributions N and E^v, i.e. $[\nabla_X Y^H]^V=0$, $[\nabla_X Y^v]^H=0$ is called an almost complex Finsler connnection.*

Theorem 4.3. *The connection*

$$(4.8)\qquad \nabla_X=\frac{1}{2}(\mathring{\nabla}_X-F\circ\mathring{\nabla}_X\circ F), \quad \forall X\in\mathcal{X}(E)$$

where $\mathring{\nabla}$ is an arbitrary fixed linear connection on E, is an almost complex connection.

Theorem 4.4. *If ∇ is an almost complex connection on E, then the set of all almost complex connections on E is given by:*

$$(4.9)\qquad \overline{\nabla}_X=\nabla_X+\Omega A_X, \quad \forall X\in\mathcal{X}(E)\,;\ \forall A\in\tau^1_2(E)$$

where $\Omega:\tau^1_1(E)\to\tau^1_1(E)$ is the Obata operator of F, given by:

$$\Omega A_X=\frac{1}{2}(A_X-F\circ A_X\circ F).$$

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INDUCED FINSLER CONNECTIONS OF A HYPERSURFACE

BY

MAKOTO MATSUMOTO

I published my first paper on Finsler geometry in 1960, so 24 years have passed since then. During the first decade I have mainly studied Finsler connections. The preface of my monograph [1] published in 1970 began with the following paragraph:

It was undesirable for the author to give the title „Differential geometry of Finsler spaces" to the present book, because the Finsler *geometry* was not yet developed satisfactorily in this book. We have been able to study, at the present stage, only the theory of general connections appearing in the theory of Finsler spaces.

Since 1970 I have gradually tended to considerations of Finsler spaces with some characteristic properties and their metrics. In particular the notion of constant curvature has excited my interest recently [2], and I was naturally attracted into the theory of hypersurfaces, because a sphere is a hypersurface of an euclidean space and a typical example of spaces of constant curvature.

Now, to our greatest regret, we can not see the father of differential geometry of Romania, M. Haimovici. Following E. Cartan, he at once developed the theory of Finslerian subspaces, not only derived the Gauss-Codazzi equations, but also showed important theorems on the existence of totally geodesic subspaces.

Let $F^n=(M^n, L(x, y), F\Gamma)$ be a Finsler space equipped with a Finsler metric $L(x, y)$ and a Finsler connection $F\Gamma$. Then a hypersurface M^{n-1} of M^n is regarded as a Finsler space with the induced metric $\underline{L}(u, v)$ which is given from $L(x, y)$ by treating only the supporting element y tangential to M^{n-1}:

$$M^{n-1}:\ x^i=x^i(u^\alpha),\quad B^i_\alpha(u)=\partial x^i/\partial u^\alpha,\quad y^i=B^i_\alpha(u)v^\alpha,$$
$$\underline{L}(u, v)=L(x(u), B^i_\alpha(u)v^\alpha).$$

We have naturally two Finsler connections on the Finslerian hypersurface $F^{n-1}=(M^{n-1}, \underline{L}(u, v))$; one is the *intrinsic* connection $F\underline{\Gamma}$, derived from $\underline{L}(u, v)$ in the way $\underline{(2)}$ which is similar to (2), and another is *induced* connection $IF\Gamma$, derived from $F\Gamma$ by the so-called projection.

$$\begin{tikzcd} & \underline{L}(x, v) \arrow[r, "\underline{(2)}"] & F\underline{\Gamma} \\ L(x, y) \arrow[ur, "(1)"] \arrow[dr, "(2)"'] & & \\ & F\Gamma \arrow[r, "(3)"] & IF\Gamma \end{tikzcd}$$

As it is well-known, $F\Gamma$ does not coincide with $IF\Gamma$ in general and this problem has given rise to complicated circumstances in Finslerian subspace theory. In fact, when we shall make a comparative reading of papers on the subspace theory written by various authors, we are sure to find cluttering up our minds owing to differences of methods and notations.

