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A NEW METHOD IN GEOMETRY OF FINSLER SUBSPACES

BY

RADU MIRON and AUREL BEJANCU

Introduction. As it is well-known, for many years, the geometry of Riemannian subspaces was the main topic in Riemannian geometry. However, this was not the case for geometry of Finsler subspaces, because of the tremendous computations which were involved with it.

It was M. Haimovici who investigated in [2] for the first time, the geometry of hypersurfaces in Finsler spaces. Since then, some interesting results have been obtained, but the geometry of Finsler subspaces of codimension greater than one was only sporadicaly investigated.

In 1983, M. Matsumoto in his excellent work [3] obtained a systematic study of geometry of hypersurfaces in Finsler spaces. The study is mainly based on a Finsler connection on the ambient space and on the induced Finsler connection on the hypersurface.

Very recently, the first author obtained in [6] a new method for the study of geometry of hypersurfaces in Finsler spaces. Roughly speaking, this method is based on an intrinsic Finsler connection on the Finsler subspace and on a lifted Finsler connection in the ambient space.

The main purpose of the present paper is to show that Miron's method can be extended to a Finsler subspace of arbitrary codimension. We start from the intrinsic Finsler connection on a Finsler subspace and obtain a lifted Finsler connection on the ambient Finsler space. In this way the induced Finsler connection on a Finsler subspace is just the intrinsic Finsler connection of the subspace. Also the fundamental equations of a Finsler subspace have a simplified form and thus they could be handled in a further study.

§1. The intrinsic Finsler connection on a Finsler subspace and the i_*- Finsler connection on the ambient space. Let $F^n=(M^n, g_{ij}(x, y))$ be a *generalized Finsler space*, where M^n is a real n-dimensional differentiable manifold and $g_{ij}(x, y)$ is a symmetric, non-degenerate and Finsler tensor field of type (0, 2) on M^n. We call $g_{ij}(x, y)$ the *generalized Finsler metric* on M^n. Denote by (y^i) the supporting element of F^n and by TM^n the tangent bundle to M^n. (For the theory of generalized Finsler spaces, defined by the first author, see [5]).

Now, let $M^m (m<n)$ be a submanifold of M^n. Denote by i the immersion of M^m in M^n and by $i_* : TM^m \to TM^n$ its differential. Then, locally the immersion i is given by

$$(1.1)\qquad \begin{cases} x^i = x^i(u^1, \dots, u^m), \\ \operatorname{rank}\,[\partial x^i/\partial u^\alpha] = m,\ i = 1, \dots, n;\ \alpha = 1, \dots, m. \end{cases}$$

Throughout the paper we use the following notations for indices: $i, j, k, \dots = 1, \dots, n$; $\alpha, \beta, \gamma, \dots = 1, \dots, m$; $a, b, c, \dots = m+1, \dots, n$.

The local field of frames on M^m is given by

$$(1.2)\qquad B^i_\alpha(u) = \frac{\partial x^i}{\partial u^\alpha}.$$

In order to introduce on M^m a Finsler structure we suppose the supporting element (y^i) of M^n satisfying

$$(1.3)\qquad y^i = B^i_\alpha(u) v^\alpha.$$

We call (v^α) the supporting element of M^m. Denoting by

$$B^i_{\alpha\beta} = \frac{\partial^2 x^i}{\partial u^\alpha \partial u^\beta}\,;\quad B^i_{0\beta} = B^i_{\alpha\beta} v^\alpha, \text{ we obtain:}$$

$$(1.4)\qquad \begin{cases} \dfrac{\partial}{\partial u^\alpha} = B^i_\alpha \dfrac{\partial}{\partial x^i} + B^i_{0\alpha} \dfrac{\partial}{\partial y^i}, \\ \dfrac{\partial}{\partial v^\alpha} = B^i_\alpha \dfrac{\partial}{\partial y^i}, \end{cases}$$

where $\{\partial/\partial x^i, \partial/\partial y^i\}$ and $\{\partial/\partial u^\alpha, \partial/\partial v^\alpha\}$ are respectively the natural local field of frames on TM^n and TM^m.

