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PONTRYAGIN CLASSES IN FINSLER GEOMETRY

BY

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In this paper we define the Pontryagin classes associated to a non-linear connection. We give some results concerning these characteristic classes in the case of the metrical Finsler structure.

Our paper is based on the new idea of vector bundle Finsler geometry, presented by professor Radu Miron in [5].

1. Let $\xi=(E, \pi, M)$ be a smooth vector bundle of fibre dimension m over the smooth n-dimensional and paracompact manifold M and denote by VE the vertical bundle Ker π^T. Suppose that N is a nonlinear connection on ξ. Let T^{qs} the space of Finsler forms of type (q, s). For a Finsler connection ∇ we denote by R the curvature tensor. It is well-known that

$$(1)\qquad (R(X, Y)Z^H)^V=(R(X, Y)Z^V)^H$$

for any $X, Y, Z \in \mathcal{X}(E)$, and then we define a map R^V by the formula

$$(2)\qquad R^V(X, Y)Z = R(X^V, Y^V)Z^V\,;$$

R^V has the following properties

$$R^V(X, Y)Z=R^V(X, Y)Z^V\,;\quad R^V(Y, X)=-R^V(X, Y),$$
$$(R^V(X, Y)Z)^H=0\,;\quad R^V(X^H, Y^V)=R^V(X^H, Y^V)=0,$$
$$R^V(fX, Y)=fR^V(X, Y),\quad f\in\mathcal{F}(E),$$

and then it may be regarded as a Finsler form of type (0, 2) with values in $gl(E^V)$.

The operator defined by

$$(d''\omega)(X_1, \ldots, X_{q+s+1})=$$
$$=\sum_{i=q+1}^{q+s+1}(-1)^{i+1}(\nabla^V_{X_i}\omega)(X^H_1, \ldots, X^V_q, X^H_{q+1}, \ldots, \widehat{X^V_i}, \ldots, X^V_{q+s+1})-$$
$$(3)\qquad -\sum_{\substack{1\leq i\leq q\\ q+1\leq j\leq q+s+1}}(-1)^{i+j}\omega(T^H(X^H_i, X^V_j), X^H_1, \ldots, \widehat{X^H_i}, \ldots, X^H_q,$$
$$X^V_{q+1}, \ldots, \widehat{X^V_j}, \ldots, X^V_{q+s+1})-\sum_{q+1\leq i<j\leq q+s+1}(-1)^{i+j}\omega(T^V, (X^V_i, X^V_j), X^H_1, \ldots, X^H_q,$$
$$X^V_{q+1}, \ldots, \widehat{X^V_i}, \ldots, \widehat{X^V_j}, \ldots, X^V_{q+s+1})$$

where $\omega \in T^{qs}$ and T is the torsion of the Finsler connection ∇, has the properties of a differential and $d''T^{qs} \subseteq T^{q,s+1}$. Now a straightforward computation involving the equalities (2) and (3) prove that R^V is d''-closed.

Let $P : \times_k gl(R^m) \to R$ be a k-linear symmetric form, invariant under change of basis in R^m and denote by $I^k(gl(R^m))$ the module of these forms. Because, for every $u \in E$, we have $E_u^V \approx R^m$, a $2k$-form on E may be defined by putting

(4) $$P(R^V)(X_1, \ldots, X_{2k}) = \sum_{\sigma \in S_{2k}} (-1)^{\varepsilon(\sigma)} P(R^V(X_{\sigma(1)}, X_{\sigma(2)}), \ldots, R^V(X_{\sigma(2k-1)}, X_{\sigma(2k)}))$$

where $P \in I^k(gl(R^m))$, S_{2k} is the set of all permutations of $\{1, 2, \ldots, 2k\}$ and $\varepsilon(\sigma)$ is the signature of $\sigma \in S_{2k}$. By a direct computation it follows that $P(R^V)$ defined in (4) is a d''-closed Finsler form of type $(0, 2k)$. Moreover we can state the following.

Theorem 1. *The d''-cohomology class defined by $P(R^V)$ is independent on the Finsler connection used to define it*

This cohomology class is called the *d''-Pontryagin class of order $2k$* corresponding to P.