In our new viewpoint of Finsler connections, however, this problem is satisfactorily solved as follows:

Theorem 1. *The induced connection $IC\Gamma$ on a hypersurface of a Finsler space $F^n=(M^n, L(x, y), C\Gamma)$ with the Cartan connection $C\Gamma$ is a generalized Cartan connection which is uniquely determined from the induced metric $\underline{L}(u, v)$ by the following 5 axioms:* (1) $g_{\alpha\beta|\gamma}=0$, (2) $T^{\alpha}_{\beta\gamma}=M^{\alpha}_{\beta}H_{\gamma}-M^{\alpha}_{\gamma}H_{\beta}$, (3) $D^{\alpha}_{\beta}=0$, (4) $g_{\alpha\beta}|_{\gamma}=0$, (5) $S^{\alpha}_{\beta\gamma}=0$.

$M_{\alpha\beta}$ is a symmetric tensor, derived from the Cartan C-tensor $g_{ijk}=\dot\partial_k g_{ij}/2$ by the projection operators B^i_{α} and the unit normal B^i: $M_{\alpha\beta}=g_{ijk}B^i_{\alpha}B^j_{\beta}B^k$, and H_{α} is the normal curvature vector: $H_{\alpha}=B_i(B^i_{0\alpha}+N^i{}_jB^j_{\alpha})$ $(B^i_{0\alpha}=v^{\beta}\partial_{\alpha}B^i_{\beta})$.

Theorem 2. *The induced connection $IB\Gamma$ from the Berwald connection $B\Gamma$ is a generalized Berwald P^1-connection which is uniquely determined from $\underline{L}(u, v)$ by the following 5 axioms:* (1) $\underline{L}_{|\alpha}=0$, (2) $T^{\alpha}_{\beta\gamma}=0$, (3) $D^{\alpha}_{\beta}=0$, (4) $P^{\alpha}_{\beta\gamma}=2H_{\beta}M^{\alpha}_{\gamma}$, (5) $C^{\alpha}_{\beta\gamma}=0$.

The following diagram is well-known:

$$\begin{tikzcd}[column sep=huge]
C\Gamma \arrow[r, "C\text{-process}"] \arrow[d, "P^1\text{-process}"'] & R\Gamma \arrow[d, "P^1\text{-process}"] \\
H\Gamma \arrow[r, "C\text{-process}"] & B\Gamma
\end{tikzcd}
\qquad
\begin{array}{l}
C\text{-process}: \ C_j{}^i{}_k \to 0 \\
P^1\text{-process}: \ F_j{}^i{}_k \to F_j{}^i{}_k + P^i{}_{kj}.
\end{array}$$

These four connections have the common nonlinear connection N^i_j, so that $IC\Gamma$, $IR\Gamma$, $IH\Gamma$ and $IB\Gamma$ have the common induced nonlinear connection $N^{\beta}_{\alpha}=B^{\beta}_i(B^i_{0\alpha}+N^i_jB^j_{\alpha})$. Then we have

$$\begin{tikzcd}[column sep=huge]
IC\Gamma \arrow[r, "IC\text{-process}"] \arrow[d, "IP^1\text{-process}"'] & IR\Gamma \arrow[d, "IP^1\text{-process}"] \\
IH\Gamma \arrow[r, "IC\text{-process}"] & IB\Gamma
\end{tikzcd}
\qquad
\begin{array}{l}
IC\text{-process}: \ \begin{cases} F_{\beta}{}^{\alpha}{}_{\gamma} \to F_{\beta}{}^{\alpha}{}_{\gamma} - M^{\alpha}_{\beta}H_{\gamma}, \\ C_{\beta}{}^{\alpha}{}_{\gamma} \to 0 \end{cases} \\
IP^1\text{-process}: \ F_{\beta}{}^{\alpha}{}_{\gamma} \to F_{\beta}{}^{\alpha}{}_{\gamma} + Q_{\beta}{}^{\alpha}{}_{\gamma} \\
\qquad Q_{\alpha\beta\gamma}=g_{ijk|0}\,B^i_{\alpha}B^j_{\beta}B^k_{\gamma}.
\end{array}$$