We choose a local field of orthonormal Finsler frames $\{B^i_a(u, v)\}$ which is normal to M^m. Thus we obtain a local field of Finsler frames $\{x^i(u, v), B^i_\alpha(u), B^i_a(u, v)\}$ on M^n along to M^m. Denote by $[B^\alpha_i(u, v), B^a_i(u, v)]$ the inverse matrix of $[B^i_\alpha(u), B^i_a(u, v)]$. Then we obtain

$$(1.5)\qquad \begin{cases} B^\alpha_i B^i_\beta = \delta^\alpha_\beta\,;\ B^\alpha_i B^i_a = 0\,;\ B^a_i B^i_\alpha = 0\,;\ B^a_i B^i_b = \delta^a_b\,; \\ B^i_\alpha B^\alpha_j + B^i_a B^a_j = \delta^i_j. \end{cases}$$

The generalized Finsler metric $g_{ij}(x, y)$ on M^n induces a generalized Finsler metric $g_{\alpha\beta}(u, v)$ on M^m by

$$(1.6)\qquad g_{\alpha\beta}(u, v) = g_{ij}(x(u), y(u, v)) B^i_\alpha(u) B^j_\beta(u).$$

Thus the manifold M^m becomes a generalized Finsler space $F^m = (M^m, g_{\alpha\beta}(u, v))$. Then we fix a non-linear connection $\underline{\overset{c}{N}}(\overset{c}{N}{}^\alpha_\beta)$ on TM^m and we have

Theorem 1.1. *There exists a unique Finsler connection* $C\Gamma(\underline{\overset{c}{N}}) = (N^\alpha_\beta, F^\alpha_{\beta\gamma}, C^\alpha_{\beta\gamma})$ *in* F^m *satisfying the conditions:*

(i) *its non-linear connection is just* $\underline{\overset{c}{N}}$,

(ii) *the (h) and (v)-covariant derivatives of* $g_{\alpha\beta}(u, v)$ *with respect to* $C\Gamma(\underline{\overset{c}{N}})$ *vanish*

(iii) $T^\alpha_{\beta\gamma} = 0$ and $S^\alpha_{\beta\gamma} = 0$.

The theorem was obtained by the first author in [5]. We call $C\Gamma(\underline{\overset{c}{N}})$ *the intrinsic Finsler connection of* F^m.

Next, we fix a non-linear connection $\overset{c}{N}(\overset{c}{N}{}^i_j)$ on F^n. By using the coefficients of both non-linear connections $\overset{c}{N}$ and $\underline{\overset{c}{N}}$ on TM^n and respectively TM^m we define

$$(1.7)\qquad \Psi^i_j(x(u), y(u, v)) = B^\alpha_j (B^h_\alpha \overset{c}{N}{}^i_h - N^\beta_\alpha B^i_\beta + B^i_{0\alpha}).$$

Thus we obtained a Finsler tensor field $\Psi(\Psi^i_j)$ on F^n defined in all points of $i_*(TM^m)$. Then locally we define

$$(1.8)\qquad N^i_j(x(u), y(u, v)) = \overset{c}{N}{}^i_j(x(u), y(u, v)) - \Psi^i_j(x(u), y(u, v))$$

Thus we have an n-dimensional differentiable distribution $N: (x, y) \to N_{(x,y)} \subset T_{(x,y)}(TM^n)$, $(x, y) \in i_*(TM^m)$, locally spanned by $\delta/\delta x^i = \partial/\partial x^i - N^j_i \partial/\partial y^j$.

Hypothesis 1. *In the present paper we suppose there exists a prolongation of the distribution* N *to* TM^n *and this prolongation will be denoted by* N, *too.*

Proposition 1.1. *The non-linear connections* $N(N^i_j)$ *and* $\underline{\overset{c}{N}}(N^\alpha_\beta)$ *are related by*

$$(1.9)\qquad B^j_\alpha N^i_j - N^\beta_\alpha B^i_\beta + B^i_{0\alpha} = 0,$$

at each point of $i_*(TM^m)$.

Remark 1.1. Then non-linear connection N on TM^n has the (H)-property with respect to $\underline{\overset{c}{N}}$ on TM^m. (For the study of non-linear connection with (H)-property see M i r o n [6]).

Now we state

Theorem 1.2. *Let* F^m *be a Finsler subspace of a Finsler space* F^n. *Then there exists a unique Finsler connection* $F\Gamma(N)_* = (N^i_j, F^i_{jk}, C^i_{jk})$ *on* F^n *satisfying conditions:*

(i) *its non-linear connection* $N(N^i_j)$ *is given by* (1.8) *and Hypothesis* 1,

(ii) *the (h) and (v)-covariant derivatives of* g_{ij} *with respect to* $F\Gamma(N)_*$ *vanish*,

(iii) $T^i_{jk} = 0$ *and* $S^i_{jk} = 0$.

The Finsler connection $F\Gamma(N)_*$ is called the *i_*-Finsler connection on* F^n. Our study will be mainly based on the intrinsic Finsler connection on F^m and the i_*-Finsler connection on F^n.

§2. The induced Finsler connection on F^m and the normal Finsler derivatives. We denote by D the covariant differentiation with respect to an arbitrary Finsler connection $F\Gamma(N) = (N^i_j, F^i_{jk}, C^i_{jk})$ on F^n. Then if $X^i(x,y)$ is a Finsler vector field on F^n we have

$$(2.1)\qquad DX^i = X^i{}_{|k} dx^k + X^i|_k \delta y^k,$$

where we put

$$(2.2)\qquad X^i{}_{|k} = \delta_k X^i + X^h F^i_{hk}\,;\ X^i|_k = \dot{\partial}_k X^i + X^h C^i_{hk} \text{ and}$$

(2.3) $$\delta y^k = dy^k + N^k_j dx^j.$$

Now, we denote by $\underline{D}$ the covariant differentiation with respect to an arbitrary Finsler connection $F\Gamma(N^\alpha_\beta, F^\alpha_{\beta\gamma}, C^\alpha_{\beta\gamma})$ on F^m. Then there are satisfied similar relations to (2.1)—(2.3).

We say that $F\Gamma$ is an *induced Finsler connection* on F^m by the Finsler connection $F\Gamma(N)$ on F^n if we have

(2.4) $$\underline{D}X^\alpha = B^\alpha_i DX^i \quad \text{and} \quad (2.5) \quad \delta v^\alpha = B^\alpha_i \delta y^i,$$

where (X^i) and (X^α) are related by

(2.6) $$X^i = B^i_\alpha(u) X^\alpha.$$

Theorem 2.1. *Let F^m be a Finsler subspace of a Finsler space F^n. Then there exists a unique induced Finsler connection on F^m by the i_*-Finsler connection $F\Gamma(N)_*$ on F^n and it is just the intrinsic Finsler connection $C\Gamma(\overset{c}{\underline{N}})$ on F^m.*

Remark 2.1. The definition of the induced Finsler connection is the same with that one given by Matsumoto in [3]. The main point is that we have a situation when the induced Finsler connection on F^m is just its intrinsic Finsler connection and thus the computations will be much simplified.

Next, we consider a Finsler vector field X^i normal to F^m, i.e., we have

(2.7) $$X^i = X^a B^i_a .$$

Then we define (h)-normal Finsler derivative by

(2.8) $$X^a_{\perp\alpha} = \delta_\alpha X^a + L^a_{b\alpha} X^b ; \quad f_{\perp\alpha} = \delta_\alpha f,$$

where $L^a_{b\alpha} = B^j_{b|\alpha} B^a_j$ and f is an arbitrary scalar field on F^m. In a similar way, the (v)-*normal Finsler derivative* is given by

(2.9) $$X^a \perp_\alpha = \dot\partial_\alpha X^a + M^a_{b\alpha} X^b ; \quad f \perp_\alpha = \dot\partial_\alpha f,$$

where $M^a_{b\alpha} = B^j_b|_\alpha B^a_j$. By means of the (h) and (v)-normal Finsler derivatives we define a new relative differentiation. Then from some commutations formulas we obtain three curvature tensors given by

(2.10) $$R^a_{b\alpha\beta} = \delta_\beta L^a_{b\alpha} - \delta_\alpha L^a_{b\beta} + L^a_{c\beta} L^c_{b\alpha} - L^a_{c\alpha} L^c_{b\beta} + R^\varphi_{\alpha\beta} M^a_{b\varphi},$$

(2.11) $$P^a_{b\alpha\beta} = \dot\partial_\beta L^a_{b\alpha} - \delta_\alpha M^a_{b\beta} + M^a_{c\beta} L^b_{b\alpha} - L^a_{c\alpha} M^c_{b\beta} + M^a_{b\varphi} \frac{\partial N^\varphi_\alpha}{\partial v^\beta},$$

(2.12) $$S^a_{b\alpha\beta} = \dot\partial_\beta M^a_{b\alpha} - \dot\partial_\alpha M^a_{b\beta} + M^a_{c\beta} M^c_{b\alpha} - M^a_{c\alpha} M^c_{b\beta} .$$

3. The fundamental equations of a generalized Finsler subspace. Let F^m be a generalized Finsler subspace of F^n. We consider the i_*-Finsler connection $F\Gamma(N)_*$ on F^n and the intrinsic Finsler connection $C\Gamma(\overset{\bullet}{\underline{N}})$ on F^m which by Theorem 2.1 is the induced Finsler connection on F^m by the i_*-Finsler connection. We define the following Finsler tensor fields

(4.1) $$\begin{cases} H^a_{\alpha\beta} = B^a_i (B^i_{\alpha\beta} + F^i_{kj} B^k_\alpha B^j_\beta), \quad K^a_{\alpha\beta} = B^a_i B^k_\alpha B^j_\beta C^i_{kj}, \\ H^\alpha_{a\beta} = g^{\alpha\varphi} H^a_{\varphi\beta} ; \quad K^\alpha_{a\beta} = g^{\alpha\varphi} K^a_{\varphi\beta} . \end{cases}$$

Then we obtain

(4.2) $$B^i_{\alpha|\beta} = H^a_{\alpha\beta} B^i_a \quad \text{and} \quad (4.3) \quad B^i_\alpha|_\beta = K^a_{\alpha\beta} B^i_a$$

We call (4.2) and (4.3) the *Gauss' formulas* and $H^a_{\alpha\beta}$, $K^a_{\alpha\beta}$ the second fundamental forms of the immersion of F^m in F^n. On the other hand, we obtain

(4.4) $$B^i_{a|\alpha} = -H^\varphi_{a\alpha} B^i_\varphi + L^b_{a\alpha} B^i_b \quad \text{and}$$

(4.5) $$B^i_a|_\alpha = -K^\varphi_{a\alpha} B^i_\varphi + M^b_{a\alpha} B^i_b ,$$

which are called the *Weingarten's formulas*.

Now we denote by $(R^i_{mjk}, P^i_{mjk}, S^i_{mjk})$ and $(R^\alpha_{\beta\gamma\delta}, P^\alpha_{\beta\gamma\delta}, S^\alpha_{\beta\gamma\delta})$ the curvature tensors of the i_*-Finsler connection on F^n and respectively the curvature tensors of the intrinsic Finsler connection on F^m. Then we get the following structure equations of F^m in F^n:

(4.6) $$\begin{aligned} &R^i_{mjk} B^m_\alpha B^j_\beta B^k_\gamma B^\varphi_i = R^\varphi_{\alpha\beta\gamma} + H^a_{\alpha\gamma} H^\varphi_{a\beta} - H^a_{\alpha\beta} H^\varphi_{a\gamma}, \quad P^i_{mjk} B^m_\alpha B^j_\beta B^k_\gamma B^\varphi_i = P^\varphi_{\alpha\beta\gamma} + \\ &+ K^a_{\alpha\gamma} H^\gamma_{a\beta} - K^a_{\alpha\beta} H^\varphi_{a\gamma}, \quad S^i_{mjk} B^m_\alpha B^j_\beta B^k_\gamma B^\varphi_i = S^\varphi_{\alpha\beta\gamma} + K^a_{\alpha\gamma} K^\varphi_{a\beta} - K^a_{\alpha\beta} K^\varphi_{a\gamma}, \end{aligned}$$

(4.7) $$\begin{aligned} &R^i_{mjk} B^m_a B^j_\gamma B^k_\delta B^\varphi_i = H^\varphi_{a\delta\perp\gamma} - H^\varphi_{a\gamma\perp\delta} - R^\psi_{\gamma\delta} K^\varphi_{a\psi}, \quad P^i_{mjk} B^m_a B^j_\gamma B^k_\delta B^\varphi_i = \\ &= K^\varphi_{a\delta\perp\gamma} - H^\varphi_{a\gamma} \perp_\delta - C^\psi_{\gamma\delta} H^\varphi_{a\psi} - P^\psi_{\gamma\delta} K^\varphi_{a\psi}, \quad S^i_{mjk} B^m_a B^j_\gamma B^k_\delta B^\varphi_i = K^\varphi_{a\delta} \perp_\gamma - K^\varphi_{a\gamma} \perp_\delta , \end{aligned}$$

(4.8) $$\begin{aligned} &R^i_{mjk} B^b_i B^m_a B^j_\gamma B^k_\delta = R^b_{a\gamma\delta} + H^\varphi_{a\delta} H^b_{\varphi\gamma} - H^\varphi_{a\gamma} H^b_{\varphi\delta}, \quad P^i_{mjk} B^b_i B^m_a B^j_\gamma B^k_\delta = P^b_{a\gamma\delta} + \\ &+ K^\varphi_{a\delta} H^b_{\varphi\gamma} - H^\varphi_{a\gamma} K^b_{\varphi\delta}, \quad S^i_{mjk} B^b_i B^m_a B^j_\gamma B^k_\delta = S^b_{a\gamma\delta} + K^\varphi_{a\delta} K^b_{\varphi\gamma} - K^\varphi_{a\gamma} K^b_{\varphi\delta} . \end{aligned}$$

The equations (4.6), (4.7) and (4.8) are called the *equations of Gauss, Codazzi* and respectively *Ricci* for the immersion of F^m in F^n. It is remarkable that, for the first time, the structure equations of F^m in F^n have a simplified form and thus they could be handled in a further study of Finsler subspaces.

Proofs and details will be given elsewhere in a forthcoming paper [7].

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University of Iaşi
Department of Mathematics
and
Polytechnic Institute of Iaşi
Department of Mathematics