If $P \in I^k(gl(R^m))$, $k > m$, then $P(R^V) = 0$ and in this manner we obtain the following:

Theorem 2. *The d''-Pontryagin classes of order $2k$ vanish for all $k > m$.*

2. We can obtained some interesting results in the case of metrical Finsler structures. For this, we first need to introduce the notion of v-Riemannian Finsler connection.

Let G be a metrical Finsler structure on the total space E of the vector bundle ξ.

Definition 1. *A Finsler connection ∇ is called v-Riemannian if $\nabla_X^V G^V = 0$ for any $X \in \mathfrak{X}(E)$, where $G^V(X, Y) = G(X^V, Y^V)$.*

It is clear from the above definition that for a v-Riemannian Finsler connection we have

(5) $$X^V(G^V(Y, Z)) = G^V(\nabla_X^V Y, Z) + G^V(Y, \nabla_X^V Z).$$

Proposition 3. *If ∇ is a v-Riemannian Finsler connection then*

$$G^V(R^V(X, Y)Z, U) = -G^V(R^V(X, Y)U, Z)$$

for any $X, Y, Z, U \in \mathfrak{X}(E)$.

If $\{X^a\}_{a \in \overline{1,m}}$ is an orthonormal basis of E^V then

(6) $$G^V(X^a, X^b) = \delta^{ab},$$

and by the proposition 3 we can write

(7) $$(R^V)_b^a(X, Y) = -(R^V)_a^b(X, Y),$$

where $((R^V)_b^a(X, Y))$ is the matrix associated to $R^V(X, Y)$ by choosing an isomorphism $E_u^V \approx R^m$. Now (7) proves that this matrix is skew-symmetric, and for $P \in I^k(gl(R^m))$ we have $P(R^V) = P((R^V)^{tr})$, where A^{tr} is the transposed matrix of A. But P is invariant and then we can state the following.

Theorem 4. *The d''-Pontryagin classes of order different from a multiple of 4 are zero.*

Note. If ∇^0 and ∇^1 are two v-Riemannian Finsler connections then $\int_0^1 P(\delta^V, (R^t)^V, \ldots, (R^t)^V)dt$ defines a d''-cohomology class. (Here R^t is the curvature form of the Finsler connection $\nabla^t = \nabla^0 + t(\nabla^1 - \nabla^0)$, $t \in [0, 1]$, and δ^V is defined by $\delta^V(X)Y = (\nabla^1{}_X^V - \nabla^0{}_X^V)Y^V$). It seems that this cohomology class is independent on the choice of the connection ∇^0 and ∇^1, but we haven't a complete proof.

3. Suppose now that the distribution N is integrable and $m > n$.

If $\omega_1, \ldots, \omega_n$ are independent Finsler forms of type $(1, 0)$ then $\Omega = \omega_1 \wedge \ldots \wedge \omega_n$ is a Finsler form of type $(n, 0)$ and $d\Omega = d''\Omega$. Moreover, from (3) it follows

(8) $$d''\Omega = \sum_{i=1}^{n} (-1)^{i-1} \omega_1 \wedge \ldots \wedge d\omega_i \wedge \ldots \wedge \omega_n,$$

and if $\theta_1, \ldots, \theta_m$ are m independent Finsler forms of type $(0, 1)$ then we have

(9) $$d\omega_i = A_i^{ja} \omega_j \wedge \theta_a + A_i^{jk} \omega_j > \omega_k$$

Now denoting by θ the Finsler form $-A_i^{ia}\theta_a$ we deduce that $d\theta \wedge \Omega = 0$ and as $d\theta = A^{ia}\omega_i \wedge \theta_a$ it follows $(d\theta)^{n+1} = 0$. In this manner we have obtained a d''-closed Finsler form $\overline{\wedge} = \theta \wedge (d\theta)^n$ of type $(n, n+1)$.

A direct computation shows that

Theorem 5. *The d''-cohomology class defined by $\overline{\wedge}$ is independent on the Finsler forms θ_a and ω_i.*

This invariant is called the *d''-Godbillon-Vey class.*

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