Corresponding to these processes, as to the second fundamental h-tensor $H_{\alpha\beta}$, we have

$$IC\text{-process}: \ H_{\alpha\beta}\to H_{\alpha\beta}-M_{\alpha}H_{\beta}, \quad M_{\alpha}=g_{ijk}B^i_{\alpha}B^jB^k,$$
$$IP^1\text{-process}: \ H_{\alpha\beta}\to H_{\alpha\beta}+Q_{\alpha\beta}, \quad Q_{\alpha\beta}=g_{ijk|0}B^i_{\alpha}B^j_{\beta}B^k.$$

From the above results it may be said that our new concept of Finsler connection is very useful in the theory of Finslerian subspaces and the well-known concept of constant curvature should be reconsidered from the wider standpoint of general Finsler connections.

I hope to speak on the existence of totally geodesic hypersurfaces afterwards.

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TOTALLY GEODESIC HYPERSURFACES AND FINSLERIAN PROJECTIVE GEOMETRY

BY

MAKOTO MATSUMOTO

In 1934 M. Haimovici found the fundamental Gauss and Codazzi equations for a hypersurface of a Finsler space and, about 5 years after, he discussed the problem of existence of totally geodesic subspaces. Next, in 1952 S. Kikuchi delt with this problem in more detailed form. Finally, in 1957 this problem was nearly finished by A. Rapcsák's paper, but I can not understand his proof of the essential lemma; according to Prof. L. Tamássy's communication, Prof. Rapcsák acknowledges a misprint in his lemma. Nevertheless his paper is very valuable.

Now, let $\underline{L}(u, v)$ and $IF\Gamma$ be the induced metric and induced connection of a hypersurface F^{n-1} of a Finsler space $F^n=(M^{n-1}, L(x, y), F\Gamma)$. We get the notion of geodesic curve with respect to a metric and the notion of path with respect to a connection. If each geodesic curve of F^n is a path, then each geodesic curve of F^{n-1} with respect to $\underline{L}(u, v)$ is a path with respect to $IF\Gamma$.

Definition 1. *If each path of F^{n-1} with respect to $IF\Gamma$ is a path of F^n with respect to $F\Gamma$, F^{n-1} is called a hyperplane of the 1-st kind.*

If we are concerned with the Cartan connection of F^{n-1}, the above notion coincides with the notion of totally geodesic hypersurface.

Next, in 1950 O. Varga defined the concept of quasigeodesic curve with respect to $C\Gamma$. I generalized this concept to a general Finsler connection about 1970; a curve $C=(x^i(s), y^i(s))$ of the tangent bundle TF^n is called an *h-path* if $y^i(s)$ is parallel along the curve $\underline{C}=(x^i(s))$ with respect to the nonlinear connection and dx^i/ds is parallel along $\underline{C}$ with respect to $y^i(s)$, i.e.

$$dy^i/ds+N^i_j(x, y)dx^j/ds=0, \quad d^2x^i/ds^2+F^i_{jk}(x, y)dx^jdx^k/ds^2=0.$$

Definition 2. *If each h-path of F^{n-1} with respect to $IF\Gamma$ is an h-path of F^n with respect to $F\Gamma$, F^{n-1} is called a hyperplane of the 2nd kind.*

Further we define another notion of hyperplane:

Definition 3. *F^{n-1} is called a hyperplane of the 3rd kind if the unit normal vector B^i of F^{n-1} is parallel along each curve $(u^{\alpha}(s), v^{\alpha}(s))$ on F^{n-1}, where $v^{\alpha}(s)$ is to be tangent to F^{n-1}.*

The relations among these three concepts of hyperplanes with respect to the connections $C\Gamma$, $B\Gamma$, $H\Gamma$ and $R\Gamma$ are as follows: